Available online at www.sciencedirect.com
ScienceDirect
Journal of Sound and Vibration 318 (2008) 965–981
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Free vibration of non-uniform Euler–Bernoulli beams with general elastically end constraints using Adomian modified decomposition method

Jung-Chang Hsu, Hsin-Yi Lai, C.K. Chen*

Department of Mechanical Engineering, National Cheng-Kung University, Tainan 70101, Taiwan, ROC

Received 13 February 2008; received in revised form 8 May 2008; accepted 10 May 2008
Handling Editor: L.G. Tham
Available online 18 June 2008

Abstract

This paper deals with free vibration problems of non-uniform Euler–Bernoulli beam under various supporting conditions. The technique we have used is based on applying the Adomian modified decomposition method (AMDM) to our vibration problems. Doing some simple mathematical operations on the method, we can obtain *i*th natural frequencies and mode shapes one at a time. The computed results agree well with those analytical and numerical results given in the literatures. These results indicate that the present analysis is accurate, and provides a unified and systematic procedure which is simple and more straightforward than the other modal analysis.

1. Introduction

The vibration problems of non-uniform Euler–Bernoulli beams with general elastically end constraints is a problem that has been extensively studied by several investigators. Mabie and Rogers [1,2] studied several cases of tapered beams with different end conditions. Laura [3,4] treated various cases of non-uniform beams with different conditions of end restraints. Naguleswaran [5–7] obtained a direct solution for the transverse vibration of Euler–Bernoulli wedge and cone beams. Goel [8] analyzed the effect of rotational restraint at either end of a linearly tapered beam. Grossi [9–11] studied various problems of tapered beams with elastically restrained ends by means of the Rayleigh–Ritz and Rayleigh–Schmidt methods. Abrate [12] obtained the exact solution for the vibration of non-uniform rods and beams. Lee et al. [13] studied the analysis of non-uniform beam vibration by a green function method in the Laplace transform domain. Ho and Chen [14] studied the analysis of general elastically end restrained non-uniform beams using differential transform. Finally, Auciello [15,16] studied the free vibrations of tapered beams with flexible ends by using Bessel's functions.

In this study, a new computed approach called Adomian modified decomposition method (AMDM) is introduced to solve the vibration problems. The concept of AMDM was first proposed by Adomian and was

*Corresponding author. Tel.: +886 6 2757575.
E-mail address: ckchen@mail.ncku.edu.tw (C.K. Chen).

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2008.05.010

applied to solve linear and nonlinear initial/boundary-value problems in physics [17–20]. In this paper, the vibration problems of non-uniform beams with various classical boundary conditions and flexible end conditions are considered. Using the AMDM, the governing differential equation becomes a recursive algebraic equation and boundary conditions become simple algebraic frequency equations which are suitable for symbolic computation. Moreover, after some simple algebraic operations on these frequency equations of any ith natural frequency, the closed-form series solution of any ith mode shape can be obtained. Finally, some problems of non-uniform beams are solved and show excellent agreement with the published results to verify the accuracy and efficiency of the present method.

2. The principle of AMDM

In order to solve vibration problems by the AMDM, the basic theory is stated in brief in this section. Consider the equation

$$Fy(x) = g(x), \tag{1}$$

where F represents a general nonlinear ordinary differential operator involving both linear and nonlinear parts, and $g(x)$ is a given function. The linear terms in Fy are decomposed into $Ly+Ry$, where L is an invertible operator, which is taken as the highest-order derivative and R is the remainder of the linear operator. Thus, Eq. (1) can be written as

$$Ly + Ry + Ny = g, \tag{2}$$

where Ny represents the nonlinear terms in Fy. Eq. (2) corresponds to an initial-value problem or a boundary-value problem. Solving for Ly, one can obtain

$$y = \Phi + L^{-1}g - L^{-1}Ry - L^{-1}Ny, \tag{3}$$

where Φ is an integration constant, and $L\Phi = 0$ is satisfied. Corresponding to an initial-value value problem, the operator L^{-1} may be regarded as a definite integration from 0 to x. In order to solve Eq. (3) by the AMDM, we decompose y into the infinite sum of convergent series

$$y = \sum_{k=0}^{\infty} c_k x^k, \tag{4}$$

and the nonlinear term Ny is decomposed as

$$Ny = \sum_{k=0}^{\infty} x^k A_k(c_0, c_1, \ldots, c_k), \tag{5}$$

where the A_k are known as Adomian coefficients. The given function $g(x)$ is also decomposed as

$$g(x) = \sum_{k=0}^{\infty} g_k x^k. \tag{6}$$

By plugging Eqs. (4)–(6) into Eq. (3) gives

$$y = \sum_{k=0}^{\infty} c_k x^k = \Phi + L^{-1}\left(\sum_{k=0}^{\infty} g_k x^k\right) - L^{-1}R\left(\sum_{k=0}^{\infty} c_k x^k\right) - L^{-1}\left(\sum_{k=0}^{\infty} x^k A_k(c_0, c_1, \ldots, c_k)\right). \tag{7}$$

The coefficients c_k of each term in series (7) can be decided by the recurrence relation, and the power series solutions of linear homogeneous differential equations in initial-value problems yield simple recurrence relations for the coefficients c_k. However, in practice all the coefficients c_k in series (7) cannot be determined exactly, and the solutions can only be approximated by a truncated series $\sum_{k=0}^{n-1} c_k x^k$.

3. Using the AMDM to analyze the free vibration problem of non-uniform beam

Let us consider the tapered beam of length l, shown in Fig. 1, with the rotational and translational flexible ends. The equation of motion for transverse vibrations of a non-uniform elastic beam is given by

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right] + \rho A(x)\frac{\partial^2 y(x,t)}{\partial t^2} = 0, \quad 0<x<l, \tag{8}$$

where $y(x, t)$ is the transverse deflection, E is Young's modulus, $A(x)$ is the cross-sectional area at the position x, $I(x)$ is the moment of inertia of $A(x)$, ρ is the mass density of the beam material and t is time.

For any mode of vibration, the lateral deflection $y(x, t)$ may be written in the form

$$y(x,t) = Y(x)h(t), \tag{9}$$

where $Y(x)$ is the modal deflection and $h(t)$ is a harmonic function of time t. If ω denotes the circular frequency of $h(t)$, then

$$\frac{\partial^2 y(x,t)}{\partial t^2} = -\omega^2 Y(x)h(t), \tag{10}$$

and the eigenvalue problem of Eq. (8) reduces to the differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[EI(x)\frac{\mathrm{d}^2 Y(x)}{\mathrm{d}x^2}\right] - \rho A(x)\omega^2 Y(x) = 0, \quad 0<x<l. \tag{11}$$

The boundary conditions, in the presence of constraints with the translational spring constants k_{TL}, k_{TR}, the rotational spring constants k_{RL}, are given by

$$EI(x)\frac{\mathrm{d}^2 Y(x)}{\mathrm{d}x^2} - k_{\mathrm{RL}}\frac{\mathrm{d}Y(x)}{\mathrm{d}x} = 0, \tag{12}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[EI(x)\frac{\mathrm{d}^2 Y(x)}{\mathrm{d}x^2}\right] + k_{\mathrm{TL}} Y(x) = 0 \tag{13}$$

at $x = 0$, and

$$EI(x)\frac{\mathrm{d}^2 Y(x)}{\mathrm{d}x^2} + k_{\mathrm{RR}}\frac{\mathrm{d}Y(x)}{\mathrm{d}x} = 0, \tag{14}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[EI(x)\frac{\mathrm{d}^2 Y(x)}{\mathrm{d}x^2}\right] - k_{\mathrm{TR}} Y(x) = 0 \tag{15}$$

at $x = l$.

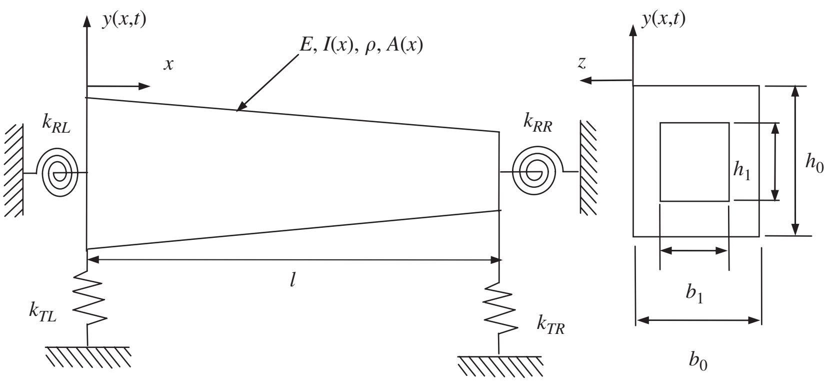

Fig. 1. A linearly tapered Euler–Bernoulli beam with rotational and translational flexible ends.

Assuming both the depth $b(x)$ and the height $h(x)$ of the cross-section can vary linearly according to the taper ratios of the beam $\alpha_b = b_1/b_0$ and $\alpha_h = h_1/h_0$, that is,

$$b(x) = b_0\left[1 + (\alpha_b - 1)\frac{x}{l}\right], \quad h(x) = h_0\left[1 + (\alpha_h - 1)\frac{x}{l}\right], \tag{16}$$

where b_0, b_1 are the cross-sectional depths at $x = 0$ and l, respectively, and h_0, h_1 are the cross-sectional heights at $x = 0$ and l, respectively, then the area and moment of inertia of the section will vary according to the following laws:

$$A(x) = b(x)h(x) = A_0\left[1 + (\alpha_b - 1)\frac{x}{l}\right]\left[1 + (\alpha_h - 1)\frac{x}{l}\right], \tag{17}$$

$$I(x) = \frac{b(x)[h(x)]^3}{12} = I_0\left[1 + (\alpha_b - 1)\frac{x}{l}\right]\left[1 + (\alpha_h - 1)\frac{x}{l}\right]^3, \tag{18}$$

where $A_0 = b_0h_0$ and $I_0 = b_0h_0^3/12$ are the cross-sectional area and the moment of inertia at $x = 0$. By setting

$$\beta_b = 1 - \alpha_b, \quad \beta_h = 1 - \alpha_h, \tag{19}$$

Eq. (11) can be written as

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[\left(1 - \beta_b\frac{x}{l}\right)\left(1 - \beta_h\frac{x}{l}\right)^3\frac{\mathrm{d}^2Y(x)}{\mathrm{d}x^2}\right] - \frac{\rho A_0\omega^2}{EI_0}\left(1 - \beta_b\frac{x}{l}\right)\left(1 - \beta_h\frac{x}{l}\right)Y(x) = 0, \tag{20}$$

and the boundary conditions of Eqs. (12)–(15) can also be written as

$$\frac{\mathrm{d}^2Y(x)}{\mathrm{d}x^2} - \frac{k_{RL}}{EI_0}\frac{\mathrm{d}Y(x)}{\mathrm{d}x} = 0, \tag{21}$$

$$\frac{\mathrm{d}^3Y(x)}{\mathrm{d}x^3} - \frac{(\beta_b + 3\beta_h)}{l}\frac{\mathrm{d}^2Y(x)}{\mathrm{d}x^2} + \frac{k_{\mathrm{TL}}}{EI_0}Y(x) = 0 \tag{22}$$

at $x = 0$, and

$$\frac{\mathrm{d}^2Y(x)}{\mathrm{d}x^2} + \frac{k_{\mathrm{RR}}}{EI_1}\frac{\mathrm{d}Y(x)}{\mathrm{d}x} = 0, \tag{23}$$

$$\frac{\mathrm{d}^3Y(x)}{\mathrm{d}x^3} - \frac{1}{l}\left(\frac{\beta_b}{1 - \beta_b} + \frac{3\beta_h}{1 - \beta_h}\right)\frac{\mathrm{d}^2Y(x)}{\mathrm{d}x^2} - \frac{k_{\mathrm{TR}}}{EI_1}Y(x) = 0 \tag{24}$$

at $x = l$, where $I_1 = \alpha_b\alpha_h^3I_0 = (1 - \beta_b)(1 - \beta_h)^3I_0$.

Without loss of generality, the following dimensionless quantities are introduced.

$$X = \frac{x}{l}, \quad Y(X) = \frac{Y(x)}{l}, \quad \lambda = \Omega^2 = \frac{\rho A_0\omega^2l^4}{EI_0}, \tag{25}$$

where $\Omega = \omega\sqrt{\rho A_0l^4/EI_0}$ is the dimensionless natural frequency of the beam, then Eq. (20) simplifies in the dimensionless form as follows:

$$\frac{\mathrm{d}^2}{\mathrm{d}X^2}\left[(1 - \beta_bX)(1 - \beta_hX)^3\frac{\mathrm{d}^2Y(X)}{\mathrm{d}X^2}\right] - \lambda(1 - \beta_bX)(1 - \beta_hX)Y(X) = 0, \quad 0 < X < 1. \tag{26}$$

Eq. (26) can be expanded as follows:

$$\frac{\mathrm{d}^4Y(X)}{\mathrm{d}X^4} - 2\left(\frac{\beta_b}{1 - \beta_bX} + \frac{3\beta_h}{1 - \beta_hX}\right)\frac{\mathrm{d}^3Y(X)}{\mathrm{d}X^3}$$
$$+ 6\left[\frac{\beta_b\beta_h}{(1 - \beta_bX)(1 - \beta_hX)} + \frac{\beta_h^2}{(1 - \beta_hX)^2}\right]\frac{\mathrm{d}^2Y(X)}{\mathrm{d}X^2} - \frac{\lambda}{(1 - \beta_hX)^2}Y(X) = 0. \tag{27}$$

It is convenient to define the following dimensionless coefficients:

$$K_{\mathrm{TL}}=\frac{k_{\mathrm{TL}}l^3}{EI_0},\quad K_{\mathrm{TR}}=\frac{k_{\mathrm{TR}}l^3}{EI_1},\quad K_{\mathrm{RL}}=\frac{k_{\mathrm{RL}}l}{EI_0},\quad K_{\mathrm{RR}}=\frac{k_{\mathrm{RR}}l}{EI_1}, \tag{28}$$

the boundary conditions of Eqs. (21)–(24) are given by the following dimensionless forms:

$$Y''(0)-K_{\mathrm{RL}}\,Y'(0)=0, \tag{29}$$

$$Y'''(0)-(\beta_b+3\beta_h)\,Y''(0)+K_{\mathrm{TL}}\,Y(0)=0, \tag{30}$$

and

$$Y''(1)+K_{\mathrm{RR}}\,Y'(1)=0, \tag{31}$$

$$Y'''(1)-\left(\frac{\beta_b}{1-\beta_b}+\frac{3\beta_h}{1-\beta_h}\right)Y''(1)-K_{\mathrm{TR}}\,Y(1)=0, \tag{32}$$

where $Y'(X)=\mathrm{d}Y(X)/\mathrm{d}X,\ Y''(X)=\mathrm{d}^2Y(X)/\mathrm{d}X^2,\ Y'''(X)=\mathrm{d}^3Y(X)/\mathrm{d}X^3$.

The deflection $Y(X)$ can be solved by the AMDM. Eq. (27) can be expressed in the following form:

$$\begin{aligned}Y(X)=\Phi(X)+L^{-1}\Bigg\{&2\left(\frac{\beta_b}{1-\beta_bX}+\frac{3\beta_h}{1-\beta_hX}\right)\frac{\mathrm{d}^3Y(X)}{\mathrm{d}X^3}\\&-6\left[\frac{\beta_b\beta_h}{(1-\beta_bX)(1-\beta_hX)}+\frac{\beta_h^2}{(1-\beta_hX)^2}\right]\frac{\mathrm{d}^2Y(X)}{\mathrm{d}X^2}+\frac{\lambda}{(1-\beta_hX)^2}Y(X)\Bigg\},\end{aligned} \tag{33}$$

where $L^{-1}=\int_0^x\int_0^x\int_0^x\int_0^x\cdots\mathrm{d}X\,\mathrm{d}X\,\mathrm{d}X\,\mathrm{d}X$. Now the decomposition $Y(X)=\sum_{k=0}^{\infty}C_kX^k$ can be put together with Eq. (33) to yield

$$\begin{aligned}Y(X)=\sum_{k=0}^{\infty}C_kX^k=\Phi(X)+L^{-1}\Bigg\{&2\left(\frac{\beta_b}{1-\beta_bX}+\frac{3\beta_h}{1-\beta_hX}\right)\sum_{k=0}^{\infty}(k+3)(k+2)(k+1)C_{k+3}X^k\\&-6\left[\frac{\beta_b\beta_h}{(1-\beta_bX)(1-\beta_hX)}+\frac{\beta_h^2}{(1-\beta_hX)^2}\right]\sum_{k=0}^{\infty}(k+2)(k+1)C_{k+2}X^k\\&+\frac{\lambda}{(1-\beta_hX)^2}\sum_{k=0}^{\infty}C_kX^k\Bigg\},\end{aligned} \tag{34}$$

where we have

$$\Phi(X)=Y(0)+Y'(0)X+\frac{Y''(0)}{2}X^2+\frac{Y'''(0)}{6}X^3 \tag{35}$$

as the initial term of the decomposition. By using the power series, one can obtain

$$\frac{1}{1-\beta_bX}=\sum_{j=0}^{\infty}(\beta_bX)^j,\quad \frac{1}{1-\beta_hX}=\sum_{j=0}^{\infty}(\beta_hX)^j,\quad \beta_b\neq0,\ \beta_h\neq0 \tag{36}$$

and using the Cauchy product, one can also obtain

$$\frac{1}{(1-\beta_bX)(1-\beta_hX)}=\sum_{j=0}^{\infty}(\beta_bX)^j\sum_{j=0}^{\infty}(\beta_hX)^j=\sum_{j=0}^{\infty}X^j\sum_{m=0}^{j}\beta_b^m\beta_h^{j-m}, \tag{37}$$

$$\frac{1}{(1-\beta_hX)^2}=\sum_{j=0}^{\infty}(\beta_hX)^j\sum_{j=0}^{\infty}(\beta_hX)^j=\sum_{j=0}^{\infty}(j+1)(\beta_hX)^j, \tag{38}$$

and

$$\frac{1}{1-\beta_b X}\sum_{k=0}^{\infty}(k+3)(k+2)(k+1)C_{k+3}X^k = \sum_{k=0}^{\infty}X^k\sum_{j=0}^{k}\beta_b^{(k-j)}(j+3)(j+2)(j+1)C_{j+3}, \tag{39}$$

$$\frac{1}{1-\beta_h X}\sum_{k=0}^{\infty}(k+3)(k+2)(k+1)C_{k+3}X^k = \sum_{k=0}^{\infty}X^k\sum_{j=0}^{k}\beta_h^{(k-j)}(j+3)(j+2)(j+1)C_{j+3}, \tag{40}$$

$$\frac{1}{(1-\beta_b X)(1-\beta_h X)}\sum_{k=0}^{\infty}(k+2)(k+1)C_{k+2}X^k = \sum_{k=0}^{\infty}X^k\sum_{j=0}^{k}\sum_{m=0}^{k-j}\beta_b^m\beta_h^{k-j-m}(j+2)(j+1)C_{j+2}, \tag{41}$$

$$\frac{1}{(1-\beta_h X)^2}\sum_{k=0}^{\infty}(k+2)(k+1)C_{k+2}X^k = \sum_{k=0}^{\infty}X^k\sum_{j=0}^{k}\beta_h^{k-j}(k-j+1)(j+2)(j+1)C_{j+2}, \tag{42}$$

$$\frac{1}{(1-\beta_h X)^2}\sum_{k=0}^{\infty}C_kX^k = \sum_{k=0}^{\infty}X^k\sum_{j=0}^{k}\beta^{k-j}(k-j+1)C_j. \tag{43}$$

By substituting Eqs. (39)–(43) into Eq. (34), one can obtain

$$\begin{aligned}\sum_{k=0}^{\infty}C_kX^k = \Phi(X) + L^{-1}\Bigg\{&\sum_{k=0}^{\infty}X^k\sum_{j=0}^{k}\Big[(j+3)(j+2)(j+1)\Big(2\beta_b^{k-j+1}+6\beta_h^{(k-j+1)}\Big)C_{j+3}\\ &-(j+2)(j+1)\left(6(k-j+1)\beta_h^{k-j+2}+6\sum_{m=0}^{k-j}\beta_b^{m+1}\beta_h^{k-j-m+1}\right)C_{j+2}+\lambda(k-j+1)\beta_h^{k-j}C_j\Bigg\}. \end{aligned}\tag{44}$$

By integrating Eq. (44), one can obtain

$$\begin{aligned}\sum_{k=0}^{\infty}C_kX^k = {} & Y(0)+Y'(0)X+\frac{Y''(0)}{2}X^2+\frac{Y'''(0)}{6}X^3+\sum_{k=0}^{\infty}\Bigg\{\frac{X^{k+4}}{(k+1)(k+2)(k+3)(k+4)}\\ &\times\sum_{j=0}^{k}\Big[(j+3)(j+2)(j+1)\Big(2\beta_b^{k-j+1}+6\beta_h^{(k-j+1)}\Big)C_{j+3}-(k-j+1)(j+2)(j+1)\\ &\times\left(6(k-j+1)\beta_h^{k-j+2}+6\sum_{m=0}^{k-j}\beta_b^{m+1}\beta_h^{k-j-m+1}\right)C_{j+2}+\lambda(k-j+1)\beta_h^{k-j}C_j\Big]\Bigg\}. \end{aligned}\tag{45}$$

Finally, equating coefficients of like powers of X, we derive the recurrence relation for the coefficients C_k

$$C_0 = Y(0),\quad C_1 = Y'(0),\quad C_2 = \frac{Y''(0)}{2},\quad C_3 = \frac{Y'''(0)}{6}, \tag{46}$$

and for $k \geqslant 4$,

$$\begin{aligned}C_k = \frac{1}{k(k-1)(k-2)(k-3)}&\sum_{j=0}^{k-4}\Big[(j+3)(j+2)(j+1)\Big(2\beta_b^{k-j-3}+6\beta_h^{k-j-3}\Big)C_{j+3}\\ &-(j+2)(j+1)\left(6(k-j-3)\beta_h^{k-j-2}+6\sum_{m=0}^{k-j-4}\beta_b^{m+1}\beta_h^{k-j-m-3}\right)C_{j+2}+\lambda(k-j-3)\beta_h^{k-j-4}C_j\Big]. \end{aligned}\tag{47}$$

Therefore, we can find the coefficients C_k from the recurrent Eqs. (46), and (47), and finally we can get the solution $Y(X)$ from Eq. (34). The series solution, of course, is $Y(X)=\sum_{k=0}^{\infty}C_kX^k$. However, in practice all the coefficients C_k in series solution cannot be determined exactly, and the solutions can only be approximated by a truncated series $\sum_{k=0}^{n-1}C_kX^k$ with n-term approximation. We can now form successive approximants, $\phi^{[n]}(X)=\sum_{k=0}^{n-1}C_kX^k$, as n increases and the boundary conditions are also met.

Thus $\phi^{[1]}(X) = C_0$, $\phi^{[2]}(X) = \phi^{[1]}(X) + C_1 X$, $\phi^{[3]}(X) = \phi^{[2]}(X) + C_2 X^2$, serve as approximate solutions with increasing accuracy as $n \to \infty$, and is also obligated to, of course, satisfy the boundary conditions.

The four coefficients $C_k (k = 0,1,2,3)$ in Eq. (46) can be decided by the BCs of Eqs. (29) and (30). In this case, the two coefficients C_0 and C_1 can be chosen as the arbitrary constants and the other two coefficients C_2 and C_3 can be expressed as the functions of C_0 and C_1. That is, from Eqs. (29), (30) and (46), by setting

$$C_2 = \frac{K_{\mathrm{RL}}}{2} C_1, \tag{48}$$

$$C_3 = \frac{(\beta_b + 3\beta_h) K_{\mathrm{RL}} C_1 - K_{\mathrm{TL}} C_0}{6}, \tag{49}$$

the initial term $\Phi(X)$ in Eq. (35) is the function of C_0, C_1 and from recurrence relation of Eq. (47), the coefficients $C_k (k \geqslant 4)$ are the function of C_0, C_1 and λ. Hence the n-term approximation $\phi^{[n]}(X) = \sum_{k=0}^{n-1} C_k X^k$ of the modal deflection $Y(x)$ is also the function of C_0, C_1 and λ. By substituting $\phi^{[n]}(X)$ into BCs of Eqs. (31) and (32), the two equations are obtained:

$$f_{r0}^{[n]}(\lambda) C_0 + f_{r1}^{[n]}(\lambda) C_1 = 0, \quad r = 1, 2. \tag{50}$$

For non-trivial solutions C_0 and C_1 the frequency equation is given as

$$\begin{vmatrix} f_{10}^{[n]}(\lambda) & f_{11}^{[n]}(\lambda) \\ f_{20}^{[n]}(\lambda) & f_{21}^{[n]}(\lambda) \end{vmatrix} = 0. \tag{51}$$

The ith estimated eigenvalue $\lambda_i^{[n]}$ corresponding to n is obtained by Eq. (51), that is the ith estimated dimensionless natural frequency $\Omega_i^{[n]} = \sqrt{\lambda_i^{[n]}}$ is also obtained and n is decided by the following equation:

$$\left| \Omega_i^{[n]} - \Omega_i^{[n-1]} \right| \leqslant \varepsilon, \tag{52}$$

where $\Omega_i^{[n-1]}$ is the ith estimated dimensionless natural frequency corresponding to $n-1$, and ε is a preset small value. If Eq. (52) is satisfied, then $\Omega_i^{[n]}$ is the ith dimensionless natural frequency Ω_i. By substituting $\Omega_i^{[n]}$ into any one of Eq. (50), one can obtain

$$C_1 = -\frac{f_{r0}^{[n]}(\Omega_i^{[n]})}{f_{r1}^{[n]}(\Omega_i^{[n]})} C_0, \quad r = 1 \text{ or } 2, \tag{53}$$

and all the other coefficients C_k can obtain from Eqs. (46) and (47). Furthermore, the ith mode shape $\phi_i^{[n]}$ corresponding to the ith eigenvalue $\Omega_i^{[n]}$ is obtained by

$$\phi_i^{[n]}(X) = \sum_{k=0}^{n-1} C_k^{[i]} X^k, \tag{54}$$

where $C_k^{[i]}(X)$ is $C_k(X)$ whose λ is substituted by λ_i, and $\phi_i^{[n]}$ is the ith eigenfunction corresponding to the ith eigenvalue λ_i. By normalizing Eq. (54), the ith normalized eigenfunction is defined as

$$\bar{\phi}_i^{[n]}(X) = \frac{\phi_i^{[n]}(X)}{\sqrt{\int_0^1 [\phi_i^{[n]}(X)]^2 \mathrm{d}X}}, \tag{55}$$

where $\bar{\phi}_i^{[n]}(X)$ is the ith mode shape function of the beam corresponding to the ith natural frequency $\omega_i^{[n]}$, $\omega_i^{[n]} = \sqrt{\lambda_i^{[n]}} \sqrt{EI/\rho A l^4} = \Omega_i^{[n]} \sqrt{EI/\rho A l^4}$.

Finally, the free transverse vibration of non-uniform Euler–Bernoulli wedge beam ($\alpha_b = 1$, $\alpha_h = \alpha$), non-uniform Euler–Bernoulli cone beam ($\alpha_b = \alpha_h = \alpha$), and uniform Euler–Bernoulli beam ($\alpha_b = \alpha_h = 1$) are, respectively, analyzed by the AMAD. Let us discuss as follows.

3.1. Non-uniform Euler–Bernoulli wedge beam ($\alpha_b = 1$, $\alpha_h = \alpha$; $\beta_b = 0$, $\beta_h = \beta$)

In the wedge beam the area and moment of inertia of the section will be obtained from Eqs. (17) and (18)

$$A(x) = A_0\left[1 + (\alpha - 1)\frac{x}{l}\right] = A_0\left(1 - \beta\frac{x}{l}\right), \tag{56}$$

$$I(x) = I_0\left[1 + (\alpha - 1)\frac{x}{l}\right]^3 = I_0\left(1 - \beta\frac{x}{l}\right)^3. \tag{57}$$

The equation of motion in dimensionless form from Eq. (27) can be written as

$$\frac{\mathrm{d}^4 Y(X)}{\mathrm{d}X^4} - \frac{6\beta}{1-\beta X}\frac{\mathrm{d}^3 Y(X)}{\mathrm{d}X^3} + \frac{6\beta^2}{(1-\beta X)^2}\frac{\mathrm{d}^2 Y(X)}{\mathrm{d}X^2} - \frac{\lambda}{(1-\beta X)^2}Y(X) = 0 \tag{58}$$

with the associated boundary conditions from Eqs. (29)–(32) as

$$Y''(0) - K_{\mathrm{RL}}Y'(0) = 0, \tag{59}$$

$$Y'''(0) - 3\beta Y''(0) + K_{\mathrm{TL}}Y(0) = 0, \tag{60}$$

and

$$Y''(1) + K_{\mathrm{RR}}Y'(1) = 0, \tag{61}$$

$$Y'''(1) - \frac{3\beta}{1-\beta}Y''(1) - K_{\mathrm{TR}}Y(1) = 0, \tag{62}$$

the recurrence relation for the coefficients C_k in Eqs. (46) and (47) can be written as

$$C_0 = Y(0), \quad C_1 = Y'(0), \quad C_2 = \frac{Y''(0)}{2}, \quad C_3 = \frac{Y'''(0)}{6}, \tag{63}$$

and for $k \geqslant 4$,

$$\begin{aligned} C_k = \frac{1}{k(k-1)(k-2)(k-3)}\sum_{j=0}^{k-4}\big[&6(j+3)(j+2)(j+1)\beta^{k-j-3}C_{j+3} \\ &-6(k-j-3)(j+2)(j+1)\beta^{k-j-2}C_{j+2} + \lambda(k-j-3)\beta^{k-j-4}C_j\big], \end{aligned} \tag{64}$$

where the two coefficients C_2 and C_3 can be obtained from Eqs. (48) and (49)

$$C_2 = \frac{K_{\mathrm{RL}}}{2}C_1, \tag{65}$$

$$C_3 = \frac{3\beta K_{\mathrm{RL}}C_1 - K_{\mathrm{TL}}C_0}{6}, \tag{66}$$

and the closed-form series solution in Eq. (54) is obtained.

3.2. Non-uniform Euler–Bernoulli cone beam ($\alpha_b = \alpha_h = \alpha$; $\beta = \beta_h = \beta$)

In the cone beam the area and moment of inertia of the section will be obtained from Eqs. (17) and (18)

$$A(x) = A_0\left[1 + (\alpha - 1)\frac{x}{l}\right]^2 = A_0\left(1 - \beta\frac{x}{l}\right)^2, \tag{67}$$

$$I(x) = I_0\left[1 + (\alpha - 1)\frac{x}{l}\right]^4 = I_0\left(1 - \beta\frac{x}{l}\right)^4. \tag{68}$$

The equation of motion in dimensionless form from Eq. (27) can be written as

$$\frac{\mathrm{d}^4 Y(X)}{\mathrm{d}X^4} - \frac{8\beta}{1-\beta X}\frac{\mathrm{d}^3 Y(X)}{\mathrm{d}X^3} + \frac{12\beta^2}{(1-\beta X)^2}\frac{\mathrm{d}^2 Y(X)}{\mathrm{d}X^2} - \frac{\lambda}{(1-\beta X)^2} Y(X) = 0 \tag{69}$$

with the associated boundary conditions from Eqs. (29)–(32) as

$$Y''(0) - K_{\mathrm{RL}} Y'(0) = 0, \tag{70}$$

$$Y'''(0) - 4\beta Y''(0) + K_{\mathrm{TL}} Y(0) = 0, \tag{71}$$

and

$$Y''(1) + K_{\mathrm{RR}} Y'(1) = 0, \tag{72}$$

$$Y'''(1) - \frac{4\beta}{1-\beta} Y''(1) - K_{\mathrm{TR}} Y(1) = 0, \tag{73}$$

the recurrence relation for the coefficients C_k in Eqs. (46) and (47) can be written as

$$C_0 = Y(0), \quad C_1 = Y'(0), \quad C_2 = \frac{Y''(0)}{2}, \quad C_3 = \frac{Y'''(0)}{6}, \tag{74}$$

and for $k \geqslant 4$,

$$C_k = \frac{1}{k(k-1)(k-2)(k-3)} \sum_{j=0}^{k-4} \big[8(j+3)(j+2)(j+1)\beta^{k-j-3} C_{j+3} \\ -12(k-j-3)(j+2)(j+1)\beta^{k-j-2} C_{j+2} + \lambda(k-j-3)\beta^{k-j-4} C_j\big], \tag{75}$$

where the two coefficients C_2 and C_3 can be obtained from Eqs. (48) and (49)

$$C_2 = \frac{K_{\mathrm{RL}}}{2} C_1, \tag{76}$$

$$C_3 = \frac{4\beta K_{\mathrm{RL}} C_1 - K_{\mathrm{TL}} C_0}{6}, \tag{77}$$

and the closed-form series solution in Eq. (54) is obtained.

3.3. Uniform Euler–Bernoulli beam ($\alpha_b = \alpha_h = 1$; $\beta = \beta_h = 0$)

In the uniform beam the area and moment of inertia of the section are constants, that is, $A(x) = A_1 = A_0$, $I(x) = I_1 = I_0$, the equation of motion in dimensionless form from Eq. (27) can be written as

$$\frac{\mathrm{d}^4 Y(X)}{\mathrm{d}X^4} - \lambda Y(X) = 0 \tag{78}$$

with the associated boundary conditions from Eqs. (29)–(32) as

$$Y''(0) - K_{\mathrm{RL}} Y'(0) = 0, \tag{79}$$

$$Y'''(0) + K_{\mathrm{TL}} Y(0) = 0, \tag{80}$$

and

$$Y''(1) + K_{\mathrm{RR}} Y'(1) = 0, \tag{81}$$

$$Y'''(1) - K_{\mathrm{TR}} Y(1) = 0, \tag{82}$$

the recurrence relation for the coefficients C_k in Eqs. (46) and (47) can be written as

$$C_0 = Y(0), \quad C_1 = Y'(0), \quad C_2 = \frac{Y''(0)}{2}, \quad C_3 = \frac{Y'''(0)}{6}, \tag{83}$$

and for $k \geqslant 4$,

$$C_k = \frac{\lambda}{k(k-1)(k-2)(k-3)} C_{k-4}, \tag{84}$$

where the two coefficients C_2 and C_3 can be obtained from Eqs. (48) and (49)

$$C_2 = \frac{K_{\mathrm{RL}}}{2} C_1, \quad C_3 = \frac{-K_{\mathrm{TL}} C_0}{6}, \tag{85}$$

and the closed-form series solution in Eq. (54) is obtained.

Hence, by using the method of AMDM, we can easily solve the vibration problem with various boundary conditions and obtain the closed-form series solutions. The proposed method is very efficient with the aid of symbolic computation.

4. Numerical results

In order to demonstrate the feasibility and the efficiency of AMDM in this paper, the previous three cases are discussed as follows. By using the formula and results of the previous cases, one can obtain the natural frequencies and mode shapes of the beam with various boundary conditions at both ends. In the particular cases, if the dimensionless spring constants are allowed to become zero or infinity, then the limiting cases of perfect constraints can be easily recovered. For example, if $K_{\mathrm{TL}} \to \infty$ and $K_{\mathrm{RL}} \to \infty$, then the beam is considered as the cantilever beam. If $K_{\mathrm{TL}} \to \infty$, $K_{\mathrm{RL}} = 0$, and $K_{\mathrm{TR}} \to \infty$, $K_{\mathrm{RR}} = 0$, then the beam is considered as the simply supported beam. If $K_{\mathrm{TL}} \to \infty$, $K_{\mathrm{RL}} \to \infty$, and $K_{\mathrm{TR}} \to \infty$, $K_{\mathrm{RR}} \to \infty$, then the beam is considered as the clamped–clamped beam. The computed results are compared with the analytical and numerical results in the literatures.

4.1. Non-uniform clamped–free wedge beam ($\alpha_b = 1$; $\alpha_h = \alpha = 0.5$)

In this case, let us consider the clamped–free beam which the area and moment of inertia of the section will be obtained from Eqs. (56) and (57)

$$A(x) = A_0\left(1 - 0.5\frac{x}{l}\right), \tag{86}$$

$$I(x) = I_0\left(1 - 0.5\frac{x}{l}\right)^3, \tag{87}$$

and the boundary conditions are given as

$$Y(0) = 0, \quad Y'(0) = 0, \tag{88}$$

$$Y''(1) = 0, \quad Y'''(1) = 0, \tag{89}$$

the beam is clamped–free, that is, $K_{\mathrm{RL}} \to \infty$, $K_{\mathrm{TL}} \to \infty$, $K_{\mathrm{RR}} = 0$, $K_{\mathrm{TR}} = 0$. Hence, from Eqs. (65) and (66), one can set $C_0 = 0$, $C_1 = 0$, and set the two coefficients C_2, C_3 as arbitrary constants, then the coefficients C_k can be obtained successively from Eq. (64). By substituting $\phi^{[n]}(X) = \sum_{k=0}^{n-1} C_k X^k$ into BC (89), the two algebraic equations of C_2 and C_3 are given as follows: when $Y''(1) = 0$,

$$[\phi^{[n]}(1)]'' = 0, \quad \sum_{k=0}^{n-3}(k+2)(k+1)C_{k+2} = f_{12}^{[n]}(\lambda)C_2 + f_{13}^{[n]}(\lambda)C_3 = 0, \tag{90}$$

and when $Y'''(1) = 0$,

$$[\phi^{[n]}(1)]''' = 0, \quad \sum_{k=0}^{n-4}(k+3)(k+2)(k+1)C_{k+3} = f_{22}^{[n]}(\lambda)C_2 + f_{23}^{[n]}(\lambda)C_3 = 0, \tag{91}$$

for the non-trivial solutions C_2 and C_3, the frequency equation can be obtained by use of Cramer's rule

$$\begin{vmatrix} f_{12}^{[n]}(\lambda) & f_{13}^{[n]}(\lambda) \\ f_{22}^{[n]}(\lambda) & f_{23}^{[n]}(\lambda) \end{vmatrix} = 0. \tag{92}$$

By solving Eq. (92) for the approximate term n, and taking real root for λ, $\Omega = \sqrt{\lambda}$, one can find that for $n = 30$,

$$\left|\Omega_1^{[30]} - \Omega_1^{[29]}\right| \leqslant \varepsilon = 0.00001. \tag{93}$$

Thus, the dimensionless natural frequency and natural frequency corresponding to $n = 30$, respectively, can be obtained as

$$\Omega_1 = \Omega_1^{[30]} = 3.8238, \tag{94}$$

$$\omega_1 = \Omega_1\sqrt{\frac{EI}{\rho A l^4}} = 3.8238\sqrt{\frac{EI_0}{\rho A_0 l^4}}, \tag{95}$$

by substituting $\Omega_1^{[30]}$ into Eqs. (64) and (54) and normalizing it by Eq. (55), the first mode shape function is given and is

$$\begin{aligned}\bar{\phi}_1^{[30]} &= 2.487898X^2 + 0.0761805X^3 - 0.253852X^4 - 0.214509X^5 \\ &\quad - 0.0354020X^6 + 0.00928653X^7 + 0.0125543X^8 + 0.00831645X^9 \\ &\quad + 0.00487125X^{10} + 0.00274446X^{11} + 0.00151636X^{12} + 0.000826246X^{13} \\ &\quad + 0.000445259X^{14} + 0.000237780X^{15} + 0.000126026X^{16} \\ &\quad + 0.0000663726X^{17} + 0.0000347681X^{18} + 0.0000181291X^{19} \\ &\quad + 9.415662 \times 10^{-6}X^{20} + 4.873425 \times 10^{-6}X^{21} + 2.514878 \times 10^{-6}X^{22} \\ &\quad + 1.294370 \times 10^{-6}X^{23} + 6.646520 \times 10^{-7}X^{24} + 3.405954 \times 10^{-7}X^{25} \\ &\quad + 1.742169 \times 10^{-7}X^{26} + 8.896778 \times 10^{-8}X^{27} + 4.536694 \times 10^{-8}X^{28} \\ &\quad + 2.310328 \times 10^{-8}X^{29} + 1.175142 \times 10^{-8}X^{30}. \end{aligned} \tag{96}$$

Following the same procedure as shown above, the other natural frequencies and mode shapes can be obtained. In Figs. 2 and 3, as the approximate term number n increases, the natural frequencies Ω_1–Ω_6 converge to 3.82378, 18.31726, 47.26483, 90.45048, 148.00174, and 219.92368 very quickly one by one without missing any frequency. Those complete natural frequencies lead to corresponding mode shapes correctly, which are shown in Fig. 4. Finally, the calculated results correspond very well with the previous works [13,14] as shown in Table 1.

4.2. Truncated clamped–free wedge and cone beams ($0<\alpha_b<1$, $0<\alpha_h<1$)

In this case of clamped–free tapered beam without translational springs and rotational springs at both ends ($K_{RL} \to \infty$, $K_{TL} \to \infty$, $K_{RR} = 0$, $K_{TR} = 0$), following the same steps as previous, the first three dimensionless natural frequencies Ω_1–Ω_3 can be obtained and listed in Tables 2 and 3. From these tables, the calculated results compared with Ref. [6] are in close agreement. The results are shown in Fig. 5, one can find that the natural frequencies of cone beam are larger than the natural frequencies of wedge beam for the same taper ratio α, and the larger the taper ratio α is, the smaller the first natural frequency Ω_1 of wedge and cone beam is, but the second and third natural frequencies Ω_2–Ω_3 of wedge and cone beam become larger as α increases.

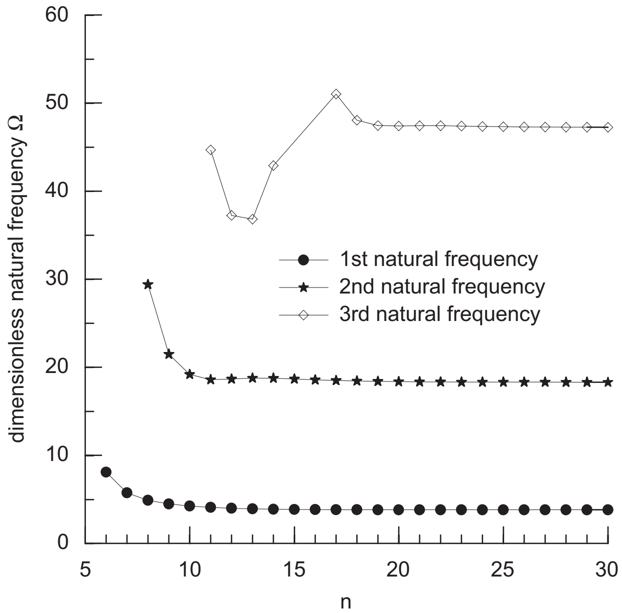

Fig. 2. The convergence of the first, second, and third dimensionless natural frequencies ($\Omega_1 = 3.82378$, $\Omega_2 = 18.31726$, $\Omega_3 = 47.26483$).

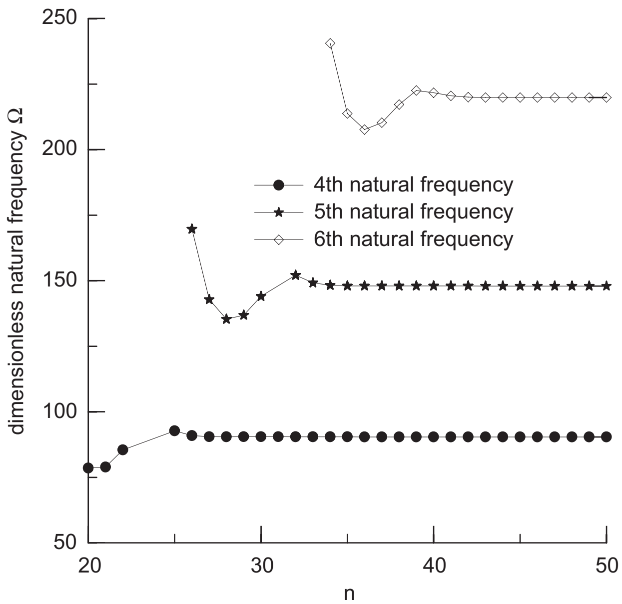

Fig. 3. The convergence of the fourth, fifth, and sixth dimensionless natural frequencies ($\Omega_4 = 90.45048$, $\Omega_5 = 148.00174$, $\Omega_6 = 219.92368$).

4.3. Wedge and cone beams with flexible constraints at both ends

In the case for $\alpha_b \geqslant 1$ and $\alpha_h > 1$, in order to obtain the calculated results of the beams by use of the previous analyses in the paper, the beam must be inverted, that is, the tapered ratio α_b must be replaced by $1/\alpha_b$, α_h by $1/\alpha_h$, and the dimensionless natural frequency Ω must be replaced by $\alpha_h \Omega$. Hence the results are obtained and listed in Tables 4–8.

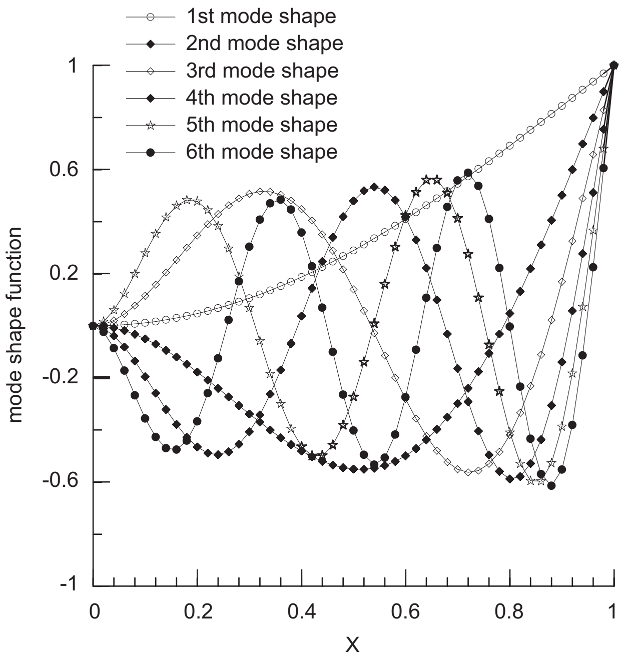

Fig. 4. The first six mode shape functions.

Table 1
The first three dimensionless natural frequencies of non-uniform wedge beam ($\alpha_b = 1$, $\alpha_h = \alpha = 0.5$)

n	Ω_1			Ω_2			Ω_3		
	(I)	(II)	Present	(I)	(II)	Present	(I)	(II)	Present
20	3.8281	3.8286	3.82809	18.3753	18.3758	18.37532	47.4212	47.4212	47.42121
40	3.8238	3.8237	3.82378	18.3173	18.3172	18.31726	47.2649	47.2648	47.26491
60	3.8238	3.8238	3.82378	18.3173	18.3172	18.31726	47.2648	47.2648	47.26483

(I) Ho's results [14]; (II) Lee's results [13].

Table 2
The first three dimensionless frequencies Ω_i, $i = 1,2,3$ of clamped–free wedge beams ($\alpha_b = 1$, $\alpha_h = \alpha$)

α	Ω_1		Ω_2		Ω_3	
	(I)	Present	(I)	Present	(I)	Present
0.1	4.6307	4.63074	14.9308	14.93080	32.8331	32.83313
0.2	4.2925	4.29250	15.7427	15.74270	36.8845	36.88456
0.3	4.0817	4.08172	16.6252	16.62525	40.5879	40.58788
0.4	3.9343	3.93428	17.4879	17.48786	44.0248	44.02481
0.5	3.8238	3.82379	18.3173	18.31726	47.2648	47.26483
0.6	3.7371	3.73708	19.1138	19.11381	50.3559	50.35366
0.7	3.6667	3.66675	19.8806	19.88061	–	53.32220
0.8	–	3.60828	–	20.62102	–	56.19228
0.9	–	3.55870	–	21.33810	–	58.97990
1[a]	3.5160	3.51601	22.0345	22.03439	61.6972	61.69644

(I) the results given by Naguleswaran [6].
[a]Uniform beam.

Table 3
The first three dimensionless frequencies Ω_i, $i = 1,2,3$ of clamped–free cone beams ($\alpha_b = \alpha_h = \alpha$)

α	Ω_1		Ω_2		Ω_3	
	(I)	Present	(I)	Present	(I)	Present
0.1	7.2049	7.20500	18.6802	18.68028	37.1238	37.12396
0.2	6.1964	6.19645	18.3855	18.38553	39.8336	39.83366
0.3	5.5093	5.50930	18.6412	18.64119	42.8104	42.81044
0.4	5.0090	5.00906	19.0649	19.06488	45.7384	45.73839
0.5	4.6252	4.62517	19.5476	19.54763	48.5789	48.57892
0.6	4.3188	4.31879	20.0500	20.05000	51.3342	51.33465
0.7	4.0669	4.06694	20.5554	20.55552	–	54.01520
0.8	–	3.85512	–	21.05676	–	56.63034
0.9	–	3.67371	–	21.55025	–	59.18864
1[a]	3.5160	3.51601	22.0345	22.03439	61.6972	61.69644

(I) the results given by Naguleswaran [6].
[a]Uniform beam.

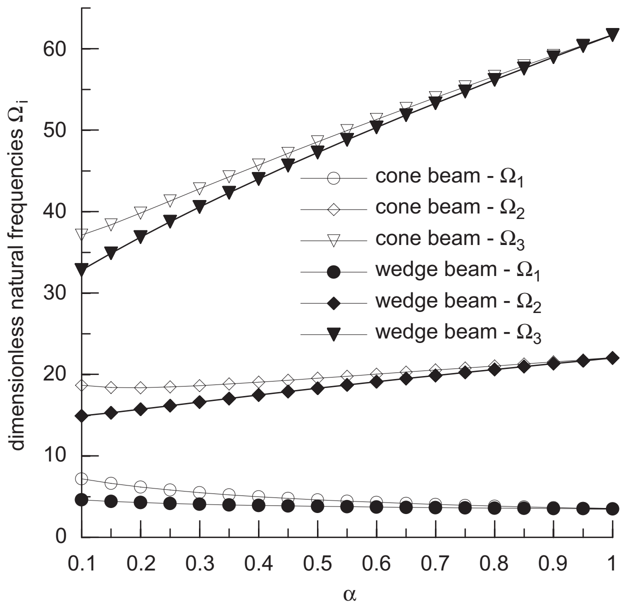

Fig. 5. The first three dimensionless natural frequencies of the wedge and cone beams for various values of taper ratio α.

In Table 4 the square root of the first dimensionless natural frequency $\sqrt{\Omega_1}$ is given for cone beam $\alpha_b = \alpha_h = 1.5$ and $K_{TL} = K_{RL} = \infty$, for various values of the dimensionless rotational spring constants K_{RL} and K_{RR}. In Table 5 the square roots of the first three dimensionless natural frequencies $\sqrt{\Omega_1} - \sqrt{\Omega_3}$ are given for cone beam $\alpha_b = \alpha_h = 2$ and $K_{TL} = K_{RL} = \infty$, for various values of K_{RL} and K_{RR}. In Table 6 the square root of the first three dimensionless natural frequencies $\sqrt{\Omega_1} - \sqrt{\Omega_3}$ are given for wedge beam $\alpha_b = 1$, $\alpha_h = 1.5$ and $K_{TL} = K_{RL} = \infty$, for various values of K_{RL} and K_{RR}. From Tables 4–6 one can find that the larger the rotational spring constants are, the larger the natural frequencies are.

In Table 7 the square roots of the first four dimensionless natural frequencies $\sqrt{\Omega_1} - \sqrt{\Omega_4}$ are given for cone beam $\alpha_b = \alpha_h = 1.4$ and $K_{TR} = K_{RR} = \infty$, for various values of the dimensionless rotational spring constant K_{RL} and translational spring constant K_{TL}. From this table one can find that the larger the translational spring constant is, the larger the natural frequencies are.

Table 4
The square root of the dimensionless fundamental frequency $\sqrt{\Omega_1}$ for $\alpha_b = \alpha_h = 1.5$ and $K_{\mathrm{TL}} = K_{\mathrm{RL}} = \infty$

K_{RL}	K_{RR}	$\sqrt{\Omega_1}$		
		(I)	(II)	Present
0	0	3.474	3.47477	3.47477
0	0.01	3.577	3.47772	3.47771
0	0.1	3.503	3.50345	3.50345
0	10	4.256	4.25615	4.25615
0	100	4.509	4.50894	4.50893
0.01	0	3.476	3.47581	3.47581
0.01	0.01	3.479	3.47875	3.47875
0.01	0.1	3.504	3.50447	3.50447
0.01	10	4.257	4.25702	4.25702
0.01	100	4.509	4.50982	4.50981
0.1	0.1	3.513	3.51356	3.51356
1	1	3.788	3.78767	3.78766
100	0.1	4.161	4.16142	4.16142

$K_{\mathrm{TL}} = k_{\mathrm{TL}}l^3/EI_0$, $K_{\mathrm{TR}} = k_{\mathrm{TR}}l^3/EI_1$, $K_{\mathrm{RL}} = k_{\mathrm{RL}}l/EI_0$, $K_{\mathrm{RR}} = k_{\mathrm{RR}}l/EI_1$; $\alpha_b = b_1/b_0$, $\alpha_h = h_1/h_0$.
(I) The results given by Grossi et al. [9]; (II) the results given by Auciello [15].

Table 5
The square root of the first three dimensionless natural frequencies $\sqrt{\Omega_i}$, $i = 1,2,3$ for $\alpha_b = \alpha_h = 2$ and $K_{\mathrm{TL}} = K_{\mathrm{RL}} = \infty$

K_{RL}	K_{RR}	$\sqrt{\Omega_1}$			$\sqrt{\Omega_2}$			$\sqrt{\Omega_3}$		
		(I)	(II)	Present	(I)	(II)	Present	(I)	(II)	Present
0	0	3.7300	3.73002	3.73003	7.4750	7.63020	7.63020	11.4201	11.42157	11.42157
0	0.01	3.7345	–	3.73454	7.4696	–	7.63167	11.4219	–	11.42247
0	0.1	3.8643	3.77372	3.77371	7.3921	7.64473	7.64473	11.4306	11.43047	11.43047
0	1	4.0635	–	4.06357	7.7619	–	7.76189	11.5038	–	11.50523
0	10	4.7625	4.75489	4.75488	8.2846	8.28460	8.28460	11.9277	11.92757	11.92757
1	0	3.7984	–	3.79840	7.6803	–	7.68029	11.4597	–	11.46031
1	0.1	3.8409	3.84092	3.84091	7.6946	7.69464	7.69464	11.4694	11.46915	11.46915
1	1	4.1249	4.12491	4.12490	7.8105	7.81050	7.81050	11.5435	11.54347	11.54347

(I) The results given by Grossi et al. [9]; (II) the results given by Auciello [15].

Table 6
The square root of the first three dimensionless natural frequencies $\sqrt{\Omega_i}$, $i = 1,2,3$ for $\alpha_b = 1$, $\alpha_h = 1.5$ and $K_{\mathrm{TL}} = K_{\mathrm{TR}} = \infty$

K_{RL}	K_{RR}	$\sqrt{\Omega_1}$			$\sqrt{\Omega_2}$			$\sqrt{\Omega_3}$		
		(I)	(II)	Present	(I)	(II)	Present	(I)	(II)	Present
0	0	3.4888	–	3.48881	6.9972	–	6.99720	10.4011	–	10.49113
0	0.01	3.4913	3.49136	3.49136	6.9983	6.99832	6.99829	10.4918	10.49194	10.49183
0	0.1	3.5136	3.51369	3.51369	6.9808	7.00805	7.00802	10.4981	10.49820	10.49810
0	1	3.6907	3.69075	3.69075	7.0894	7.09609	7.09605	10.5569	10.55703	10.55692
0	10	4.2027	4.20276	4.20276	7.5143	7.51439	7.51435	10.9020	10.90277	10.90264
1	0	3.5912	3.59124	3.59124	7.0610	7.06107	7.06104	10.5369	10.53703	10.53692
1	0.1	3.5936	3.61516	3.61516	7.0621	7.07170	7.07167	10.5377	10.54394	10.54383
1	1	3.7865	3.78654	3.78655	7.1583	7.15831	7.15828	10.6022	10.60229	10.60218

(I) The results given by Grossi et al. [9]; (II) the results given by Auciello [15].

Table 7
The square root of the first four dimensionless natural frequencies $\sqrt{\Omega_i}$, $i = 1,2,3,4$ for $\alpha_b = \alpha_h = 1.4$ and $K_{TL} = K_{RL} = \infty$

K_{RL}	K_{TL}	$\sqrt{\Omega_1}$		$\sqrt{\Omega_2}$		$\sqrt{\Omega_3}$		$\sqrt{\Omega_4}$	
		(I)	Present	(I)	Present	(I)	Present	(I)	Present
0	0	2.3766	2.37661	5.3739	5.37387	8.7264	8.72637	12.1135	12.11351
0	0.01	–	2.37730	–	5.37394	–	8.72639	–	12.11352
0	0.1	2.3834	2.38344	5.3745	5.37454	8.7265	8.72653	12.1136	12.11357
0	1	2.4420	2.44201	5.3805	5.38055	8.7280	8.72798	12.1141	12.11412
0	10	2.8554	2.85543	5.4414	5.44142	8.7426	8.74258	12.1196	12.11961
0	100	–	3.91909	–	6.03037	–	8.89809	12.17640	12.17640
0	1000	4.3755	4.37550	7.4772	7.47721	10.2215	10.22153	–	12.85719
0	∞	4.4329	4.43289	7.8008	7.80081	11.2061	11.20607	14.6219	14.62191

(I) The results given by De Rosa et al. [16].

Table 8
The square root of the first four dimensionless natural frequencies $\sqrt{\Omega_i}$, $i = 1,2,3,4$ for $\alpha_b = \alpha_h = 1.4$ and $K_{TL} = K_{RL} = 0$

K_{TL}	K_{TR}	$\sqrt{\Omega_1}$		$\sqrt{\Omega_2}$		$\sqrt{\Omega_3}$		$\sqrt{\Omega_4}$	
		(I)	Present	(I)	Present	(I)	Present	(I)	Present
0	0	–	0	–	0	–	5.19176	–	8.59573
0.001	0.001	–	0.21656	–	0.31795	–	5.19178	–	8.59574
0.01	0.01	–	0.38510	–	0.56539	–	5.19196	–	8.59578
0.1	0.1	–	0.68462	–	1.00528	–	5.19381	–	8.59619
1	1	1.2140	1.21404	1.7851	1.78509	5.2122	5.21223	8.6003	8.60028
10	10	2.1010	2.10096	3.1302	3.13023	5.3938	5.39376	8.6415	8.64148
100	100	3.0724	3.07241	5.0667	5.06670	6.7115	6.71152	9.0709	9.07091
1000	1000	3.3755	3.37553	6.5696	6.56963	9.2888	9.28876	11.5626	11.56263
∞	∞	3.4159	3.41595	6.8687	6.86864	10.2978	10.29775	13.7260	13.72576

(I) The results given by De Rosa et al. [16].

In Table 8 the square root of the first four dimensionless natural frequencies $\sqrt{\Omega_1} - \sqrt{\Omega_4}$ are given for cone beam $\alpha_b = \alpha_h = 1.4$ and $K_{RL} = K_{RR} = \infty$, for various values of K_{TL} and K_{TR}. From this table one can find that the larger the spring constants are, the larger the natural frequencies are.

From these tables one can find that the calculated results in the study compared with the results of the other literatures are in close agreement.

5. Conclusion

By the method proposed in this study, the closed-form series solutions of the free vibrations of tapered beams with various elastically supported conditions can be obtained. This paper presents an effective method to solve vibration problems of non-uniform beams with various flexible ends. By using the proposed method, any ith natural frequency and mode shape function can be obtained one at a time. The larger the approximate term n is giving, more natural frequency can be found at the same time. The computed results are compared closely with the results obtained by using other analytical and numerical methods. This study provides a unified and systematic procedure which is seemingly simpler and more straightforward than the other methods.

References

[1] H.H. Mabie, C.B. Rogers, Transverse vibrations of tapered cantilever beams with end supports, *Journal of the Acoustical Society of America* 44 (4) (1968) 1739–1741.

[2] H.H. Mabie, C.B. Rogers, Transverse vibrations of double-tapered cantilever beams with end support and with end mass, *Journal of the Acoustical Society of America* 55 (1974) 986–991.
[3] P.A.A. Laura, R.H. Guitierrez, Vibrations of a elastically restrained cantilever beam of varying cross section with tip mass of finite length, *Journal of Sound and Vibration* 108 (1986) 123–131.
[4] P.A.A. Laura, B.V. De Greco, J. Utjes, R. Carnicer, Numerical experiments on free and forced vibrations of beams of non-uniform cross section, *Journal of Sound and Vibration* 120 (1988) 587–596.
[5] S. Naguleswaran, Vibration of an Euler–Bernoulli beam of constant depth and with linearly varying breadth, *Journal of Sound and Vibration* 153 (3) (1992) 509–522.
[6] S. Naguleswaran, A direct solution for the transverse vibration of Euler–Bernoulli wedge and cone beams, *Journal of Sound and Vibration* 172 (3) (1994) 289–304.
[7] S. Naguleswaran, comments on "Vibration of non-uniform rods and beams," *Journal of Sound and Vibration* 195 (2) (1996) 331–337.
[8] R.P. Goel, Transverse vibrations of tapered beams, *Journal of Sound and Vibration* 47 (1) (1976) 1–7.
[9] R.O. Grossi, R.B. Bhat, A note on the vibrating tapered beams, *Journal of Sound and Vibration* 147 (1) (1991) 174–178.
[10] R.O. Grossi, A. Aranda, R.B. Bhat, Vibration of tapered beams with one end spring hinged and the other end with tip mass, *Journal of Sound and Vibration* 160 (1) (1993) 175–178.
[11] R.O. Grossi, B. Del, V. Arenas, A variational approach to the vibration of tapered beams with elastically restrained ends, *Journal of Sound and Vibration* 195 (3) (1996) 507–511.
[12] S. Abrate, Vibration of non-uniform rods and beams, *Journal of Sound and Vibration* 185 (4) (1995) 703–716.
[13] S.Y. Lee, H.Y. Ke, Y.H. Kuo, Exact solutions for the analysis of general elastically restrained non-uniform beams, *Transactions of the ASME: Journal of the Applied Mechanics* 59 (1992) s205–s212.
[14] S.H. Ho, C.K. Chen, Analysis of general elastically restrained non-uniform beams using differential transform, *Applied Mathematical Modelling* 22 (1998) 219–234.
[15] N.M. Auciello, A comment on "A note on vibrating tapered beams," *Journal of Sound and Vibration* 187 (4) (1995) 724–726.
[16] M.A. De Rosa, N.M. Auciello, Free vibrations of tapered beams with flexible ends, *Computers & Structures* 60 (2) (1996) 197–202.
[17] G. Adomian, *Solving Frontier Problems of Physics: The Decomposition Method*, Kluwer Academic Publishers, Dordrecht, 1994.
[18] G. Adomian, R. Rach, Modified decomposition solution of nonlinear partial differential equations, *Computers & Mathematics with Applications* 23 (1992) 17–23.
[19] G. Adomian, R. Rach, Modified decomposition solution of nonlinear partial differential equations, *Applied Mathematics Letters* 5 (1992) 29–30.
[20] G. Adomian, R. Rach, Nonlinear transformation of series, *Applied Mathematics Letters* 4 (1991) 69–71.