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Sensitivity of friction-induced vibration in idealised systems

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Abstract

Friction-induced vibration occurs in many contexts: vehicle brake squeal in particular remains surprisingly unpredictable and poorly understood. Testing theory against measurements has been hindered by the difficulty in obtaining repeatable results suggesting that the phenomenon is sensitive to small changes in parameters. This paper explores highly idealised cases as a starting point to understanding sensitivity. Using a stability criterion based on the roots of the characteristic equations of the system, the sensitivity of predictions to parameter changes is studied, focussing on a single-mode model. The effects of contact stiffness, non-proportional damping and a velocity-dependent coefficient of friction are considered. It is found that each physical effect can significantly alter predictions; each physical effect can lead to extreme sensitivity; and high sensitivity can sometimes occur when modal amplitudes are small such that they might normally be considered insignificant. With a large body of literature focussing on reduced-order models this study provides an important warning when interpreting their results.
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1. Introduction

Friction-induced vibration due to a sliding contact between two systems is a phenomenon that arises across a diverse range of scales and contexts, including musical instruments, machine tool vibration, railway wheel noise, earthquakes and vehicle brake squeal. The literature on brake squeal has moved from very simple lumped parameter models to ever more sophisticated finite element models, but testing theory against measurements has always proved difficult. This research has been summarised by North [1], Ibrahim and Rivin [2], Kinkaid et al. [3] and Ouyang et al. [4]. A recurring theme is the difficulty in obtaining repeatable experimental results that correlate with theoretical models: to date, there is no validated predictive model of friction-induced vibration. This difficulty in validating theoretical models suggests that friction-coupled systems are highly sensitive to parameter changes beyond an experimenter's control. The study of sensitivity in such systems has now begun to attract some attention (e.g. Guan et al. [5] and Huang et al. [6]).

This research follows and extends a theoretical model developed by Duffour and Woodhouse [7,8] which allows for direct testing against experimental data. The model, summarised in the next section, is based on a linear transfer function analysis of two subsystems coupled by a sliding point contact. The predicted system

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stability depends on the values of the complex zeros of two characteristic equations. Although a simplified analysis, it is general enough for the theory to be directly applicable to results from a test rig. Obtaining repeatable results is still difficult, so this research focusses on exploring the reasons for the sporadic nature of squeal. Initial results show that even measurement uncertainty is sometimes enough to significantly affect predictions, and that the sensitivity of predictions is highly dependent on the system parameters (see Butlin and Woodhouse [9]).

Despite the simple problem formulation, understanding the reasons for the effects of parameter changes remains a serious challenge as predictions are an intricate function of these parameters. This is a result of the familiar trade-off when modelling any system between realism and acuity, and is reflected in the development of the literature. Added to this is the difficulty in knowing what is important to include in the model.

Much of the literature is dominated by numerical analysis as this is often the only way that the behaviour of a given model can be explored. While such analysis is valuable, the model complexity can make it difficult to understand underlying reasons for results. It is therefore logical to explore the behaviour of those models that can be studied analytically—the substantial literature dedicated to analytical reduced-order models supports the uncontroversial nature of this approach. Simplified models can be analysed in detail and with reasonable transparency, helping to understand the effect of parameters and model choices on predictions and to identify potential limitations of models: it is precisely because simplified models can be studied analytically that they are of interest. A thorough understanding of them allows meaningful conclusions to be drawn from simple models and highlights circumstances under which larger scale models may give misleading results. Sensitivity is a particularly pertinent issue that may underly the difficulty in obtaining repeatable results and provides insight into the reliability of predictions. Care should be taken in drawing conclusions where high sensitivity to either parameter changes or modelling details occurs. This paper describes some key results from a systematic analysis of reduced-order models that can be studied analytically taking into account a range of physical effects often found in the literature, such as contact stiffness or a velocity-dependent coefficient of friction.

The complexity of a model can be subdivided under two headings: the number of natural modes that the system is modelled with, and the range of physical effects that are taken into account. Most studies take a model of some degree of complexity and are based on a fixed set of assumptions, which may be relaxed progressively so that the complexity steadily increases. However, a clear presentation of the various effects of different types of models has not been found in the literature, so the aim of this article is to clarify some of the most basic routes to instability and causes of high sensitivity.

A number of independent issues are raised:

- What range of behaviours is predicted by simplified models?
- Under what conditions can these models give highly sensitive predictions?
- Over what bandwidth (if any) do these models give a useful approximation to more complex systems?
- What is the convergence behaviour of predictions as more modes are included in the system model?
- Are there phenomena which are fundamentally absent from the simplified models, which only appear in more complex systems?

This paper mainly addresses the first two of these points by describing a systematic exploration of highly idealised cases. Only a few modes are included in the model and the effects of including different physical elements are compared. Of particular interest are conditions that result in high sensitivity, which may begin to explain the sporadic nature of friction-induced vibration. With much of the literature focussing on reduced-order models (recent examples include Hoffmann et al. [10], Kinkaid [11], von Wagner [12] and Emira [13]) this work on sensitivity provides greater insight into the interpretation of results from such models. It should be noted that although the work will be described in the language of brake squeal, the theory applies to any sliding point contact system (e.g. certain types of machine tool vibration).

The analysis presented within this paper is in the context of further theoretical studies of the other questions raised, and a thorough comparison of model predictions with experimental results (see Butlin [14]). While in many cases it is possible to identify minimal reduced-order models that give good approximations to a larger model, under certain conditions of high sensitivity the convergence behaviour is non-trivial. In addition, there is compelling experimental evidence that including a combination of the physical effects considered within this paper is essential to predicting observed instabilities.

2. Theoretical framework

2.1. Basic model

The system to be analysed is sketched in Fig. 1. A 'disc' is driven at constant velocity, V_0, and a 'brake' is pushed against it with a dynamically varying normal force, N, composed of a steady equilibrium pre-load, N_0, plus a small fluctuating component, N', such that $N = N_0 + N'$. Similarly, the force tangential to the sliding direction due to friction, F, can be expressed as a steady equilibrium force, F_0, plus a fluctuating component, F', such that $F = F_0 + F'$. With a Coulomb friction law the normal and tangential forces are related by $F = \mu_0 N$ where μ_0 is the coefficient of friction. The normal and tangential displacements from equilibrium of the disc are denoted u_1 and v_1, respectively, and u_2 and v_2 for the brake. The normal and tangential displacements from equilibrium of the point of contact are denoted u_3 and v_3. The springs of stiffness k_n and k_t represent the linearised contact stiffness in the normal and tangential directions, respectively. Any damping that may result from the contact has been ignored.

The dynamics of the disc and brake can be described in terms of transfer functions:

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} G_{11}(\omega) & G_{12}(\omega) \\ G_{21}(\omega) & G_{22}(\omega) \end{bmatrix} \begin{bmatrix} N' \\ F' \end{bmatrix}, \tag{1}$$

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} H_{11}(\omega) & H_{12}(\omega) \\ H_{21}(\omega) & H_{22}(\omega) \end{bmatrix} \begin{bmatrix} N' \\ F' \end{bmatrix}, \tag{2}$$

where $G_{ij}(\omega)$ are the transfer functions representing the disc's response and $H_{ij}(\omega)$ represent the equivalent set of responses for the brake. These transfer functions can be determined using standard vibration measurement techniques. Note that this paper follows the convention of the vibration literature by using transfer functions defined as the Fourier transform of an impulse response, rather than the Laplace transform. For readers more familiar with the Laplace formalism the complex ω-plane used here should be rotated anticlockwise by 90° to correspond to the complex s-plane as $s = i\omega$.

The stability of the coupled system can be determined from the transfer functions of the system: all the poles of the transfer functions from any possible input to any output must be stable to predict overall stability. There are several possible inputs: surface roughness $r = u_1 + u_3$, normal force perturbations P, and tangential force perturbations Q. The outputs of the system can be considered as variations in the normal force, N', or the relative tangential displacement of the disc and brake, $v_1 + v_3$. Physically, normal and tangential force perturbations could be the result of any kind of external disturbance, for example random fluctuations in the coefficient of friction as the disc rotates would cause tangential perturbations.

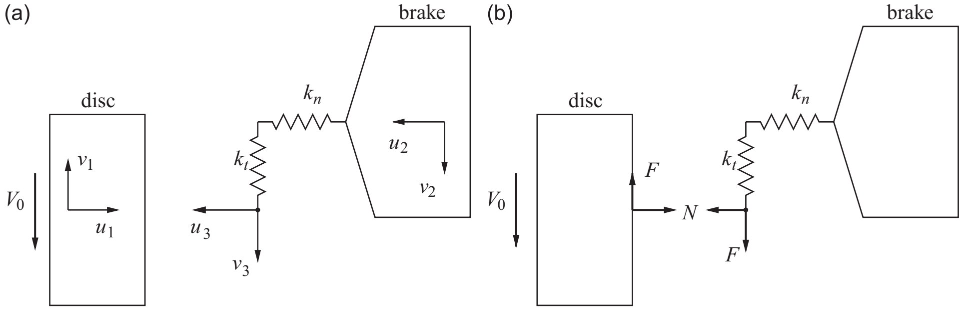

Fig. 1. Two linear subsystems coupled by a single point sliding contact, with definition of variables for (a) displacements and (b) forces.

Assuming a constant coefficient of friction the transfer function from surface roughness, r, to changes in the normal force, N', can be readily shown to be

$$\frac{N'}{r} = \frac{1}{G_{11} + \mu_0 G_{12} + H_{11} + \mu_0 H_{12} + 1/k_n}. \tag{3}$$

Considering the transfer functions from the other possible inputs to all the outputs results in no additional poles which could be unstable. Therefore the steady sliding solution of this system will be asymptotically stable if and only if all zeros of the characteristic function $D(\omega)$ lie in the upper half-plane, where

$$D = G_{11} + \mu_0 G_{12} + H_{11} + \mu_0 H_{12} + 1/k_n \tag{4}$$

as derived by Duffour and Woodhouse [7]. The corresponding condition for stability in the Laplace formalism would require all the zeros to lie in the left half-plane.

2.2. Model extension

If a coefficient of friction that varies with sliding velocity is included, the relationship between F and N can be linearised such that

$$F \approx [\mu_0 - \mathrm{i}\omega\varepsilon(v_1 + v_3)]N \tag{5}$$

and only first-order terms are kept on expansion. The factor $\mathrm{i}\omega$ converts the displacements v_1 and v_3 into velocities and ε is the linearised gradient of the friction-velocity function. More generally, if ε is allowed to become complex and frequency dependent then Eq. (5) could describe any linearised relationship between F and N, for example a friction law based on contact temperature or asperity deformation history.

Overall system stability must be determined by considering the poles of all the possible transfer functions. Changes in the tangential force, F', are related to changes in the normal force, N', by

$$F' = \mu_0 N' - \mathrm{i}\omega\varepsilon N_0(v_1 + v_3), \tag{6}$$

where only first-order terms have been kept. If P and Q represent external perturbations to the normal and tangential force, respectively, then the inputs and outputs can be expressed by the matrix equation:

$$\begin{bmatrix} u_1 + u_3 \\ v_1 + v_3 \end{bmatrix} = \begin{bmatrix} G_{11} + H_{11} + 1/k_n & G_{12} + H_{12} \\ G_{21} + H_{21} & G_{22} + H_{22} + 1/k_t \end{bmatrix} \begin{bmatrix} N' + P \\ F' + Q \end{bmatrix}. \tag{7}$$

Substituting Eq. (6) into the second line gives

$$\begin{aligned} v_1 + v_3 = {} & (G_{21} + H_{21})(N' + P) \\ & + (G_{22} + H_{22} + 1/k_t)[\mu_0(N' + P) + Q - \mathrm{i}\omega\varepsilon N_0(v_1 + v_3)]. \end{aligned} \tag{8}$$

Rearranging this leads to an expression for the relative tangential displacement, $v_1 + v_3$, in terms of N' and the inputs P and Q. Using the same method, the transfer function from the surface roughness, $r = u_1 + u_3$, to changes in the normal force, N', can be derived. The relative tangential displacement can then be written in terms of all the inputs as

$$\begin{aligned} v_1 + v_3 = {} & \frac{G_{21} + H_{21} + \mu_0(G_{22} + H_{22} + 1/k_t)}{E_1(\omega)} r \\ & + \frac{G_{21} + H_{21} + \mu_0(G_{22} + H_{22} + 1/k_t)}{1 + \mathrm{i}\omega\varepsilon N_0(G_{22} + H_{22} + 1/k_t)} P \\ & + \frac{G_{22} + H_{22} + 1/k_t}{1 + \mathrm{i}\omega\varepsilon N_0(G_{22} + H_{22} + 1/k_t)} Q. \end{aligned} \tag{9}$$

All of these functions have numerators consisting of linear combinations of stable transfer functions. Thus unstable poles can only arise from the zeros of the denominators. There are two distinct denominators and therefore two characteristic functions of this system. The first of these, $E_1(\omega)$, was derived by Duffour and

Woodhouse [8]:

$$E_1 = D + \mathrm{i}\omega\varepsilon N_0[(G_{11} + H_{11} + 1/k_n)(G_{22} + H_{22} + 1/k_t) - (G_{12} + H_{12})^2]. \tag{10}$$

The system will be unstable if the function $E_1(\omega)$ has at least one zero with a negative imaginary part. Overall stability is determined by also considering the zeros of the second distinct denominator:

$$E_2(\omega) = 1 + \mathrm{i}\omega\varepsilon N_0(G_{22} + H_{22} + 1/k_t). \tag{11}$$

This characteristic function has a more immediately intuitive interpretation than $E_1(\omega)$. It can be written as a standard negative feedback loop where the system transfer function is $(G_{22} + H_{22} + 1/k_t)$ and the feedback gain is $(\mathrm{i}\omega\varepsilon N_0)$. Classical feedback theory results can then be applied directly, for example stability could be determined using the Nyquist stability criterion. It can also be seen how increasing εN_0 will at some point lead to instability: this is formalised by the Nyquist criterion but can be argued intuitively as increasing the negative damping until the overall damping of the system is negative.

It can now be said that this system will be unstable if and only if all the zeros of both $E_1(\omega)$ and $E_2(\omega)$ lie in the upper half-plane. The relationship between these zeros and the system parameters will now be explored for highly idealised cases.

3. Single-mode analysis

Each term of $D(\omega)$ in Eq. (4) can be written as a sum of modal contributions using a standard expression (e.g. Skudrzyk [15]). Thus $D(\omega)$ can be written explicitly:

$$D = \sum_{\text{all } j(\text{disc})} \frac{g_{u_1}^{(j)} g_{u_1}^{(j)} + \mu_0 g_{u_1}^{(j)} g_{v_1}^{(j)}}{\omega_j^2 + 2\mathrm{i}\zeta_j\omega_j\omega - \omega^2} + \sum_{\text{all } j(\text{brake})} \frac{h_{u_2}^{(j)} h_{u_2}^{(j)} + \mu_0 h_{u_2}^{(j)} h_{v_2}^{(j)}}{\omega_j^2 + 2\mathrm{i}\zeta_j\omega_j\omega - \omega^2} + \frac{1}{k_n}, \tag{12}$$

where ω_j and ζ_j are the natural frequency and damping factor of the jth mode, respectively, of either the disc or the brake (if the uncoupled modes of both subsystems are sorted by ascending frequency, then the jth mode could correspond to either the disc or the brake). Modal amplitudes that correspond to the disc subsystem are denoted g, and those for the brake h. As there is no mathematical distinction between the modal sums of the disc and the brake they can be combined. For convenience we define the modal coefficient $a_{pq}^{(j)}$ to represent the relevant product of modal amplitudes of either subsystem for the jth mode. To simplify the notation further we define $a_j = a_{11}^{(j)} + \mu_0 a_{12}^{(j)}$, such that we can rewrite $D(\omega)$ more simply:

$$D = \sum_{\text{all } j} \frac{a_j}{\omega_j^2 + 2\mathrm{i}\zeta_j\omega_j\omega - \omega^2} + \frac{1}{k_n}. \tag{13}$$

It is a common assumption that a given system can be approximated by a finite set of modes spanning the bandwidth of interest. Resonances far enough outside this bandwidth are considered as having an insignificant effect. Consequently models that include just a few neighbouring modes would be expected to work reasonably well in a narrow bandwidth surrounding these modes. Taken to the extreme, a single-mode model might be expected to approximate the system in the locality of its natural frequency. This paper is mainly concerned with the analysis of precisely such a single-degree-of-freedom system. It should be kept in mind that the intention is to locally approximate a multi-degree-of-freedom system. The focus is on the sensitivity of solutions and to provide a clear presentation of the model behaviour when different physical effects are included.

The transfer function for a single-mode model can be described by the natural frequency ω_1, modal coefficient a_1, and damping factor ζ_1:

$$G(\omega) = \frac{a_1}{\omega_1^2 + 2\mathrm{i}\zeta_1\omega_1\omega - \omega^2}. \tag{14}$$

There are two poles, symmetric about the imaginary axis of the complex ω-domain, and so the transfer function can also be written:

$$G(\omega) = \frac{c_1}{\omega - \bar{\omega}_1} + \frac{-c_1^*}{\omega + \bar{\omega}_1^*}, \tag{15}$$

where * denotes the complex conjugate, $c_1 = -a_1/2\omega_1$ and is purely real if a_1 is real, and $\bar{\omega}_1 = \omega_1\left(\sqrt{1-\zeta_1^2} + \mathrm{i}\zeta_1\right) \approx \omega_1(1+\mathrm{i}\zeta_1)$. The approximate expression applies if damping is light, $\zeta_1 \ll 1$.

Thus a single-mode model can be represented by two simple poles, one at negative frequency and one positive. If the negative frequency pole is discarded by considering it as a 'distant' mode, as is often done (e.g. Skudrzyk [15]), then the analysis simplifies considerably. All other 'distant' modes are neglected in such a local model so it would seem a reasonable approximation. However, discarding the symmetry has deeper implications as will be seen later in this section.

3.1. Effect of individual parameters

If contact stiffness, non-proportional damping, and the velocity-dependence of friction are all neglected (i.e. $k_n \to \infty$, a_1 is real and $\varepsilon = 0$) then the characteristic equation can be written as

$$D(\omega) = H_{11} + \mu_0 H_{12} = \frac{a_1}{\omega_1^2 + 2\mathrm{i}\zeta_1\omega_1\omega - \omega^2} = 0. \tag{16}$$

Recall that the modal coefficient a_1 is a combination of driving point and cross-terms, such that $a_1 = a_{11}^{(1)} + \mu_0 a_{12}^{(1)}$. Clearly Eq. (16) has no non-trivial solutions and thus cannot result in an unstable response. This has led previous authors (e.g. Blok [16]) to conclude that a necessary condition for the onset of squeal for such a simple system is a velocity-dependent coefficient of friction. While it is true that this can lead to squeal, there are other possible routes to instability.

3.1.1. Contact stiffness

If the contact stiffness, k_n, is considered to be finite then the characteristic equation becomes

$$D(\omega) = \frac{a_1}{\omega_1^2 + 2\mathrm{i}\zeta_1\omega_1\omega - \omega^2} + \frac{1}{k_n} = 0, \tag{17}$$

where a_1 can be either positive or negative since it includes the modal coefficients of H_{12}, which can be of arbitrary sign. Eq. (17) can be written as

$$a_1 k_n + \omega_1^2 + 2\mathrm{i}\zeta_1\omega_1\omega - \omega^2 = 0 \tag{18}$$

with solutions:

$$\omega_z = \mathrm{i}\zeta_1\omega_1 \pm \sqrt{a_1 k_n + \omega_1^2(1-\zeta_1^2)}, \tag{19}$$

which can either be purely imaginary or a complex pair. If the solution is a complex pair then the system must be stable as the imaginary part is positive, but if it is purely imaginary then the stability depends on the system parameters. If unstable, this implies that the impulse response would grow purely exponentially until it is limited by nonlinearity. Within this linearised model there is no 'frequency of squeal', but the effect may relate to the initiation of 'judder' or 'groan' (e.g. Jacobson [17]) or 'sprag-slip' (Spurr [18]).

Of particular interest here is not directly the conditions for stability but the sensitivity of the prediction. The partial derivatives of the solution with respect to each of the system parameters are as follows:

$$\frac{\partial\omega_z}{\partial w_1} = \mathrm{i}\zeta_1 \mp \frac{2\omega_1(1-\zeta_1^2)}{2\sqrt{a_1 k_n + \omega_1^2(1-\zeta_1^2)}}, \tag{20}$$

$$\frac{\partial\omega_z}{\partial a_1} = \mp \frac{k_n}{2\sqrt{a_1 k_n + \omega_1^2(1-\zeta_1^2)}}, \tag{21}$$

$$\frac{\partial\omega_z}{\partial \zeta_1} = \mathrm{i}\omega_1 \mp \frac{-2\omega_1^2\zeta_1}{2\sqrt{a_1 k_n + \omega_1^2(1-\zeta_1^2)}}, \tag{22}$$

$$\frac{\partial \omega_z}{\partial k_n} = \mp \frac{a_1}{2\sqrt{a_1 k_n + \omega_1^2(1-\zeta_1^2)}}. \tag{23}$$

A singularity occurs in each of these derivatives at the bifurcation point when

$$a_1 = -\frac{\omega_1^2(1-\zeta_1^2)}{k_n}. \tag{24}$$

One implication of Eq. (24) is that if the contact stiffness is high compared with the equivalent modal stiffness then the system is sensitive to small amplitude modes, a significant point if this result remains true for multiple-mode systems.

The sensitivity of the roots can be plotted as a surface as two parameters vary: we choose to illustrate results for variation of the natural frequency, ω_1, and the modal coefficient, a_1. The damping factor has been held constant at $\zeta = 0.03$ throughout this section and, when included, the contact stiffness has been chosen to be $k_n = 10$. This set of choices is somewhat arbitrary but is one which facilitates comparison with results of Duffour and Woodhouse [7] and illustrates a range of qualitative behaviour. Fig. 2 shows the magnitude of the derivative of one of the roots with respect to (a) the natural frequency ω_1 (Eq. (20)), (b) the modal coefficient a_1 (Eq. (21)), (c) the damping factor ζ_1 (Eq. (22)), and (d) the contact stiffness k_n (Eq. (23)). The solid line in each represents the bifurcation condition from Eq. (24). Apart from Fig. 2 (c) it can be seen that this shows up

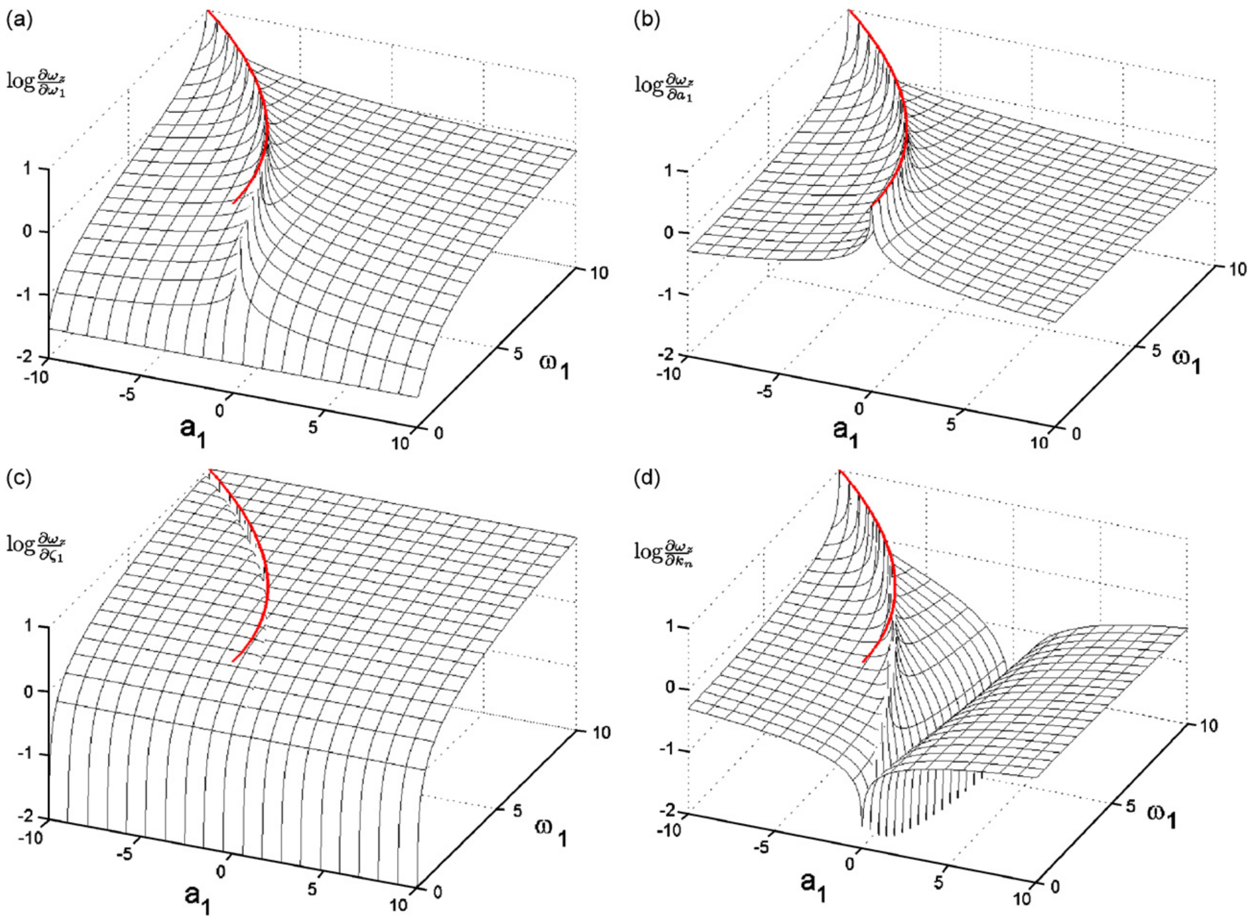

Fig. 2. Sensitivity of characteristic root with respect to (a) ω_1, (b) a_1, (c) ζ_1 and (d) k_n for a single-mode model that includes contact stiffness. Solid curve shows bifurcation condition of Eq. (24).

as a singular ridge in each of the plots (note that the maximum amplitude has been limited in order to make the figures clearer). On closer inspection it is seen to be true, though less clear, in Fig. 2 (c): in this case the singularity is very close to a zero so there is almost a pole-zero cancellation and the ridge is nearly invisible. If the damping had been greater then the singular ridge would have been more obvious.

It is convenient to define a combined measure of sensitivity to more than one parameter, for example the following definition provides a measure of sensitivity with respect to the natural frequency, ω_1, and modal coefficient, a_1:

$$|\nabla\omega_z| = \sqrt{\left|\omega_1 \frac{\partial\omega_z}{\partial\omega_1}\right|^2 + \left|a_1 \frac{\partial\omega_z}{\partial a_1}\right|^2}. \tag{25}$$

The factor in front of each partial derivative serves to normalise them such that each term represents the derivative with respect to a fractional change in one parameter.

Fig. 3 shows the variation of this combined measure of sensitivity—the singular ridge is still clearly visible though the rest of the surface has changed. There are a number of other possible related measures for sensitivity, including differentiating the imaginary part of the roots or their phase, quantities that are sometimes defined as 'squeal propensity' (e.g. Cao et al. [19]). The method used here is based on absolute root displacement (as in Eq. (25)) to consider sensitivity in its purest sense. These results show that in this case the root is only sensitive to parameter changes when they are close to the bifurcation condition in Eq. (24) and then, as is intuitively plausible, it is sensitive to *any* parameter changing. The introduction of contact compliance into the model has added both a mechanism for instability and a region in the parameter space which leads to high sensitivity.

The expression for sensitivity is valid for the whole parameter space, both near to and far from bifurcations. The bifurcation of roots is a special case that is often discussed in connection with stability, and for simple models it is frequently true that they also represent the boundary between stable and unstable predictions. For more complicated systems where bifurcation conditions are non-trivial they do not correspond so directly to stability, though they still give rise to high sensitivity.

As discussed above, a simpler analysis is possible if just the positive frequency pole of the mode is considered. This leads to the characteristic equation:

$$\omega - \omega_1\left(\sqrt{1-\zeta_1^2} + \mathrm{i}\zeta_1\right) - k_n \frac{a_1}{2\omega_1} = 0. \tag{26}$$

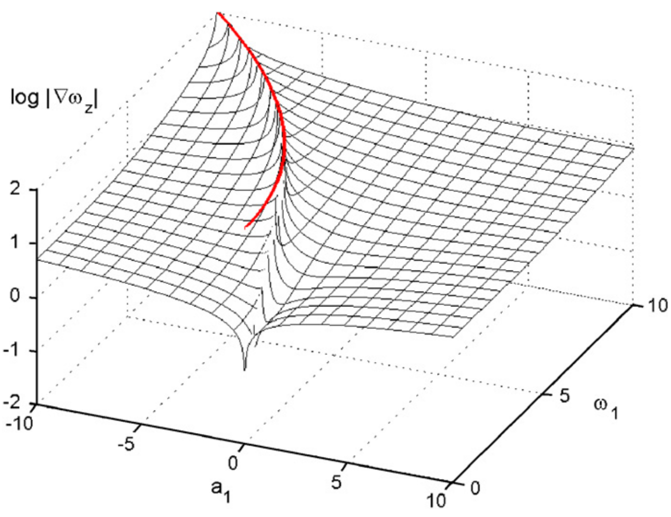

Fig. 3. Sensitivity of characteristic root with respect to proportional changes in natural frequency, ω_1, and modal coefficient, a_1.

The solution to this equation converges on the quadratic solution of Eq. (19) if damping is assumed to be light and $\omega_1 \gg \sqrt{a_1 k_n}$. Under these conditions, the positive frequency solution tends to:

$$\omega_z \approx \omega_1(1 + i\zeta_1) + k_n \frac{a_1}{2\omega_1}. \tag{27}$$

It is expected that only predicted zeros in the region of the positive frequency pole will give good approximations to the global prediction. Therefore it is no surprise that Eq. (27) only approximates the full single-mode analysis when the solution is a long way from the imaginary axis, as the uncoupled poles of a single mode are symmetric about this axis. Since in this case bifurcations, and high sensitivity due to the square root, only occur when the solutions lie on the imaginary axis, the approximation is not useful for predicting the region of high sensitivity here.

3.1.2. Non-proportional damping

There is no physical reason to assume that the damping matrix of a general system is proportional, i.e. simultaneously diagonalisable with the mass and stiffness matrices. Proportional damping is commonly assumed for mathematical convenience as it simplifies system analysis enormously by making it possible to find a set of decoupled equations that lead to the natural modes of a system. Usually this approximation is adequate for describing the system behaviour but it is not known how significant the introduction of non-proportional damping may be with regard to predicting the stability of sliding-coupled systems.

A detailed analysis of single-mode systems with non-proportional damping has revealed that it is not particularly significant if contact stiffness is also included in the model. Without contact stiffness, the introduction of non-proportional damping changes the order of the characteristic polynomial and therefore changes the number of solutions—the solutions are then extremely sensitive when the degree of non-proportional damping is small. This points towards the importance of including contact stiffness for a robust model.

3.1.3. Velocity-dependent coefficient of friction

A much studied model for frictional instability is based on an assumption that the coefficient of friction is velocity-dependent. In this case two characteristic functions determine the stability of the system: $E_1(\omega)$ and $E_2(\omega)$ from Eqs. (10) and (11), respectively. Contact stiffness is at first neglected. It turns out that the characteristic equation from $E_1(\omega)$ then has no solutions.

The second characteristic equation becomes

$$E_2(\omega) = 1 + i\omega\varepsilon N_0 \left[\frac{a_{22}}{\omega_1^2 + 2i\zeta_1\omega_1\omega - \omega^2}\right] = 0, \tag{28}$$

which can also be written as

$$\omega^2 - i(2\zeta_1\omega_1 + \varepsilon N_0 a_{22})\omega - \omega_1^2 = 0, \tag{29}$$

which has solutions:

$$\omega_z = i\left(\zeta_1\omega_1 - \frac{\varepsilon N_0 a_{22}}{2}\right) \pm \sqrt{\omega_1^2(1 - \zeta_1^2) - \omega_1\zeta_1\varepsilon N_0 a_{22} - \frac{\varepsilon^2 N_0^2 a_{22}^2}{4}}. \tag{30}$$

This result differs from the previous case. In Section 3.1.1 it was seen that when only contact stiffness was included only purely imaginary predictions could be unstable (see Eq. (19)). The solution above (Eq. (30)) can give unstable poles that are not purely imaginary and could therefore lie in the vicinity of the uncoupled mode. This can occur when the square root term is close to ω_1, for example if damping is light and the product $\varepsilon N_0 a_{22}$ is small. This implies that the local model may be particularly applicable in this case, both for stable and unstable solutions. In a similar way to Eq. (19) this solution bifurcates when the determinant equals zero:

$$a_{22} = -\frac{2\omega_1(\zeta_1 \pm 1)}{\varepsilon N_0}. \tag{31}$$

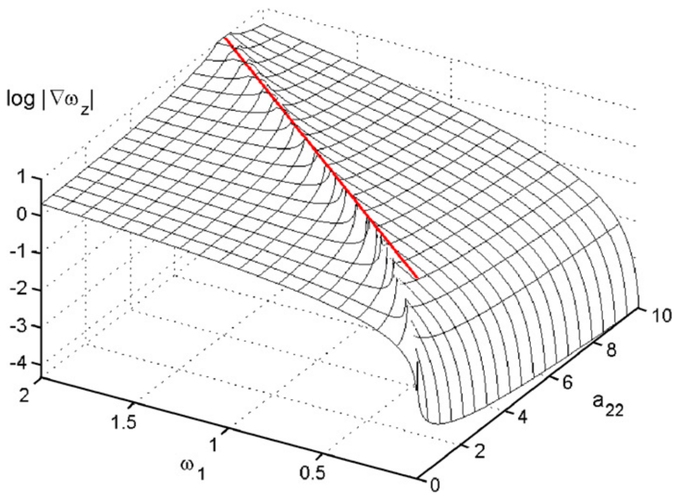

Fig. 4. Sensitivity of characteristic root when including one mode and a velocity-dependent coefficient of friction in the model.

A driving point modal coefficient can only be positive, so for cases where $\zeta_1 < 1$ only one of these conditions is possible, remembering that N_0 is negative. Fig. 4 shows the variation in sensitivity as the modal coefficient and natural frequency vary with $\varepsilon N_0 = -0.5$ (chosen arbitrarily). The condition for bifurcation, and high sensitivity due to the square root, given by Eq. (31) can be seen as a ridge. This condition suggests that if εN_0 is large, then the system becomes more sensitive to small amplitude modes; or as the normal pre-load is increased, low amplitude modes become significant. If this is true of multiple-mode systems this could be of value in understanding the sporadic nature and contact-force dependence of brake squeal.

An analysis of this case neglecting negative frequency poles does not yield a good approximation near the bifurcation point suggesting that this method is not particularly helpful when analysing sensitivity. If the negative frequency pole is regarded as a distant mode it may be hinting that neglecting other modes could adversely affect predictions.

Friction is an extremely complicated phenomenon to model, depending on many physical properties of the sliding contact in question. Including a velocity-dependent coefficient of friction in the model only scratches the surface of the level of complexity that has been uncovered in the friction modelling literature. However, as Duffour and Woodhouse [8] discuss, if the value of the velocity dependence, ε, is allowed to become complex then it can be used to describe *any* system linearised about a particular operating point, whatever internal state variables it may involve.

The simplest possible case that gives a real-valued impulse response is to assume that ε is constant over all positive frequencies and the conjugate of that constant over all negative frequencies. Interestingly this does not affect the condition for high sensitivity given in Eq. (31). This reduces the maximum sensitivity: a singularity can no longer occur for any parameter combination as the modal coefficient, a_1, and natural frequency, ω_1, are both purely real. Note that a constant but complex value of ε does not guarantee causality (see Bracewell [21]) and in reality it is likely to be a more complicated function of frequency, but this constant approximation can be considered to be valid over small frequency ranges and useful for local models with few modes.

3.2. *Effect of combining model parameters*

If both contact stiffness and a state-dependent coefficient of friction are included in the model then the characteristic functions $E_1(\omega)$ and $E_2(\omega)$ are each cubic.

The general solution can be written as a series of substitutions in a fairly compact form which contain nested roots of polynomials of the coefficients of the original equation. It is therefore possible to obtain bifurcation conditions that themselves bifurcate. Parameters that describe such points in the parameter space are strong candidates for extreme prediction sensitivity. However, the solution is a long-winded expression that sheds little light on the effect of changing parameters, or when bifurcations and high sensitivity occur. It is more

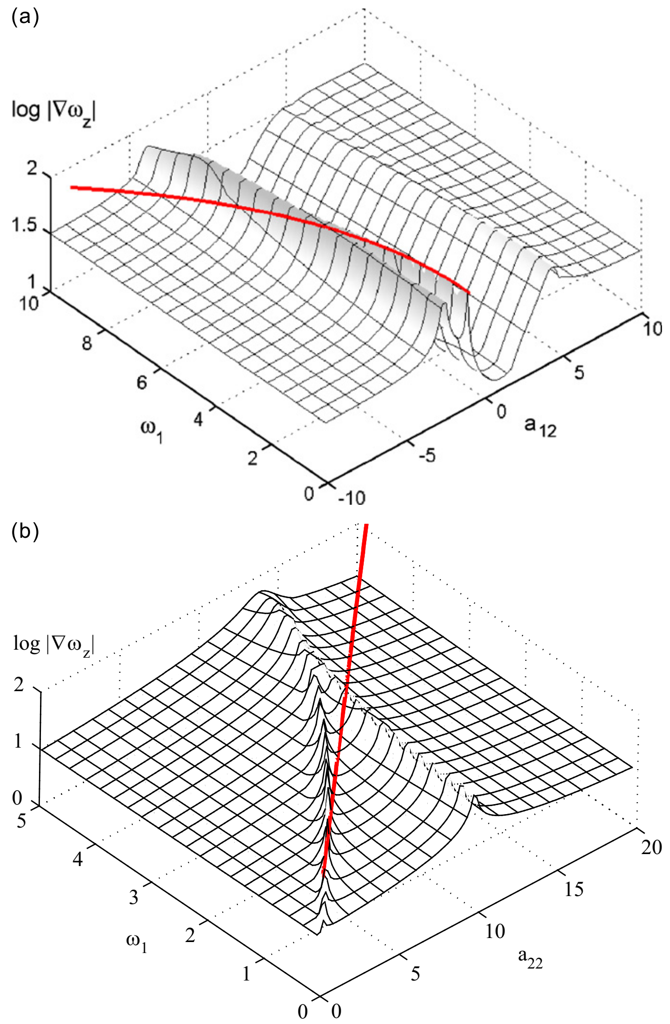

Fig. 5. Sensitivity of characteristic root when including one mode, contact stiffness and a velocity-dependent coefficient of friction in the model: numerical estimate of the maximum root sensitivity for each characteristic function: (a) $E_1(\omega)$ and (b) $E_2(\omega)$.

insightful to observe the behaviour of the solutions numerically as parameters change, rather than proceed with analytic study.

Fig. 5(a) shows a numerical estimate of the maximum sensitivity of the roots of $E_1(\omega)$ (as defined in Eq. (25)) while varying the natural frequency, ω_1, and the modal coefficient, a_{12} (with a_{11} held constant and $a_{22} = a_{12}^2/a_{11}$). In this case $k_n = 10$ as before and $\varepsilon N_0 = -0.5$. The numerical algorithms used have been validated by comparing results with the simpler cases of the previous section that could also be treated analytically. The critical parameter combinations that result in bifurcations are visible as ridges and it can be seen that they are more complicated than the cases studied so far. The solid line shows the bifurcation condition for the case when only contact stiffness was considered superimposed on the combined analysis. It can be seen that there is some correlation for low values of ω_1 but the independent case diverges for higher values. This is not surprising as a_{22} varies with a_{12}^2, amplifying the effect of the velocity-dependent coefficient of friction (as the term involving ε is multiplied by a_{22}).

Fig. 5(b) shows a similar plot for the maximum sensitivity of the roots of $E_2(\omega)$. Superimposed is the critical line from the case when only ε was included in the model. It can be seen that one of the ridges is predicted to an extent but that the second ridge is not accounted for.

4. Extension to include multiple modes

A similar analysis can be carried out on two and three mode systems, although very quickly the order of the polynomials increase with the model complexity. This can be seen in Table 1 which shows the order of the characteristic polynomials E_1 and E_2 (defined in Eqs. (10) and (11)) for different cases. The items of the table that are dashed out correspond to cases where there are no solutions.

Clear analytic results are only obtainable from linear or quadratic equations, so only a few more cases could be explored in this manner. For $E_1(\omega)$: the two-mode case without contact stiffness or velocity-dependent coefficient of friction (with or without negative frequency poles), the two-mode case considering the contact stiffness but discarding the negative frequency poles, and the simplest three-mode case discarding negative-frequency poles. For $E_2(\omega)$: the non-proportionally damped one-mode case with ε and the two-mode case with only ε.

Each of these cases has been analysed but a full description is not given here: only a few highlighted examples will be discussed to illustrate the kind of effects observed. The two-mode system that includes negative frequency poles has a quadratic characteristic equation, the solutions of which are highly sensitive at the two bifurcation conditions. If damping is light, these are:

$$a_1 \approx -a_2, \tag{32}$$

$$a_1 \approx \frac{-a_2\omega_1^2}{\omega_2^2}. \tag{33}$$

If the sensitivity is plotted as a function of the modal coefficient, a_n, and natural frequency, ω_n, two corresponding ridges can be identified. This shows that the interaction of the two modes can also give rise to high sensitivity. This case is particularly interesting as high sensitivity can occur when one modal amplitude is much smaller than the other and if they are well separated in frequency. This shows that it can be dangerous to regard small amplitude modes as insignificant.

Table 1
Effect of number of modes included and model complexity upon order of the characteristic polynomials, E_1 and E_2

No. of modes	Model	Polynomial order					
		With −ve freq poles		Non-prop. damping		Without −ve freq poles	
		$E_1(\omega)$	$E_2(\omega)$	$E_1(\omega)$	$E_2(\omega)$	$E_1(\omega)$	$E_2(\omega)$
1	No k or ε	–	–	1[a,b]	–	–	–
	Only k	2[a]	–	2[a]	–	1[a]	–
	Only ε	–	2[a]	3[b,c]	2[a]	–	1[a]
	k and ε	3[b,c]	3[b,c]	3[b,c]	3[b,c]	2[c]	2[a]
2	No k or ε	2[a]	–	3[b,c]	–	1[a]	–
	Only k	4[c]	–	4[c]	–	2[a]	–
	Only ε	6	4[c]	7[b]	4[c]	3[c]	2[a]
	k and ε	9[b]	5[b]	9[b]	5[b]	5	3[c]
3	No k or ε	4[c]	–	5[b]	–	2[a]	–
	Only k	6	–	6	–	3[c]	–
	Only ε	10	6	11[b]	6	5	3[c]
	k and ε	13[b]	7[b]	13[b]	7[b]	7	4[c]
N	No k or ε	$2N-2$	–	$2N-1$[b]	–	$N-1$	–
(if $N>1$)	Only k	$2N$	–	$2N$	–	N	–
	Only ε	$4N-2$	$2N$	$4N-1$[b]	$2N$	$2N-1$	N
	k and ε	$4N+1$[b]	$2N+1$[b]	$4N+1$[b]	$2N+1$[b]	$2N+1$	$N+1$

[a]Useful analytic solutions can be derived.
[b]Odd degree polynomial that has one root on imaginary axis.
[c]Analytic solutions possible but shed little light.

An important issue is whether the critical lines indicating conditions for high sensitivity and bifurcation derived for the simplest cases can be applied to more complex systems by superposition. This has been shown to be true to a very limited extent in Section 3.2 when the single-mode case was made more complicated. Numerical analysis is required to study the two-mode case that includes contact stiffness, as the characteristic equation becomes a quartic. Results indicate that the critical conditions from the independent cases (given by Eqs. (24), (32) and (33)) are robust for the parameter space considered, and superposition quite accurately predicts high sensitivity for this more complicated case. However, when a velocity-dependent coefficient of friction is also included in the model then this technique breaks down and ridges of high sensitivity do not clearly correlate with those of the simpler cases. This remains true for the other more complicated cases studied.

A further class of cases can be treated analytically if damping is neglected. This leads to characteristic equations in integer powers of ω^2, allowing quartics to be solved easily. Hoffmann et al. [10] discuss extensively the two-mode case with contact stiffness but without damping or a velocity-dependent coefficient of friction. The model consists of a mass with two degrees of freedom in sliding contact with a belt via a linearised contact stiffness, the mass being supported by two springs at arbitrary angles (see Fig. 6). It is designed as a minimal model to study the 'mode-coupling' route to instability commonly thought to be a primary mechanism that leads to squeal. The discussion is valuable in providing some intuition to the phenomenon of friction-induced vibration, but the reduced-order models described in this article flag some warnings. The theoretical framework of Section 2 can be applied to the particular case discussed by Hoffmann et al., when the characteristic equation becomes

$$D(\omega) \approx \frac{0.777 + \mu_0 0.416}{0.363^2 - \omega^2} + \frac{0.223 - \mu_0 0.416}{1.592^2 - \omega^2} + \frac{3}{4} = 0. \quad (34)$$

In this case the four complex solutions are:

$$\omega_z = \pm \left[2 \pm \sqrt{1 - \frac{4\mu_0}{3}} \right]^{1/2}. \quad (35)$$

Two bifurcation points exist for the two conditions when the terms in the square bracket go to zero. These occur when $\mu_0 = 0.75$ and $\mu_0 = -2.25$. Note that the latter negative value corresponds to a reverse in the direction of the steady velocity, V_0, and though its magnitude is somewhat implausible it does not detract from the illustration. For each bifurcation, two purely real solutions split into one stable and one unstable mode. Such bifurcations have also been shown to be points of high sensitivity with respect to many parameters, so care should be taken in drawing conclusions near the thresholds of stability. Further from these boundaries the model is more robust to parameter changes, though not necessarily to model changes (such as additional modes). It is notable that in the presence of a small amount of damping no bifurcation is apparent and the onset of instability can less clearly be linked to bifurcations or 'mode-coupling'. In a subsequent paper

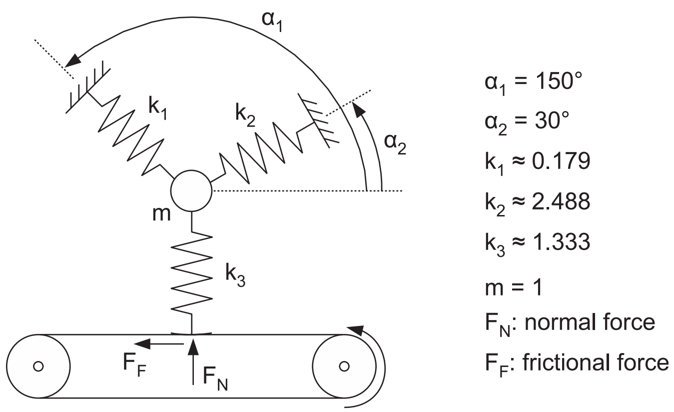

Fig. 6. Reduced-order model analysed by Hoffmann et al. [10] with parameter values used for case study.

Hoffmann and Gaul [20] include damping and the results show some of the more complicated behaviour that can arise. Further analysis will form the basis of future work—of most importance here is that reduced-order models such as this must be treated with care.

It is of interest to conclude with a series of examples developing a three-mode model. The characteristic function E_1 becomes

$$
\begin{aligned}
E_1 = {} & \sum_{n=1}^{3} \frac{a_{11}^{(n)} + \mu_0 a_{12}^{(n)}}{\omega_n^2 + 2\mathrm{i}\zeta_n\omega_n\omega - \omega^2} + \frac{1}{k_n} \\
& + \mathrm{i}\omega\varepsilon N_0 \left\{ \left(\sum_{n=1}^{3} \frac{a_{11}^{(n)}}{\omega_n^2 + 2\mathrm{i}\zeta_n\omega_n\omega - \omega^2} + \frac{1}{k_n} \right) \left(\sum_{n=1}^{3} \frac{a_{22}^{(n)}}{\omega_n^2 + 2\mathrm{i}\zeta_n\omega_n\omega - \omega^2} + \frac{1}{k_t} \right) \right. \\
& \left. - \left(\sum_{n=1}^{3} \frac{a_{12}^{(n)}}{\omega_n^2 + 2\mathrm{i}\zeta_n\omega_n\omega - \omega^2} \right)^2 \right\}
\end{aligned} \tag{36}
$$

Firstly the three-mode case ignoring negative frequency poles is considered without contact stiffness or a velocity-dependent coefficient of friction. Secondly the equivalent case with negative frequency poles and finally a closely related three-mode case that also includes contact stiffness and a velocity-dependent coefficient of friction. These results are shown in Fig. 7(a–c). Bifurcation conditions for these cases have not been superimposed as analytic expressions cannot be derived for (b) and (c). For each of these cases, the properties of the second and third modes are held constant with natural frequencies of $\omega_2 = 1$ and $\omega_3 = 1.2$, corresponding modal coefficients $a_{11}^{(2)} = a_{11}^{(3)} = 1$ with cross-terms $a_{12}^{(2)} = a_{12}^{(3)} = 0$, and damping factors $\zeta_2 = \zeta_3 = 0.01$. The damping factor of the first mode is fixed at $\zeta_1 = 0.1$ (chosen arbitrarily but with a view to representing a more heavily damped mode of the brake subsystem). The equivalent figures for E_2 have not been shown as this region of the parameter space turns out not to be of interest. In Fig. 7(a and b) the modal coefficient a_1 is varied, which could represent a change in either $a_{11}^{(1)}$ or $a_{12}^{(1)}$. It can be seen that case (a) which ignores negative frequency poles is a good approximation to case (b) over parts of the parameter space. The ridge of case (a) approximates the central part of the diagonal ridge in case (b), and the sharp peak near $a_1 \approx 0.4$ and $\omega_1 \approx 1.1$ is evident in both plots. Analysis using only positive frequency poles for more realistic systems may therefore prove to be somewhat more valuable than first suspected. This is an area of ongoing work.

Fig. 7(c) shows an example of the sensitivity variation for E_1 obtained for the three-mode case that includes both contact stiffness ($k_n = 10$) and a velocity-dependent coefficient of friction ($\varepsilon N_0 = -0.01$). The natural frequency of the first mode, ω_1, and the cross-term of the modal coefficient, $a_{12}^{(1)}$ are allowed to vary but with the direct term fixed at $a_{11}^{(1)} = 0.1$. This value was deliberately chosen to be an order of magnitude smaller than the other two modes to represent the case where a cluster of three modes is found during modal analysis, where one of the modes is much smaller than the others (and is often considered to be insignificant). The modal coefficients a_{22} for each mode are still kept consistent with the other coefficients such that $a_{22} = a_{12}^2/a_{11}$. Since $a_{11}^{(1)}$ is small, it is still possible to compare results with the previous two cases (as $a_1 = a_{11}^{(1)} + \mu_0 a_{12}^{(1)}$). Therefore varying $\mu_0 a_{12}^{(1)}$ is the same as varying a_1 over a very similar range.

It can be seen in Fig. 7(c) that features of the two preceeding cases are recognisable but modified. The main ridge bifurcates at around $a_{12} \approx -2$ and $\omega_1 \approx 1.1$. Two peaks are visible when $a_{12} \approx -1.5$ with $\omega_1 \approx 1$ and $\omega_1 \approx 1.2$. This suggests that sensitivity can be increased when modes are close in natural frequency, as these correspond to the natural frequencies of the fixed modes. One more very sharp peak can be identified at $\omega_1 \approx 1.1$ and $a_{12} \approx 0.25$, a feature that is evident in all three cases. Though none of the details of these features can be predicted using the idea of superposition of critical conditions, the simpler models do seem to broadly highlight the regions of the parameter space where high sensitivity occurs in more complicated models. The most interesting result from this example is that high sensitivity occurs under all kinds of conditions. Sometimes this is counter-intuitive, for example in this case when one of the modal amplitudes is significantly smaller than the others. Also non-singular maxima can occur in the sensitivity for these more complicated systems: it can be seen that not all of the ridges in Fig. 7(c) are truncated.

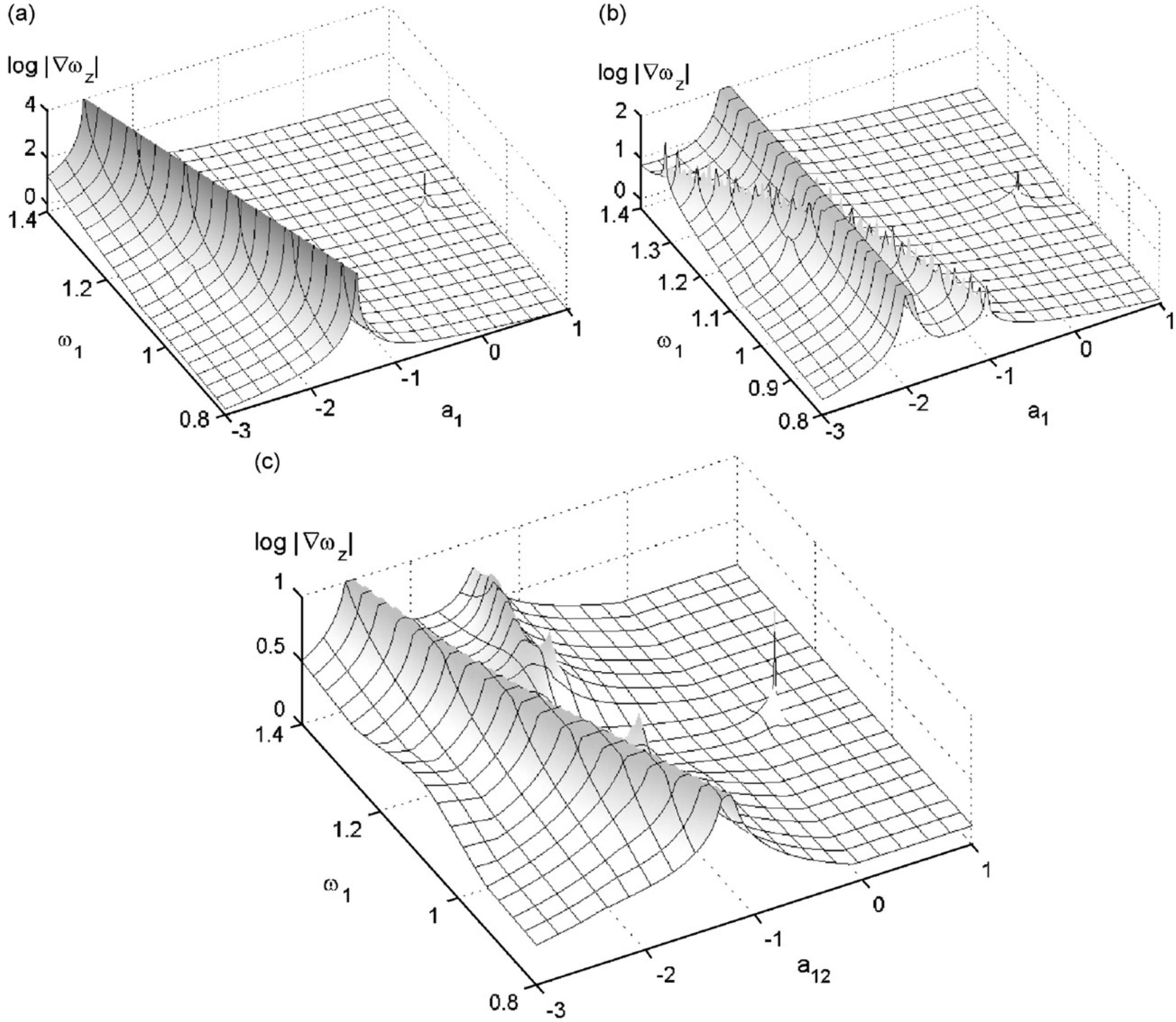

Fig. 7. Sensitivity of characteristic root of $E_1(\omega)$ when including three modes under different modelling assumptions: (a) without contact stiffness or velocity-dependent coefficient of friction and ignoring negative frequency poles, (b) without contact stiffness or velocity-dependent coefficient of friction but including negative frequency poles and (c) with contact stiffness and a velocity-dependent coefficient of friction and including negative frequency poles. Figure (a) is the maximum analytical sensitivity of the roots, and figures (b) and (c) use a numerical estimate of the maximum root sensitivity.

5. Conclusions

Friction-induced vibration occurs in many contexts: vehicle brake squeal in particular remains surprisingly unpredictable and poorly understood. A recurring theme in the literature is the difficulty in obtaining repeatable experimental results that correlate with theoretical models, which suggests that the phenomenon is sensitive to small changes in parameters. This paper has explored reduced-order models as a starting point to understanding this sensitivity, and to better understand the validity of the multitude of other studies that focus on reduced-order models (Hoffmann et al. [10], Kinkaid [11], von Wagner [12] and Emira [13]).

Firstly, a new stability criterion has been derived which extends the model previously developed by Duffour and Woodhouse [7,8]. This accounts for all linear routes to instability within the framework of Duffour's single-point contact model. Secondly, this study has examined highly idealised models of friction-coupled systems in order to address fundamental questions of modelling details and causes of high sensitivity. Predicting the effect of parameter changes of real systems is difficult as the characteristic polynomials are an intricate function of these parameters: reduced-order models allow a more thorough analysis to be carried out which provides clearer insight into this relationship. Therefore this study focussed on the behaviour of systems with very few modes for which the effects of including a range of physical elements in the model were considered.

Even with an extremely simple model that includes just a single natural mode of vibration of one of the subsystems, a rich variety of possible behaviours exist and several independent modelling assumptions can lead to the prediction of instability. It has been shown that the inclusion of contact stiffness, non-proportional damping, or a velocity-dependent coefficient of friction can each lead to instability and/or extreme sensitivity to parameter variations under particular conditions. Consequently all of these are very likely to play a significant part in the overall system stability. Conversely if any of these factors are neglected in a model there is a danger that predictions will poorly approximate real systems. Significantly, high sensitivity can occur at the boundary of stability and when small amplitude modes are included in the model that might normally be considered insignificant. High sensitivity can either be due to a bifurcation point or result from simple singularities in the partial derivative of a root with respect to some parameter.

Not surprisingly the conditions derived for high sensitivity from studies that include physical effects independently cannot in general be superimposed to provide an estimate of conditions for high sensitivity when combining several effects. The conditions are also unlikely to correlate in a clear way to larger systems, but as the polynomials increase in order it is clear that the number of possible conditions for high sensitivity multiply.

If non-proportional damping is included in a model in isolation then the order of the characteristic polynomial is raised by one, such that the extra root must lie on the imaginary ω-axis. However, considered together with contact stiffness, non-proportional damping has a less significant effect as the order of the characteristic polynomial remains unchanged. In this case the coefficients of the polynomial are only perturbed slightly.

Analysing systems by discarding negative frequency poles significantly simplifies the characteristic polynomial but has been found to be of limited value for sensitivity considerations of reduced-order models as the approximate solution is often invalid when the sensitivity becomes large. Interestingly the method seems to yield better approximations for more complicated systems (e.g. with three or more modes included).

Including a velocity-dependent coefficient of friction introduces a route to instability, which otherwise does not exist in the single-mode case without contact stiffness or non-proportional damping. It also introduces a highly sensitive region of the parameter space. Allowing a complex term to describe a more general state-dependent coefficient of friction does not significantly affect sensitivity analyses and reduces the maximum sensitivity.

This paper has revealed that even very simple systems behave in surprisingly complicated ways and that predictions are strongly affected by physical effects that are often neglected. This highlights the importance of carefully developing models, and points towards significant reasons for the sporadic nature of friction-induced vibration. With a great deal of literature focussing on reduced-order models this study gives an important warning for the interpretation of conclusions from such work. It has been clearly shown that sensitivity is highly dependent upon the dynamics of the system. This suggests that it may be possible to design more robust friction-coupled systems in future by including a sensitivity measure in the cost function for a design optimisation.

Acknowledgements

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