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The forced vibration of one-dimensional multi-coupled periodic structures: An application to finite element analysis

Denys J. Mead*

Institute of Sound and Vibration Research, University of Southampton, Southampton SO17 1BJ, UK

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Abstract

A general theory for the forced vibration of multi-coupled one-dimensional periodic structures is presented as a sequel to a much earlier general theory for free vibration. Starting from the dynamic stiffness matrix of a single multi-coupled periodic element, it derives matrix equations for the magnitudes of the characteristic free waves excited in the whole structure by prescribed harmonic forces and/or displacements acting at a single periodic junction. The semi-infinite periodic system excited at its end is first analysed to provide the basis for analysing doubly infinite and finite periodic systems. In each case, total responses are found by considering just one periodic element. An already-known method of reducing the size of the computational problem is reexamined, expanded and extended in detail, involving reduction of the dynamic stiffness matrix of the periodic element through a wave-coordinate transformation. Use of the theory is illustrated in a combined periodic structure + finite element analysis of the forced harmonic in-plane motion of a uniform flat plate. Excellent agreement between the computed low-frequency responses and those predicted by simple engineering theories validates the detailed formulations of the paper. The primary purpose of the paper is not towards a specific application but to present a systematic and coherent forced vibration theory, carefully linked with the existing free-wave theory.

1. Introduction

A periodic structure (PS) is well known as comprising a number of identical structural components joined together repetitively in identical ways to form the whole structure. Each repeated component constitutes a 'periodic element' which itself may be composed of sub-elements which can be identical or different from one another. A general theory of free harmonic wave propagation in linear multi-coupled PSs was first published by the author in the early 1970s [1–3] and has been widely used since then. More recently others have incorporated it into particular finite element studies of both free and forced high-frequency vibration along rails and other structural waveguides. This paper draws together, extends and explains some of the methods used in these particular forced vibration studies and presents a general theory of the forced vibration of one-dimensional (1D) PSs. It therefore constitutes a sequel to the author's early paper.

*Tel.: +44 23 8055 7357.
E-mail address: denysmead@aol.com

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Nomenclature

$\mathbf{D}$ dynamic stiffness matrix of a periodic element
$\mathbf{D}_{LL}$, $\mathbf{D}_{LR}$, $\mathbf{D}_{RR}$, $\mathbf{D}_{Li}$, etc. sub-matrices of $\mathbf{D}$
$\mathbf{D}_{N_{el}}$ dynamic stiffness of a line of N_{el} periodic elements
$\mathbf{E}^{\mu}$ diagonal matrix of all e^{μ}'s
$^{+}\mathbf{E}^{\mu}$, $^{-}\mathbf{E}^{\mu}$ diagonal matrix of the e^{μ}'s for positive- or negative-going waves
$\mathbf{F}$ column vector of all coupling forces acting on a periodic element
$\mathbf{F}_{ext}$ vector of external forces applied at a junction
$\mathbf{F}_{0ext}$ vector of externally applied forces at junction 0
$\mathbf{F}_L$, $\mathbf{F}_R$ coupling force vectors on the left- or right-hand ends of an element
$\mathbf{F}_i$ vector of forces acting at the internal coordinates of the element
$\mathbf{F}_{L\text{ext(gen)}}$ column vector of the generalised forces on the sub-modes generated by the externally applied forces
$\mathbf{F}_{L\text{gen}}$, $\mathbf{F}_{R\text{gen}}$ generalised coupling forces acting on the left- and right-hand ends of an element in the reduced analysis
$^{+}\mathbf{F}_{L(\text{LHend})}$, $^{+}\mathbf{F}_{L(\text{RHend})}$ forces at the left- or right-hand end due to a positive-going wave reflected from a left-hand end
$^{-}\mathbf{F}_{L(\text{LHend})}$, $^{-}\mathbf{F}_{L(\text{RHend})}$ forces at the left- or right-hand end due to a negative-going wave reflected from a right-hand end
$\mathbf{F}_{L\text{sub},j}$ vector of generalised coupling forces of wave j of the reduced system
J total number of participating waves in the original system
J_{red} total number of participating waves in a reduced analysis
n_c number of coupling coordinates between an adjacent pair of periodic elements
n_{cw} number of complex waves contributing to the reduced analysis
n_i number of internal coordinates within a periodic element
n_{new} complex propagation constant of a wave found from the reduced analysis
n_{red} number of sub-modes used in the reduced analysis
n_w number of complex waves used in the reduced analysis
N_{FR}, N_{FL} number of periodic elements between the excitation point and the right- or left-hand end of the finite system
N_k number of a selected element within a finite array of periodic elements
N_{st} number of finite elements in each stack
$\mathbf{q}$ column vector of initial set of all initial coordinates of a periodic element
$\mathbf{q}_i$ column vector of the internal displacement coordinates of the element
$\mathbf{q}_L$, $\mathbf{q}_R$ column vectors of displacement coordinates on the left- and right-hand ends of an element
$\mathbf{q}_{L,j}$ displacement coordinate vector at junction j of the periodic array
$^{-}\mathbf{q}_{L,j}$ total displacement vector of all negative-going waves at junction j of a left-hand semi-infinite system
$^{+}\mathbf{q}_{R,j}$ displacement vector due to all positive-going waves at junction j of a right-hand semi-infinite system
$\mathbf{q}_{L_0}$, $\mathbf{q}_{R_0}$ column vectors of displacements at the left- and right-hand ends of an internally loaded element
$\mathbf{q}_{L\text{sub}}$, $\mathbf{q}_{R\text{sub}}$ column vectors of all the sub-mode generalised complex displacements at the left- and right-hand ends of an element
$^{+}\mathbf{q}_{L(\text{LHend})}$, $^{+}\mathbf{q}_{R(\text{RHend})}$ displacement vectors at the extreme left- and right-hand ends of a finite system due to all positive-going waves from the left-hand end
$^{-}\mathbf{q}_{L(\text{LHend})}$, $^{-}\mathbf{q}_{R(\text{RHend})}$ displacement vectors at the extreme left- and right-hand ends of a finite system due to all negative-going waves from the right-hand end
$\boldsymbol{\alpha}$ receptance matrix of a periodic element
$^{0}\boldsymbol{\alpha}$ complete receptance matrix of the internally loaded periodic element
$\boldsymbol{\alpha}_{N_{el}}$ overall receptance matrix relating the $\mathbf{q}$'s and $\mathbf{F}$'s at the two ends of a finite system of N_{el} periodic elements
$\boldsymbol{\alpha}_{N\text{semi}}$ receptance matrix relating the displacements $\mathbf{q}_R$ at junction N to the forces applied at the left-hand end of a semi-infinite system
μ complex propagation constant
$\boldsymbol{\theta}_j$ normalised eigenvector of coupling coordinates q_L for wave j
$\boldsymbol{\theta}_{\text{new},j}$ column matrix of the $\mathbf{q}_{L\text{sub}}$ contributions to new wave j in the reduced analysis
$\boldsymbol{\Theta}$ matrix of all the ϕ_j's

$\boldsymbol{\Theta}_{L\text{red}}$, $\boldsymbol{\Theta}_{R\text{red}}$, $\boldsymbol{\Theta}_{\text{red}}$ real parts of the reduced original $\boldsymbol{\Theta}$ matrix
$\boldsymbol{\Theta}_{\text{new}}$ matrix of all $\theta_{\text{new},j}$'s
$\boldsymbol{\Theta}_{\text{free}}$, $\boldsymbol{\Phi}_{\text{free}}$ sub-matrices of $\boldsymbol{\Theta}$ and $\boldsymbol{\Phi}$ corresponding to the free displacements and external forces at the loading location
$\boldsymbol{\Theta}_{\text{pres}}$,$\boldsymbol{\Phi}_{\text{pres}}$ sub-matrices of $\boldsymbol{\Theta}$ and $\boldsymbol{\Phi}$ corresponding to the prescribed displacements and external forces at the loading location
$\boldsymbol{\phi}_j$ normalised eigenvector of the coupling forces corresponding to wave j
$\boldsymbol{\Phi}$ matrix of all the ϕ_j's
$\boldsymbol{\phi}_{\text{new},j}$ coupling force vector corresponding to $\theta_{\text{new},j}$
$\boldsymbol{\Phi}_{\text{new}}$ matrix of all the $\phi_{\text{new},j}$'s
$\boldsymbol{\Phi}_{\text{pres}}$ sub-matrix of $\boldsymbol{\Phi}$ corresponding to the prescribed forces at the loading location
ψ_j complex magnitude of wave θ_j
$\psi_{\text{new},\,j}$ complex magnitude of new wave $\theta_{\text{new},j}$
$\boldsymbol{\Psi}$ column vector of all ψ_j's in original system
$\boldsymbol{\Psi}_{\text{new}}$ column vector of all $\psi_{\text{new},j}$'s in the reduced analysis
$^+\boldsymbol{\Psi}_{\text{new}}$ complex magnitudes of the n_{red} positive-going waves of the reduced set
$^+\boldsymbol{\Psi}_{\text{inf}}$ $^-\boldsymbol{\Psi}_{\text{inf}}$ vectors of wave magnitudes to the right or left of the single exciting force on an infinite periodic system
$^+\boldsymbol{\Psi}_{\text{refl}}$, $^-\boldsymbol{\Psi}_{\text{refl}}$ vector of positive- or negative-going reflected wave magnitudes

The general theory is concerned with the free harmonic *wave* motion which can propagate in the whole system, rather than with motion expressed in terms of free normal modes. At any frequency a number of distinctly different waves can exist, some of which propagate with no attenuation if the structure is undamped, while others decay as they travel. The total number of such waves which can exist is twice the number of coupling coordinates linking the adjacent pairs of periodic elements. Each wave is characterised by a unique complex propagation constant, its real part being the 'attenuation constant' or 'decay index' quantifying the exponential decay (or growth) rate of the decaying (or growing) wave as it traverses a single element. Its imaginary part has been called the 'phase constant' and represents the phase difference between the motions of the wave at the two ends of an element. If the structure is undamped, the propagation constants may be purely imaginary and correspond to waves which propagate energy, unlike the evanescent decaying waves. They are not necessarily (nor often) purely spatially sinusoidal but actually consist of wave groups which have both positive- and negative-going sinusoidal components of different wave speed. This feature is particularly significant if the structure is excited by distributed non-uniform pressure fluctuations.

In general, external harmonic forces acting on the structure excite all the possible free waves, and the total generated structural motion can be expressed as the sum of their contributions. The forced vibration theory of this paper presents a matrix method by which this total contribution can be determined when point harmonic forces act on the structure. It starts from the free-wave propagation theory which yields the characteristic wave displacements and their propagation constants. These are the eigenvectors and eigenvalues of an equation derived from the dynamic stiffness matrix of a single periodic element. (An alternative method utilises the transfer matrix rather than the dynamic stiffness matrix.) From this, one finds the complex magnitude of each wave excited in the structure by an arbitrary set of external forces at a single periodic junction. The sum of their contributions yields the total response at the point of excitation. The known propagation constants for each contributing wave are then used to find the response at any other point in the whole structure. General equations are first derived for semi-infinite systems excited at one end. This particular system is seen as the key to handling both doubly infinite and finite periodic systems excited at any periodic junction or within a periodic element.

Different ways have been used in the past to determine the waves, their propagation constants and the forced responses but they lack the generality of the method presented here. Early work concentrated on the free and forced vibrations of periodic beam and plate configurations [4,5] and utilised closed-form solutions of the Euler–Bernoulli or sandwich-beam flexural wave equations. Excitations by single-point, multi-point harmonic forces or distributed harmonic and random pressure fields were considered. With the early computational facilities the calculations were often very tedious so approximate methods were developed to deal with beam and plate responses to distributed pressure field excitation. A Rayleigh–Ritz approach was first used, pairs or series of complex approximate modes being combined to represent the flexural motion [6,7]. The so-called 'space-harmonic method' [7,8] was very convenient for studying responses to random pressure field excitation. A later method for point-excited uniform periodic beams and stiffened plates [9,10] made use of the closed-form response function for an infinite, uninterrupted continuous structure under a single-point load.

The finite element method (FEM) has been used since 1974 in the study of periodic beams, regularly stiffened plates and stiffened cylindrical shells [11–14]. In Refs. [11,12], the basic periodic element was subdivided into standard FEs, with the individual element displacements being expressed in terms of real polynomial functions. In Refs. [13,14] the hierarchical FEM was used, the displacements of the whole periodic element being expressed in terms of real orthogonal polynomial functions. In all of these studies (and in many others) FEM was being used as a tool in PS analysis, allowing increasingly complicated periodic structural problems to be handled.

Latterly, the reverse process has taken place with PS theory being used to assist in FE calculations [15–23]. This has been required in studies of high-frequency wave propagation and sound radiation from 1D uniform rails, bars and thin-walled stiffeners, all on periodic supports. At low frequencies, exact closed-form methods are adequate, with the motion of each periodic element being represented by the exact solution of the appropriate (approximate) wave equation. However, when the rail or stiffener has a complicated cross section and the frequency is high, these equations are no longer appropriate as the motion can involve coupled flexural, torsional and longitudinal wave motion, together with that of cross-sectional distortions. Such waves are best analysed by FEM with numerous small elements, but if the standard computational methods are used, enormous matrices and computer times are required due to the sheer number of elements in the lengths of rail which have to be considered.

The forced vibration theory of this paper is readily applied as a remedy to this problem by reducing it to the analysis of a single 'slice' of elements across the rail section. Extending the author's previous free vibration theory [1] in a systematic way, it is based on the following concepts. A system of harmonic forces acting at one position on the PS generates free waves which travel outwards from the source in both directions. The wave amplitudes can be related to the applied force vector through the wave eigenvectors and propagation constants. If the PS extends indefinitely in both directions from these forces, only the outward-going waves are generated and these combine to produce the total motion. When the structure is finite, they are reflected back from the ultimate boundaries and the whole set of outgoing and reflected waves combine in proportions which satisfy all the boundary conditions of the structure. Viewed like this, the response calculation involves a *wave motion* study, rather than one of *normal modes*. It can yield the harmonic structural response both at the forcing point and at any other point remote from the source.

This paper shows how this can be accomplished by a series of matrix operations and should be regarded as a sequel to the author's free-wave theory of 1973 [1]. it applies fundamentally to general 1D PSs which are not necessarily being analysed by FEMs. However, by drawing together and expanding on concepts already used by other authors, it does lay the foundation for combining PS theory with the FE analysis of forced wave motion in uniform bars, beams and structural sections. Its use is illustrated and the detailed theory validated in some FE computations of forced in-plane vibrations of infinite and finite flat plates.

Studies of wave propagation in transport vehicle rails have already used the combined periodic-structure/finite-element analysis, usually with an emphasis on the actual rail dynamics. This paper focuses principally on the basic forced vibration theory, developing it in a systematic and coherent manner and carefully linking it with the former free-wave theory.

2. The matrix theory of forced harmonic vibration of multi-coupled periodic structures

2.1. The general approach

Fig. 1a illustrates several multi-coupled periodic elements joined together end-to-end. Let N_{el} denote the number of periodic elements in the whole length of the structure. Each element is connected to its neighbour on either side through n_c coupling coordinates and forces. The nature of these coordinates depends on the deflection forms allowed within the elements, different forms having been used by different investigators over the years. The analysis of this paper applies to any of these.

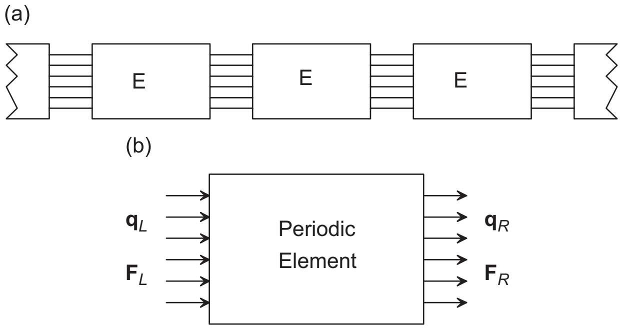

Fig. 1. (a) Diagram illustrating several multi-coupled periodic elements joined together end-to-end. (b) A single periodic element and the coupling coordinates and forces at its ends.

The whole object of this forced vibration theory is to determine the dynamic stiffness and receptance matrices relating the displacements at one location in the whole PS to the externally applied forces at another location. These locations may be separated by many elements, and this can lead to very long calculations when traditional methods are used. The methods of PS analysis reduce the calculation dramatically because these matrices can be found by analysing just one periodic element. It starts from the dynamic stiffness matrix of the single periodic element, but it is not the purpose of this paper to discuss ways in which this matrix can be derived.

The first stage of the theory is to reduce the size of the element dynamic stiffness matrix to the minimum required and then to find all the propagation constants of the wave motions which can occur in the whole PS. The displacement eigenvectors corresponding to each of these must be found (i.e. the characteristic complex modes of displacement of a single element) and their force eigenvectors (i.e. the forces acting on that element to produce these displacements). The basic free-wave theory of Mead [1] is used for this but the data it yields needs to be assembled in appropriate matrix forms for the forced vibration analysis. Further matrices are required which relate the displacements and forces of a particular wave at one location to those of that wave at another location. This is a very simple relationship for just one wave but a single matrix equation is required to express it when all the possible waves are simultaneously present.

Since data from the free-wave theory is essential to the forced wave theory, it is considered in some detail before being used in the later sections. First of all, these establish matrix equations for the response anywhere in a semi-infinite periodic system excited by an arbitrary set of harmonic forces acting at its finite end. These equations are then used to deal with a 'doubly' infinite system with externally applied forces acting at a single junction between two elements. Considered next are forces which act actually within a single element, and this is followed by an analysis of the finite system with forces acting anywhere within it.

2.2. The dynamic stiffness of the periodic element

The deflections of a single periodic element are represented (as in Ref. [1]) by the column vector of generalised coordinates $\mathbf{q} = \lfloor \mathbf{q}_L \quad \mathbf{q}_i \quad \mathbf{q}_R \rfloor^{\mathrm{T}}$. $\mathbf{q}_L$ are the n_c coupling coordinates which link the element to its left-hand neighbour; $\mathbf{q}_R$ are the coupling coordinates linking it to the right (see Fig. 1b). $\mathbf{q}_i$ are n_i interior coordinates which are not indicated on the figure. Corresponding to $\mathbf{q}$ is the vector of generalised forces $\mathbf{F} = (\mathbf{F}_L \quad \mathbf{F}_i \quad \mathbf{F}_R)^{\mathrm{T}}$.

$\mathbf{q}$ and $\mathbf{F}$ are related through the element dynamic stiffness matrix $\mathbf{D}$ such that $\mathbf{F} = \mathbf{D}\mathbf{q}$. In expanded partitioned form (as in Ref. [1]) this is

$$\begin{Bmatrix} \mathbf{F}_L \\ \mathbf{F}_i \\ \mathbf{F}_R \end{Bmatrix} = \begin{bmatrix} \mathbf{D}_{LL} & \mathbf{D}_{Li} & \mathbf{D}_{LR} \\ \mathbf{D}_{iL} & \mathbf{D}_{ii} & \mathbf{D}_{iR} \\ \mathbf{D}_{RL} & \mathbf{D}_{Ri} & \mathbf{D}_{RR} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_L \\ \mathbf{q}_i \\ \mathbf{q}_R \end{Bmatrix} \tag{1}$$

When a free-wave propagates through the whole system, no external forces act on the periodic elements apart from the coupling forces. This motion is therefore governed by Eq. (1) without all the terms and sub-matrices associated with $\mathbf{F}_i$ and $\mathbf{q}_i$ so the relevant equation for the free-wave motion has the simpler minimum form

$$\begin{Bmatrix} \mathbf{F}_L \\ \mathbf{F}_R \end{Bmatrix} = \begin{bmatrix} \mathbf{D}_{LL} & \mathbf{D}_{LR} \\ \mathbf{D}_{RL} & \mathbf{D}_{RR} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_L \\ \mathbf{q}_R \end{Bmatrix} \tag{2}$$

The complete matrix equation Eq. (1) is only used in this paper when the system response to external force(s) acting within a single periodic element is considered. When they only act at the junction of two elements, the simpler form of Eq. (2) is sufficient.

2.3. The relationship between the propagation constants and the element dynamic stiffness matrix

Consider an arbitrary reference element and its nodal displacements and forces as in Fig. 1b. Denote the displacement coordinates on its left-hand and right-hand sides by the column matrices $\mathbf{q}_L$ and $\mathbf{q}_R$, respectively, and the forces acting at these coordinates by $\mathbf{F}_L$ and $\mathbf{F}_R$. The corresponding coordinate and force vectors are

$$\mathbf{q} = \begin{Bmatrix} \mathbf{q}_L \\ \mathbf{q}_R \end{Bmatrix} \text{ and } \mathbf{F} = \begin{Bmatrix} \mathbf{F}_L \\ \mathbf{F}_R \end{Bmatrix} \tag{3a,b}$$

Consider the wave which propagates through the whole system with propagation constant μ. Making use of the Floquet relationships between the displacements and forces at the ends of the element (as in Ref. [1]), one derives the well-known quadratic eigenvalue equation for e^{μ} in the form

$$[\mathbf{D}_{RL} + [\mathbf{D}_{LL} + \mathbf{D}_{RR}]e^{\mu} + \mathbf{D}_{LR}\, e^{2\mu}]\mathbf{q}_L = 0 \tag{4}$$

In the process one finds the following important relationship between $\mathbf{q}_L$ and $\mathbf{F}_L$:

$$\mathbf{F}_L = [\mathbf{D}_{LL} + \mathbf{D}_{RR}\, e^{\mu}]\mathbf{q}_L \tag{5}$$

Eq. (4) yields $2n_c$ different eigenvalues in reciprocal pairs, so the corresponding values of μ occur in positive and negative pairs. The equation can be reduced to the linear eigenvalue equation in e^{μ}

$$\left[\begin{bmatrix} \mathbf{D}_{RL} & \mathbf{D}_{RR} \\ 0 & \mathbf{I} \end{bmatrix} - e^{\mu} \begin{bmatrix} \mathbf{D}_{RL} & \mathbf{D}_{RR} \\ 0 & \mathbf{I} \end{bmatrix}\right] \begin{Bmatrix} \mathbf{q}_L \\ \mathbf{q}_R \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{6}$$

in which $\mathbf{I}$ is the identity matrix. Thompson [19] showed that this equation can be further reduced when the exterior coordinates of the periodic element fall into the Class A and Class B categories of Ref. [3]. This is always true for periodic elements which are length-wise symmetrical.

2.4. Displacements and forces at different points in a periodic system

A general free harmonic motion in the array of N_{el} elements consists of all the waves thus determined. Associated with the jth eigenvalue μ_j is a particular eigenvector of $\mathbf{q}_L$ which, in a suitably normalised form, will be denoted by $\boldsymbol{\theta}_j$. Corresponding to this is the normalised force vector $\boldsymbol{\phi}_j$ related to $\boldsymbol{\theta}_j$ through Eq. (5) by

$$\boldsymbol{\phi}_j = [\mathbf{D}_{LL} - \mathbf{D}_{LR} e^{\mu j}]\boldsymbol{\theta}_j \tag{7}$$

Let the generalised 'wave' coordinate of wave vector $\boldsymbol{\theta}_j$ at the left-hand side of the element be ψ_j. The corresponding actual displacements and forces along this side due to this one wave are $\boldsymbol{\theta}_j\psi_j$ and $\boldsymbol{\phi}_j\psi_j$, respectively. The total displacement and force vectors on the left-hand side of the element due to all J waves in the total motion are

$$\mathbf{q}_L = \sum_{j=1}^{J} \boldsymbol{\theta}_j\psi_j, \quad \mathbf{F}_L = \sum_{j=1}^{J} \boldsymbol{\phi}_j\psi_j \tag{8a,b}$$

For the most accurate calculations J must be taken to be $2n_c$, every possible positive and negative participating wave being included. If it is known in advance that only a few identifiable waves contribute significantly to the total response, these series may be truncated to include only those waves.

Along the left-hand side of the next element in the system to the right, the amplitude of each of these terms is multiplied by e^{μ_j}. Continuity and equilibrium at the junction of elements 1 and 2 require the displacements and forces along the right-hand side of element 1 to be given by

$$\mathbf{q}_{R,1} = \mathbf{q}_{L,2} = \sum_{j=1}^{J} e^{\mu_j}\boldsymbol{\theta}_j\psi_j \text{ and } \mathbf{F}_{R,1} = -\mathbf{F}_{L,2} = \sum_{j=1}^{J} e^{\mu_j}\boldsymbol{\phi}_j\psi_j \tag{9a,b}$$

Note the negative sign in front of $\mathbf{F}_{L,2}$. It now follows that the displacements and forces along the right-hand side of element N_k in the whole periodic array are given by

$$\mathbf{q}_{R,N_k} = \sum_{j=1}^{J} e^{N_k\mu_j}\boldsymbol{\theta}_j\psi_j \text{ and } \mathbf{F}_{R,N_k} = \sum_{j=1}^{J} e^{N_k\mu_j}\boldsymbol{\phi}_j\psi_j \tag{10a,b}$$

All of these summations (Eqs. (9a,b) and (10a,b)) can be expressed in the following matrix forms:

$$\mathbf{q}_{L,1} = \sum_{j=1}^{J}\boldsymbol{\theta}_j\psi_j = \begin{bmatrix} \theta_{1,1} & \theta_{2,1} & \cdots & \theta_{J,1} \\ \theta_{1,2} & \theta_{2,2} & & \theta_{J,2} \\ \vdots & & \ddots & \vdots \\ \theta_{1,j} & \theta_{2,j} & \cdots & \theta_{J,j} \end{bmatrix} \begin{Bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_J \end{Bmatrix} = \boldsymbol{\Theta\Psi} \tag{11}$$

$$\mathbf{F}_{L,1} = \sum_{j=1}^{J}\boldsymbol{\phi}_j\psi_j = \begin{bmatrix} \phi_{1,1} & \phi_{2,1} & \cdots & \phi_{J,1} \\ \phi_{1,2} & \phi_{2,2} & & \phi_{J,2} \\ \vdots & & \ddots & \vdots \\ \phi_{1,J} & \phi_{2,J} & \cdots & \phi_{J,J} \end{bmatrix} \begin{Bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_J \end{Bmatrix} = [\boldsymbol{\Phi}]\{\boldsymbol{\Psi}\} \tag{12}$$

$$\mathbf{q}_{R,N_k} = -\sum_{j=1}^{J}\boldsymbol{\theta}_j e^{N_k\mu_j}\psi_j = \begin{bmatrix} \theta_{1,1} & \theta_{2,1} & \cdots & \theta_{J,1} \\ \theta_{1,2} & \theta_{2,2} & & \theta_{J,2} \\ \vdots & & \ddots & \vdots \\ \theta_{1,j} & \theta_{2,j} & \cdots & \theta_{J,j} \end{bmatrix} \begin{bmatrix} e^{N_k\mu_1} & 0 & \cdots & 0 \\ 0 & e^{N_k\mu_z} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & e^{N_k\mu_J} \end{bmatrix} \begin{Bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_J \end{Bmatrix} = \boldsymbol{\Theta} \boldsymbol{E}^{N_k\mu}\boldsymbol{\Psi} \tag{13}$$

$$\mathbf{F}_{R,N_k} = -\sum_{j=1}^{J}\boldsymbol{\phi}_j e^{N_k\mu_j}\psi_j = -\begin{bmatrix} \phi_{1,1} & \phi_{2,1} & \cdots & \phi_{J,1} \\ \phi_{1,2} & \phi_{2,2} & & \phi_{J,2} \\ \vdots & & \ddots & \vdots \\ \phi_{1,j} & \phi_{2,j} & \cdots & \phi_{J,j} \end{bmatrix} \begin{bmatrix} e^{N_k\mu_1} & 0 & \cdots & 0 \\ 0 & e^{N_k\mu_z} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & e^{N_k\mu_J} \end{bmatrix} \begin{Bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_J \end{Bmatrix} = -\boldsymbol{\Theta} \boldsymbol{E}^{N_k\mu}\boldsymbol{\Psi} \tag{14}$$

where

$$\mathbf{E}^{N_k\mu} = \begin{bmatrix} e^{N_k\mu_1} & 0 & \cdots & 0 \\ 0 & e^{N_k\mu_1} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & e^{N_k\mu_J} \end{bmatrix} \tag{15}$$

In these, $\mathbf{\Theta}$ and $\mathbf{\Phi}$ are matrices of the displacement vectors and corresponding force vectors, respectively, both of order $n_c \times 2n_c$. $\mathbf{\Psi}$ is a column matrix of the $2n_c$ values of ψ_j. These matrix equations are fundamental to the calculation of forced responses of both infinite and finite arrays. Notice, however, that if the two series of Eqs. (10a,b) are truncated to include only those J_{red} waves which make significant response contributions, the orders of the $\mathbf{\Theta}$ and $\mathbf{\Phi}$ matrices become $J_{\text{red}} \times 2n_c$ and that of $\mathbf{E}^{\mu}$ becomes $J_{\text{red}} \times J_{\text{red}}$. Use of these features can make enormous savings in subsequent computation times.

3. The response of periodic systems to applied harmonic forces

3.1. The semi-infinite system excited at the finite end

Consider the system extending indefinitely to the right and excited at the left-hand end of element 1 (junction 0) by a prescribed set of forces/moments $\mathbf{F}_{L\text{ext}}$. In the first place, restrict the problem to that in which all the displacements $\mathbf{q}_{L,0}$ are unspecified and no other displacements, externally applied forces or constraints are specified elsewhere in the system. In the absence of any discontinuity or termination in the whole system to the right of element 1, no reflected waves are reflected back to the source. The only admissible wave motions in this system are positive-going, i.e. those n_c waves whose μ's have negative real parts or negative purely imaginary parts. The displacement(s) at any point in the system can be found when the ψ values of all of these waves have been determined from the wave vectors $\boldsymbol{\theta}$ and $\boldsymbol{\phi}$ as found in the last section.

Now the externally applied force system at the left-hand end can be decomposed into a series of the force vectors of all the induced n_c waves. Each force vector in the series can only generate its own free wave, and no other. To find the total motion in the given PS, it is necessary firstly to determine the ψ value for each of the terms in the force vector series of Eq. (12), i.e. in $\mathbf{F}_{L\text{ext}} = \mathbf{\Phi}\boldsymbol{\psi}$. In general, of course, $\mathbf{\Phi}$ is of order $n_c \times 2n_c$ but when only the n_c positive-going waves are excited (as in this example) this reduces to $n_c \times n_c$ and so is invertible. Hence

$$^{+}\mathbf{\Psi} = {}^{+}\mathbf{\Phi}^{-1}\mathbf{F}_{L\text{ext}} \tag{16}$$

The + prefix has been inserted before $\mathbf{\Psi}$ and $\mathbf{\Phi}$ (and will also appear before other symbols) to associate them with positive-going waves in the system extending to $+\infty$. From Eq. (13), the displacements at junctions 0 and N_k from the end are evidently

$$^{+}\mathbf{q}_{L,0} = {}^{+}\mathbf{\Theta}^{+}\mathbf{\Psi} \text{ and } {}^{+}\mathbf{q}_{L,N_k} = {}^{+}\mathbf{\Theta}^{+}\mathbf{E}^{N_k\mu+}\mathbf{\Psi} \tag{17a,b}$$

The forces at junction N_k are

$$^{+}\mathbf{F}_{L,N_k} = \pm{}^{+}\mathbf{\Phi}^{+}\mathbf{E}^{N_k\mu+}\mathbf{\Psi} \tag{18}$$

Substituting for $^{+}\mathbf{\Psi}$ into these from Eq. (16) one obtains

$$^{+}\mathbf{q}_{L,N_k} = {}^{+}\mathbf{\Theta}^{+}\mathbf{E}^{N_k\mu+}\mathbf{\Phi}^{-1}\mathbf{F}_{L\text{ext}} \text{ and } {}^{+}\mathbf{F}_{L,N_k} = \pm{}^{+}\mathbf{\Phi}^{+}\mathbf{E}^{N_k\mu+}\mathbf{\Phi}^{-1}\mathbf{F}_{L\text{ext}} \tag{19a,b}$$

The $\pm$ sign in both Eqs. (18) and (19b) indicate that the forces on the contiguous elements at a junction are equal and opposite. The receptance matrices relating the applied forces at the left-hand end of the system to the displacements at junction 0 and N_k to the right are now seen from Eq. (19a) to be

$$^{+}\boldsymbol{\alpha}_{\text{semi}0} = {}^{+}\mathbf{\Theta}^{+}\mathbf{\Phi}^{-1} \text{ and } {}^{+}\boldsymbol{\alpha}_{\text{semi}N_k} = {}^{+}\mathbf{\Theta}^{+}\mathbf{E}^{N_k\mu}\mathbf{\Phi}^{-1} \tag{20a,b}$$

If the semi-infinite system extends indefinitely to the left (i.e. to $-\infty$) it is analysed in exactly the same way. The only participating waves are now those which propagate or decay away from the source in the negative direction so the relevant $\mathbf{\Psi}$ and $\mathbf{\Phi}$ values are those associated with values of μ having positive real parts or positive purely imaginary parts. The external forces now act on the right-hand end of the first periodic element and will be denoted by $\mathbf{F}_{R\text{ext}}$ and the displacements to be determined are those at the right-hand ends of the elements. All the matrices and vectors for the left-hand system are related in a similar way to those of the right-hand system. Identifying these by a negative prefix, the junction displacements, forces and receptances are found to be

$$^{-}\mathbf{q}_{R,0} = {}^{-}\mathbf{\Theta}^{-}\mathbf{\Psi} \text{ and } {}^{-}\mathbf{q}_{R,N_k} = {}^{-}\mathbf{\Theta}^{-}\mathbf{E}^{N_k\mu-}\mathbf{\Psi} \tag{21a,b}$$

$$^{-}\mathbf{F}_{R,N} = \pm{}^{-}\mathbf{\Phi}^{-}\mathbf{E}^{N_k\mu-}\mathbf{\Psi} \tag{22}$$

$$^{-}\boldsymbol{\alpha}_{\text{semi0}} = {}^{-}\mathbf{\Theta}^{-}\mathbf{\Phi}^{-1} \text{ and } {}^{-}\boldsymbol{\alpha}_{\text{semi}N_k} = {}^{-}\mathbf{\Theta}^{-}\mathbf{E}^{N_k\mu-}\mathbf{\Phi}^{-1} \tag{23a,b}$$

Now suppose the semi-infinite system is excited at its left-hand end by a prescribed set of external forces acting at some (but not all) of the left-hand coordinate locations. The displacements at these locations may be said to be 'free' or 'unconstrained', i.e. free to take up the values generated by the forces. At the remaining locations the displacements are prescribed to be rigidly constrained or (perhaps) excited by inexorable motions (displacements). The forces at the first set of locations are prescribed while at the other set the displacements are prescribed. A simple example of this is a semi-infinite beam excited by an external harmonic moment at its finite end where it is constrained by a simple support. At that end the beam is free to rotate under the action of the moment but its transverse displacement is prevented inexorably by an unknown transverse force from the support.

Clearly where the forces are prescribed, the displacements are free (unprescribed) and where the displacements are prescribed the forces are free. The $\mathbf{\Theta}$ and $\mathbf{\Phi}$ matrices can be partitioned into sets corresponding to these as follows:

$$\mathbf{\Theta} = \begin{Bmatrix} \mathbf{\Theta}_{\text{pres}} \\ \mathbf{\Theta}_{\text{free}} \end{Bmatrix} \text{ and } \mathbf{\Phi} = \begin{Bmatrix} \mathbf{\Phi}_{\text{free}} \\ \mathbf{\Phi}_{\text{pres}} \end{Bmatrix} \tag{24a,b}$$

The order of $\mathbf{\Theta}_{\text{pres}}$ is the same as that of $\mathbf{\Phi}_{\text{free}}$ and the order of $\mathbf{\Theta}_{\text{free}}$ is that of $\mathbf{\Phi}_{\text{pres}}$ These matrices are related through Eq. (7) by

$$\mathbf{\Phi}_{\text{free}} = [\mathbf{D}_{LL} + \mathbf{D}_{LR}\mathbf{E}^{\mu}]\mathbf{\Theta}_{\text{pres}} \text{ and } \mathbf{\Phi}_{\text{press}} = [\mathbf{D}_{LL} + \mathbf{D}_{LR}\mathbf{E}^{\mu}]\mathbf{\Theta}_{\text{free}} \tag{25}$$

Now the prescribed coordinate displacements and forces at the finite end of this periodic system are related to the generalised positive-going wave vector $^{+}\mathbf{\Psi}$ (which is of order n_c) by

$$\mathbf{q}_{L\text{pres}} = \mathbf{\Theta}_{\text{pres}}{}^{+}\mathbf{\Psi} \text{ and } \mathbf{F}_{L\text{pres}} = \mathbf{\Phi}_{\text{pres}}{}^{+}\mathbf{\Psi}$$

In a single matrix equation, the whole set of n_c prescribed quantities is

$$\begin{Bmatrix} \mathbf{q}_{L\text{pres}} \\ \mathbf{F}_{L\text{pres}} \end{Bmatrix} = \begin{Bmatrix} \mathbf{\Theta}_{\text{pres}} \\ \mathbf{\Phi}_{\text{pres}} \end{Bmatrix}{}^{+}\mathbf{\Psi} = \mathbf{X}_{\text{pres}}{}^{+}\mathbf{\Psi} \tag{26}$$

This defines the matrix $\mathbf{X}_{\text{pres}}$ which is square and invertible, of order $n_c \times n_c$, so $^{+}\mathbf{\Psi}$ is given by

$$^{+}\mathbf{\Psi} = \mathbf{X}_{\text{pres}}^{-1}\begin{Bmatrix} \mathbf{q}_{L\text{pres}} \\ \mathbf{F}_{L\text{pres}} \end{Bmatrix} \tag{27}$$

This constitutes the set of n_c equations required to find the n_c terms in $^{+}\mathbf{\Psi}$. The two sets of equations required for all values of $\mathbf{q}_{L,0}$ and $\mathbf{F}_{L\text{ext}}$ at the left-hand end of the system are therefore

$$\mathbf{q}_{L,0} = \begin{Bmatrix} \mathbf{\Theta}_{\text{pres}} \\ \mathbf{\Theta}_{\text{free}} \end{Bmatrix}\mathbf{\Psi} = \begin{Bmatrix} \mathbf{\Theta}_{\text{pres}} \\ \mathbf{\Theta}_{\text{free}} \end{Bmatrix}\mathbf{X}_{\text{pres}}^{-1}\begin{Bmatrix} \mathbf{q}_{L\text{pres}} \\ \mathbf{F}_{L\text{pres}} \end{Bmatrix} \tag{28}$$

and

$$\mathbf{F}_{L} = \begin{Bmatrix} \mathbf{\Phi}_{\text{free}} \\ \mathbf{\Phi}_{\text{pres}} \end{Bmatrix}\mathbf{\Psi} = \begin{Bmatrix} \mathbf{\Phi}_{\text{free}} \\ \mathbf{\Phi}_{\text{pres}} \end{Bmatrix}\mathbf{X}_{\text{pres}}^{-1}\begin{Bmatrix} \mathbf{q}_{L\text{pres}} \\ \mathbf{F}_{L\text{pres}} \end{Bmatrix} \tag{29}$$

3.2. The doubly infinite system excited at a single junction

This system can be considered as two contiguous semi-infinite systems extending to infinity on either side of the loaded junction. At the junction, all the connected coordinates of the two systems must be identical while the total force from each system at each coordinate must be equal to the external force applied at that coordinate. Some of the coordinate displacements may in fact be zero, but it may not be obvious *a priori* which these will be although under some external loading conditions it is clear. The simplest example is the infinite beam excited by a transverse force when symmetry of the transverse response demands that the rotational displacement at the junction must vanish. In more complicated doubly infinite periodic systems, similar deductions may be made if all the exciting forces are in one of the two classes, "Class A" or "Class B", as defined in Ref. [3]. If they are all of Class A, then all the Class B displacements will be zero and *vice versa*. The Class A forces and the Class B displacements are then the 'prescribed' forces and displacements, respectively, as categorised in the last section. The Class A displacements and Class B forces are the 'free' ones. The doubly infinite system response can then be found directly from Eqs. (28) and (29) except that the external forces in those equations must be one half of the forces applied to the doubly infinite system, simply because the applied forces are shared equally by the left-hand and right-hand semi-infinite systems. When the external forces are all of Class A, the Class A displacements at equal distances from the loaded junction on either side are identical, and the Class B displacements are equal and opposite.

When these conditions are not readily identifiable (and this will occur when the periodic element does not possess x-wise symmetry) the doubly infinite system response can still be found from the general expressions in the previous section by considering the two adjacent semi-infinite systems, as follows.

Firstly, find separate ${}^{+}\boldsymbol{\Theta}$ and ${}^{-}\boldsymbol{\Theta}$ matrices for the left- and right-hand systems. Find also the corresponding ${}^{+}\boldsymbol{\Phi}$ and ${}^{-}\boldsymbol{\Phi}$ matrices, equate the displacements at the junctions of the left- and right-hand systems (junction 0) and use Eqs. (17a), (20a)) to yield

$$ {}^{+}\mathbf{q}_{L,0} = {}^{+}\boldsymbol{\Theta}{}^{+}\boldsymbol{\Psi} = {}^{-}\mathbf{q}_{R,0} = {}^{-}\boldsymbol{\Theta}{}^{-}\boldsymbol{\Psi} \tag{30} $$

So

$$ {}^{+}\boldsymbol{\Theta}{}^{+}\boldsymbol{\Psi} - {}^{-}\boldsymbol{\Theta}{}^{-}\boldsymbol{\Psi} = 0 \tag{31} $$

Next equate the sum of the forces on the two systems at junction 0 to the externally applied forces at that junction, $\mathbf{F}_{L\mathrm{ext}}$. Eqs. (18), (21) now lead to

$$ {}^{+}\mathbf{F}_{L,0} + {}^{-}\mathbf{F}_{R,0} = {}^{+}\boldsymbol{\Phi}{}^{+}\boldsymbol{\Psi} - {}^{-}\boldsymbol{\Phi}{}^{-}\boldsymbol{\Psi} = \mathbf{F}_{L\mathrm{ext}} \tag{32} $$

Finally combine Eqs. (31), (32) into the single equation for ${}^{+}\boldsymbol{\Psi}$ and ${}^{-}\boldsymbol{\Psi}$

$$ \begin{bmatrix} {}^{+}\boldsymbol{\Theta} & -{}^{-}\boldsymbol{\Theta} \\ {}^{+}\boldsymbol{\Phi} & -{}^{-}\boldsymbol{\Phi} \end{bmatrix} \begin{Bmatrix} {}^{+}\boldsymbol{\Psi} \\ {}^{-}\boldsymbol{\Psi} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{F}_{L\mathrm{ext}} \end{Bmatrix} \tag{33} $$

so

$$ \begin{Bmatrix} {}^{+}\boldsymbol{\Psi} \\ {}^{-}\boldsymbol{\Psi} \end{Bmatrix} = \begin{bmatrix} {}^{+}\boldsymbol{\Theta} & -{}^{-}\boldsymbol{\Theta} \\ {}^{+}\boldsymbol{\Phi} & -{}^{-}\boldsymbol{\Phi} \end{bmatrix}^{-1} \begin{Bmatrix} \mathbf{0} \\ \mathbf{F}_{L\mathrm{ext}} \end{Bmatrix} \tag{34} $$

From these values of ${}^{+}\boldsymbol{\Psi}$ and ${}^{-}\boldsymbol{\Psi}$ one finds the displacements and forces at any point within the whole infinite system.

3.3. The finite system of N_{el} elements excited at its ends

All $2n_c$ characteristic waves, positive- and negative-going, participate in the total motion so $\boldsymbol{\Psi}$ is now of order $2n_c$. The matrix equations for $\mathbf{q}_{L,0}$ and $\mathbf{q}_{R,N_{\mathrm{el}}}$, $\mathbf{F}_{L,0}$ and $\mathbf{F}_{R,N_{\mathrm{el}}}$ for the system can be stacked to relate the whole set of end displacements and end forces to $\mathbf{\Psi}$. Hence

$$\left\{\begin{array}{c}\mathbf{q}_{L,0}\\ \mathbf{q}_{R,N_{\mathrm{el}}}\end{array}\right\}=\left[\begin{array}{c}\mathbf{\Theta}\\ \mathbf{\Theta}\mathbf{E}^{\mu N_{\mathrm{el}}}\end{array}\right]\mathbf{\Psi}\ \text{and}\ \left\{\begin{array}{c}\mathbf{F}_{L,0}\\ \mathbf{F}_{R,N_{\mathrm{el}}}\end{array}\right\}=\left[\begin{array}{c}\mathbf{\Phi}\\ \mathbf{\Phi}\mathbf{E}^{\mu N_{\mathrm{el}}}\end{array}\right]\mathbf{\Psi} \tag{35a,b}$$

in which the matrices $\mathbf{\Theta}$ and $\mathbf{\Phi}$ are of order $n_c \times n_c$. The two matrices containing them in the above equation are therefore square and invertible so together they yield

$$\mathbf{\Psi}=\left[\begin{array}{c}\mathbf{\Theta}\\ \mathbf{\Theta}\mathbf{E}^{\mu N_{\mathrm{el}}}\end{array}\right]^{-1}\left\{\begin{array}{c}\mathbf{q}_{L,1}\\ \mathbf{q}_{R,N_{\mathrm{el}}}\end{array}\right\}\ \text{and}\ \mathbf{\Psi}\left[\begin{array}{c}\mathbf{\Phi}\\ \mathbf{\Phi}\mathbf{E}^{\mu N_{\mathrm{el}}}\end{array}\right]^{-1}\left\{\begin{array}{c}\mathbf{F}_{L,1}\\ \mathbf{F}_{R,N_{\mathrm{el}}}\end{array}\right\} \tag{36a,b}$$

Altogether these constitute $4n_c$ equations for the $2n_c$ terms within $\mathbf{\Psi}$. To start with, only those which express the known boundary conditions of displacement and/or force at the two ends of the system ($2n_c$ of them) are useable but are sufficient for the determination of $\mathbf{\Psi}$. Thereafter the remaining $2n_c$ are used to find the unknown displacements or forces at the ends.

Consider now the system which is excited by known forces at the left-hand and/or the right-hand end, but whose end displacements are not otherwise constrained. $\mathbf{F}_{L,0}$ and $\mathbf{F}_{R,N_{\mathrm{el}}}$ are therefore known (i.e. prescribed) and will be identified by symbols already used in Section 3.1, $\mathbf{F}_{L\mathrm{ext}}$ and $\mathbf{F}_{R\mathrm{ext}}$, $\mathbf{q}_{L,0}$ and $\mathbf{q}_{R,N_{\mathrm{el}}}$ are all unknown (i.e. free). Hence

$$\mathbf{\Psi}=\left[\begin{array}{c}\mathbf{\Phi}\\ \mathbf{\Phi}\mathbf{E}^{\mu N_{\mathrm{el}}}\end{array}\right]^{-1}\left\{\begin{array}{c}\mathbf{F}_{L\mathrm{ext}}\\ \mathbf{F}_{R\mathrm{ext}}\end{array}\right\} \tag{37}$$

and

$$\left\{\begin{array}{c}\mathbf{q}_{L,0}\\ \mathbf{q}_{R,N_{\mathrm{el}}}\end{array}\right\}=\left[\begin{array}{c}\mathbf{\Theta}\\ \mathbf{\Theta}\mathbf{E}^{\mu N_{\mathrm{el}}}\end{array}\right]\mathbf{\Psi}=\left[\begin{array}{c}\mathbf{\Theta}\\ \mathbf{\Theta}\mathbf{E}^{\mu N_{\mathrm{el}}}\end{array}\right]\left[\begin{array}{c}\mathbf{\Phi}\\ -\mathbf{\Phi}\mathbf{E}^{\mu N_{\mathrm{el}}}\end{array}\right]^{-1}\left\{\begin{array}{c}\mathbf{F}_{L\mathrm{ext}}\\ \mathbf{F}_{R\mathrm{ext}}\end{array}\right\}. \tag{38}$$

The overall receptance matrix relating all the values of $\mathbf{q}$ and $\mathbf{F}$ at the two ends of the system of N_{el} elements is therefore

$$\boldsymbol{\alpha}_{1,N_{\mathrm{el}}}=\left[\begin{array}{c}\mathbf{\Theta}\\ \mathbf{\Theta}\mathbf{E}^{\mu N_{\mathrm{el}}}\end{array}\right]\left[\begin{array}{c}\mathbf{\Phi}\\ -\mathbf{\Phi}\mathbf{E}^{\mu N_{\mathrm{el}}}\end{array}\right]^{-1}. \tag{39}$$

Computational difficulties arise with this when the terms in $\mathbf{E}^{\mu N_{\mathrm{el}}}$ with positive exponents are raised to large enough powers of N_{el} to cause computer overflow. This must be circumvented by splitting $\mathbf{\Psi}$ into its positive and negative-going components, ${}^{+}\mathbf{\Psi}$ and ${}^{-}\mathbf{\Psi}$ and $\mathbf{E}^{\mu N_{\mathrm{el}}}$ into the corresponding matrices $\mathbf{E}^{\mu N_{\mathrm{el}}}$ and ${}^{-}\mathbf{E}^{\mu N_{\mathrm{el}}}$. ${}^{+}\mathbf{\Psi}$ represents the wave-coordinate values at the left-hand end of the system of the waves which propagate to the right, while ${}^{-}\mathbf{\Psi}$ represents the wave coordinates at the right-hand end of the system and which propagate to the left. The μN_{el}'s of μN_{el}'s for positive-going waves cannot cause overflow. The negative-going waves travel backwards from the right-hand end so their μ's in ${}^{-}\mathbf{E}^{\mu N_{\mathrm{el}}}$ are multiplied by $-N_{\mathrm{el}}$, so the terms of $\mathbf{E}^{\mu N_{\mathrm{el}}}$ cannot overflow either. If corresponding positive and negative waves are represented in ${}^{+}\mathbf{\Psi}$ and ${}^{-}\mathbf{\Psi}$ in the same order, then ${}^{-}\mathbf{E}^{\mu N_{\mathrm{el}}}$ and ${}^{+}\mathbf{E}^{\mu N_{\mathrm{el}}}$ are identical.

The total motion at the two ends is now expressed by

$$\mathbf{q}_{L,0}={}^{+}\mathbf{\Theta}{}^{+}\mathbf{\Psi}+{}^{-}\mathbf{\Theta}{}^{-}\mathbf{E}^{\mu N_{\mathrm{el}}}{}^{-}\mathbf{\Psi}\ \text{and}\ \mathbf{q}_{R,N_{\mathrm{el}}}={}^{+}\mathbf{\Theta}{}^{+}\mathbf{E}^{\mu N_{\mathrm{el}}}{}^{+}\mathbf{\Psi}+{}^{-}\mathbf{\Theta}{}^{-}\mathbf{\Psi} \tag{40a,b}$$

and the end displacements are given by the single equation

$$\left\{\begin{array}{c}\mathbf{q}_{L,0}\\ \mathbf{q}_{R,N_{\mathrm{el}}}\end{array}\right\}=\left[\begin{array}{cc}{}^{+}\mathbf{\Theta} & {}^{-}\mathbf{\Theta}{}^{-}\mathbf{E}^{\mu N_{\mathrm{el}}}\\ {}^{+}\mathbf{\Theta}{}^{+}\mathbf{E}^{\mu N_{\mathrm{el}}} & {}^{-}\mathbf{\Theta}\end{array}\right]^{-1}\left\{\begin{array}{c}{}^{+}\mathbf{\Psi}\\ {}^{-}\mathbf{\Psi}\end{array}\right\} \tag{41}$$

The associated force vectors at the ends are expressed in similar form by

$$\left\{\begin{array}{c}\mathbf{F}_{L\mathrm{ext}}\\ \mathbf{F}_{R\mathrm{ext}}\end{array}\right\}=\left[\begin{array}{cc}{}^{+}\mathbf{\Phi} & {}^{-}\mathbf{\Phi}{}^{-}\mathbf{E}^{\mu N_{\mathrm{el}}}\\ -{}^{+}\mathbf{\Phi}{}^{+}\mathbf{E}^{\mu N_{\mathrm{el}}} & {}^{-}\mathbf{\Phi}\end{array}\right]^{-1}\left\{\begin{array}{c}{}^{+}\mathbf{\Psi}\\ {}^{-}\mathbf{\Psi}\end{array}\right\} \tag{42a}$$

so that

$$\begin{Bmatrix} {}^{+}\boldsymbol{\Psi} \\ {}^{-}\boldsymbol{\Psi} \end{Bmatrix} = \begin{bmatrix} {}^{+}\boldsymbol{\Phi} & {}^{-}\boldsymbol{\Phi}{}^{-}\mathbf{E}^{\mu N_{\text{el}}} \\ -{}^{+}\boldsymbol{\Phi}{}^{+}\mathbf{E}^{\mu N_{\text{el}}} & -{}^{-}\boldsymbol{\Phi} \end{bmatrix}^{-1} \begin{Bmatrix} \mathbf{F}_{\text{ext}L} \\ \mathbf{F}_{\text{ext}R} \end{Bmatrix}. \tag{42b}$$

and

$$\begin{Bmatrix} \mathbf{q}_{L,0} \\ \mathbf{q}_{R,N_{\text{el}}} \end{Bmatrix} = \begin{bmatrix} {}^{+}\boldsymbol{\Theta} & {}^{-}\boldsymbol{\Theta}{}^{-}\mathbf{E}^{\mu N_{\text{el}}} \\ {}^{+}\boldsymbol{\Theta}{}^{+}\mathbf{E}^{\mu N_{\text{el}}} & {}^{-}\boldsymbol{\Theta} \end{bmatrix} \begin{bmatrix} {}^{+}\boldsymbol{\Phi} & {}^{-}\boldsymbol{\Phi}{}^{-}\mathbf{E}^{\mu N_{\text{el}}} \\ -{}^{+}\boldsymbol{\Phi}{}^{+}\mathbf{E}^{\mu N_{\text{el}}} & -{}^{-}\boldsymbol{\Phi} \end{bmatrix}^{-1} \begin{Bmatrix} \mathbf{F}_{\text{ext}L} \\ \mathbf{F}_{\text{ext}R} \end{Bmatrix} \tag{43}$$

The receptance matrix across the whole N_{el} array is

$$\boldsymbol{\alpha}_{N_{\text{el}}} = \begin{bmatrix} {}^{+}\boldsymbol{\Theta} & {}^{-}\boldsymbol{\Theta}{}^{-}\mathbf{E}^{\mu N_{\text{el}}} \\ {}^{+}\boldsymbol{\Theta}{}^{+}\mathbf{E}^{\mu N_{\text{el}}} & {}^{-}\boldsymbol{\Theta} \end{bmatrix} \begin{bmatrix} {}^{+}\boldsymbol{\Phi} & {}^{-}\boldsymbol{\Phi}{}^{-}\mathbf{E}^{\mu N_{\text{el}}} \\ -{}^{+}\boldsymbol{\Phi}{}^{+}\mathbf{E}^{\mu N_{\text{el}}} & -{}^{-}\boldsymbol{\Phi} \end{bmatrix}^{-1}. \tag{44}$$

3.4. The system with one end fully constrained

This can easily be analysed by using Eq. (40b) and the top line of Eq. (42a). If the right-hand end is fixed, set $\mathbf{q}_{R,N_{\text{el}}}$ to zero and equate $\mathbf{F}_{\text{ext}L}$ for the left-hand end to the externally applied forces. This leads to

$$[{}^{+}\boldsymbol{\Theta}{}^{+}\mathbf{E}^{\mu N_{\text{el}}} \quad {}^{-}\boldsymbol{\Theta}]\begin{Bmatrix} {}^{+}\boldsymbol{\Psi} \\ {}^{-}\boldsymbol{\Psi} \end{Bmatrix} = 0 \text{ and } \mathbf{F}_{L\text{ext}} = [{}^{+}\boldsymbol{\Phi} \quad {}^{-}\boldsymbol{\Phi}{}^{-}\mathbf{E}^{\mu N_{\text{el}}}]\begin{Bmatrix} {}^{+}\boldsymbol{\Psi} \\ {}^{-}\boldsymbol{\Psi} \end{Bmatrix}$$

which combine into the simple equation

$$\begin{bmatrix} {}^{+}\boldsymbol{\Theta}{}^{-}\mathbf{E}^{\mu N_{\text{el}}} & {}^{-}\boldsymbol{\Theta} \\ {}^{+}\boldsymbol{\Phi} & {}^{-}\boldsymbol{\Phi}{}^{-}\mathbf{E}^{\mu N_{\text{el}}} \end{bmatrix} \begin{Bmatrix} {}^{+}\boldsymbol{\Psi} \\ {}^{-}\boldsymbol{\Psi} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \mathbf{F}_{\text{ext}L} \end{Bmatrix}. \tag{45}$$

The generalised coordinate values at the right-hand end of element N_m are given by

$$\mathbf{q}_{L_{N_J}} = [{}^{+}\boldsymbol{\Theta} \quad {}^{-}\boldsymbol{\Theta}] \begin{bmatrix} {}^{-}\mathbf{E}^{\mu N_m} & \mathbf{0} \\ \mathbf{0} & {}^{-}\mathbf{E}^{\mu(N_{\text{el}}-N_m)} \end{bmatrix} \begin{Bmatrix} {}^{+}\boldsymbol{\Psi} \\ {}^{-}\boldsymbol{\Psi} \end{Bmatrix} \tag{46}$$

and the forces acting at the same coordinate are

$$\mathbf{q}_{L_{N_J}} = [{}^{+}\boldsymbol{\Phi} \quad {}^{-}\boldsymbol{\Phi}] \begin{bmatrix} {}^{-}\mathbf{E}^{\mu N_m} & \mathbf{0} \\ \mathbf{0} & {}^{-}\mathbf{E}^{\mu(N_{\text{el}}-N_m)} \end{bmatrix} \begin{Bmatrix} {}^{+}\boldsymbol{\Psi} \\ {}^{-}\boldsymbol{\Psi} \end{Bmatrix} \tag{47}$$

3.5. The finite system excited at a single intermediate junction

Let N_{FL} and N_{FR}, respectively, be the number of elements between the excited junction and the left- and right-hand ends of the system so the total number of elements in the whole system is $N_{\text{el}} = N_{FL} + N_{FR}$. There are $2n_c$ characteristic waves of unknown amplitude to the left of the excited junction and another set of $2n_c$ waves of different amplitude to its right. Although there are $4n_c$ unknowns altogether, the computational problem may be solved by a method which uses some of the previous equations but involves only $2n_c$ unknowns at a time. The general approach has often been used for solving uniform-beam vibration problems by utilising known flexural wave motions [9,10].

The total motion in the finite periodic system can be regarded as the superposition of the motion generated by the given exciting forces when they act on a doubly infinite system (the 'infinite system motion') together with the motions reflected back into the system from the finite ends. Denote the magnitude of 'infinite system motion' to the right and left of the exciting forces by the vectors ${}^{+}\boldsymbol{\Psi}_{\text{inf}}$, ${}^{+}\boldsymbol{\Psi}_{\text{inf}}$, respectively. They can be evaluated by the methods of Section 3.2. The corresponding displacements and forces at the right-hand end of the finite system due to ${}^{+}\mathbf{\Psi}_{\text{inf}}$ are (from Eqs. (17) and (18))

$$\mathbf{q}_{R,N_{FR}} = {}^{+}\mathbf{\Theta}^{+}\mathbf{E}^{\mu N_{FR}+}\mathbf{\Psi}_{\text{inf}} \text{ and } \mathbf{F}_{R,N_{FR}} = {}^{+}\mathbf{\Phi}^{+}\mathbf{E}^{\mu N_{FR}+}\mathbf{\Psi}_{\text{inf}} \tag{48a,b}$$

At the left-hand end of the system, due to ${}^{-}\mathbf{\Psi}_{\text{inf}}$ they are (from Eqs. (21b) and (22))

$${}^{-}\mathbf{q}_{L,N_{FL}} = {}^{-}\mathbf{\Theta}^{-}\mathbf{E}^{\mu N_{FL}-}\mathbf{\Psi}_{\text{inf}} \text{ and } {}^{-}\mathbf{F}_{R,N_{FL}} = {}^{-}\mathbf{\Phi}^{-}\mathbf{E}^{\mu N_{FL}-}\mathbf{\Psi}_{\text{inf}} \tag{49a,b}$$

Now the motion reflected from the left-hand end is the same as that excited by a set of forces acting at the left-hand end of a semi-infinite system (see Section 4.1) and will be represented here by the vector ${}^{+}\mathbf{\Psi}_{\text{refl}}$ of positive-going waves. Due to this, the corresponding displacements and forces at the left-hand end are

$${}^{+}\mathbf{q}_{L(\text{LHend})} = {}^{+}\mathbf{\Theta}^{+}\mathbf{\Psi}_{\text{refl}} \text{ and } {}^{+}\mathbf{F}_{L(\text{LHend})} = {}^{+}\mathbf{\Phi}^{+}\mathbf{\Psi}_{\text{refl}} \tag{50a,b}$$

and at the right-hand end of the finite system they are

$${}^{+}\mathbf{q}_{R(\text{RHend})} = {}^{+}\mathbf{\Theta}^{-}\mathbf{E}^{\mu N_{\text{el}}+}\mathbf{\Psi}_{\text{refl}} \text{ and } {}^{+}\mathbf{F}_{R(\text{RHend})} = {}^{+}\mathbf{\Phi}^{-}\mathbf{E}^{\mu N_{\text{el}}+}\mathbf{\Psi}_{\text{refl}} \tag{51a,b}$$

The motion reflected from the right-hand end is the same as that excited by a set of forces acting at the right-hand end of the other semi-infinite system (see Section 4.1). Represented by the vector ${}^{-}\mathbf{\Psi}_{\text{refl}}$ of negative-going reflected waves, it generates displacements and forces at the right-hand end given by

$${}^{-}\mathbf{q}_{R(\text{RHend})} = {}^{-}\mathbf{\Theta}^{-}\mathbf{\Psi}_{\text{refl}} \text{ and } {}^{-}\mathbf{F}_{R(\text{RHend})} = -{}^{-}\mathbf{\Phi}^{-}\mathbf{\Psi}_{\text{refl}} \tag{52a,b}$$

At the left-hand end they are

$${}^{-}\mathbf{q}_{R(\text{LHend})} = {}^{-}\mathbf{\Theta}^{-}\mathbf{E}^{\mu N_{\text{el}}-}\mathbf{\Psi}_{\text{refl}} \text{ and } {}^{-}\mathbf{F}_{R(\text{LHend})} = -{}^{-}\mathbf{\Phi}^{-}\mathbf{E}^{\mu N_{\text{el}}-}\mathbf{\Psi}_{\text{refl}} \tag{53a,b}$$

The total motion due to the two sets of reflected waves and the infinite system motion must satisfy the boundary conditions at the two ends of the finite system.

Consider the special case when both ends of the whole system are free and unconstrained. The total force vector at each end must vanish so

$${}^{-}\mathbf{F}_{L(\text{LHend})} + {}^{+}\mathbf{F}_{L(\text{LHend})} + {}^{-}\mathbf{F}_{\text{inf(LHend)}} = 0 \quad \text{at the left-hand end} \tag{54a}$$

and

$${}^{-}\mathbf{F}_{R(\text{RHend})} + {}^{+}\mathbf{F}_{R(\text{RHend})} + {}^{+}\mathbf{F}_{\text{inf(RHend)}} = 0 \quad \text{at the right hand end} \tag{54b}$$

In terms of the $\mathbf{\Psi}$, $\mathbf{\Phi}$, $\mathbf{\Theta}$ and $\mathbf{E}$ matrices, these are

$${}^{-}\mathbf{\Phi}^{-}\mathbf{E}^{\mu N_{\text{el}}-}\mathbf{\Psi}_{\text{refl}} + {}^{+}\mathbf{\Phi}^{+}\mathbf{\Psi}_{\text{refl}} + {}^{-}\mathbf{\Phi}^{-}\mathbf{E}^{\mu N_{FL}-}\mathbf{\Psi}_{\text{inf}} = 0, \tag{55a}$$

and

$${}^{-}\mathbf{\Phi}^{-}\mathbf{\Psi}_{\text{refl}} + {}^{+}\mathbf{\Phi}^{+}\mathbf{E}^{\mu N_{\text{el}}+}\mathbf{\Psi}_{\text{refl}} + {}^{+}\mathbf{\Phi}^{+}\mathbf{E}^{\mu N_{FR}+}\mathbf{\Psi}_{\text{inf}} = 0. \tag{55b}$$

Combined into a single equation, these become

$$\begin{bmatrix} {}^{-}\mathbf{\Phi}^{-}\mathbf{E}^{\mu N_{\text{el}}} & {}^{+}\mathbf{\Phi} \\ {}^{-}\mathbf{\Phi} & {}^{+}\mathbf{\Phi}^{+}\mathbf{E}^{\mu N_{\text{el}}} \end{bmatrix} \begin{Bmatrix} {}^{-}\mathbf{\Psi} \\ {}^{+}\mathbf{\Psi} \end{Bmatrix} = \begin{bmatrix} {}^{-}\mathbf{\Phi}^{-}\mathbf{E}^{\mu N_{FL}} & 0 \\ 0 & {}^{+}\mathbf{\Phi}^{+}\mathbf{E}^{\mu N_{FR}} \end{bmatrix} \begin{Bmatrix} {}^{-}\mathbf{\Psi}_{\text{inf}} \\ {}^{+}\mathbf{\Psi}_{\text{inf}} \end{Bmatrix} \tag{56a}$$

Hence

$$\begin{Bmatrix} {}^{-}\mathbf{\Psi}_{\text{refl}} \\ {}^{+}\mathbf{\Psi}_{\text{refl}} \end{Bmatrix} = \begin{bmatrix} {}^{-}\mathbf{\Phi}^{-}\mathbf{E}^{\mu N_{FL}} & {}^{+}\mathbf{\Phi} \\ {}^{-}\mathbf{\Phi} & {}^{+}\mathbf{\Phi}^{+}\mathbf{E}^{\mu N_{\text{el}}} \end{bmatrix}^{-1} \begin{bmatrix} {}^{-}\mathbf{\Phi}^{-}\mathbf{E}^{\mu N_{FL}} & 0 \\ 0 & {}^{+}\mathbf{\Phi}^{+}\mathbf{E}^{\mu N_{FR}} \end{bmatrix} \begin{Bmatrix} {}^{-}\mathbf{\Psi}_{\text{inf}} \\ {}^{+}\mathbf{\Psi}_{\text{inf}} \end{Bmatrix}. \tag{56b}$$

These constitute $2n_c$ equations for the n_c unknown values of ${}^{-}\mathbf{\Psi}_{\text{refl}}$ and the n_c unknown values ${}^{+}\mathbf{\Psi}_{\text{refl}}$ from which are found the displacements and forces at any coordinate location in the whole system.

3.6. The doubly infinite system excited by a force acting within an element

The forced element is connected to a semi-infinite system on each of its sides, so the analysis of the whole system involves the linking of just three dynamic subsystems, end-to-end, as indicated in Fig. 2. Now the dynamic stiffnesses and receptances of the two semi-infinite subsystems have been determined in Section 3.1 from the reduced dynamic stiffness matrix of Eq. (2). When the loaded element is linked with these two subsystems, it must be represented analytically by the complete matrix of Eq. (1) which includes the interior coordinate contributions. The so-called 'external force' which is acting on the whole system is now one of the interior forces acting on the loaded element.

The analysis is conducted more concisely by using the receptance matrices of the three systems rather than with their dynamic stiffness matrices. Let the loaded element be numbered "0". Use this suffix on its displacements and forces which are then related by

$$\begin{Bmatrix} \mathbf{q}_{L_0} \\ \mathbf{q}_{i_0} \\ \mathbf{q}_{R_0} \end{Bmatrix} = {}^0\boldsymbol{\alpha} \begin{Bmatrix} \mathbf{F}_{L_0} \\ \mathbf{F}_{i_0} \\ \mathbf{F}_{R_0} \end{Bmatrix} \tag{57}$$

${}^0\boldsymbol{\alpha}$ is the receptance matrix of the loaded element and can be partitioned as

$${}^0\boldsymbol{\alpha} = \mathbf{D}^{-1} = \begin{bmatrix} {}^0\boldsymbol{\alpha}_{LL} & {}^0\boldsymbol{\alpha}_{Li} & {}^0\boldsymbol{\alpha}_{LR} \\ {}^0\boldsymbol{\alpha}_{iL} & {}^0\boldsymbol{\alpha}_{ii} & {}^0\boldsymbol{\alpha}_{iR} \\ {}^0\boldsymbol{\alpha}_{RL} & {}^0\boldsymbol{\alpha}_{Ri} & {}^0\boldsymbol{\alpha}_{RR} \end{bmatrix} \tag{58}$$

The displacement and force vectors at the two ends of the loaded element which link it with the adjacent semi-infinite systems are

$$\mathbf{q}_{R_0} = \begin{bmatrix} {}^0\boldsymbol{\alpha}_{RL} & {}^0\boldsymbol{\alpha}_{Ri} & {}^0\boldsymbol{\alpha}_{RR} \end{bmatrix} \begin{Bmatrix} \mathbf{F}_{L_0} \\ \mathbf{F}_{i_0} \\ \mathbf{F}_{R_0} \end{Bmatrix} \text{ and } \mathbf{q}_{L_0} = \begin{bmatrix} {}^0\boldsymbol{\alpha}_{LL} & {}^0\boldsymbol{\alpha}_{Li} & {}^0\boldsymbol{\alpha}_{LR} \end{bmatrix} \begin{Bmatrix} \mathbf{F}_{L_0} \\ \mathbf{F}_{i_0} \\ \mathbf{F}_{R_0} \end{Bmatrix} \tag{59a,b}$$

Continuity of displacement requires that $\mathbf{q}_{L_0}$ be equal to the displacement vector ${}^-\mathbf{q}_{R,1}$ at the finite end of the left-hand semi-infinite system which is now excited by the force vector $-\mathbf{F}_{L\text{ext}}$. Likewise, $\mathbf{q}_{R_0}$ must be equal to the displacement vector ${}^+\mathbf{q}_{L,1}$ at the finite end of the right-hand semi-infinite system which is excited by $-\mathbf{F}_{R\text{ext}}$. (Negative and positive prefixes are assigned to these $\mathbf{q}$ vectors as before to show they pertain to the left-hand or right-hand semi-infinite systems, respectively.) Use of Eqs. (19a), (20a) and (21a), (23a) for the right-hand and left-hand semi-infinite systems, respectively, now leads to

$$\mathbf{q}_{R_0} = {}^+\mathbf{q}_{L,1} = -\boldsymbol{\alpha}_{0\text{semi}}\mathbf{F}_{\text{ext}} \text{ and } \mathbf{q}_{L_0} = {}^-\mathbf{q}_{R,1} = -\boldsymbol{\alpha}_{0\text{semi}}\mathbf{F}_{L\text{ext}}. \tag{60a,b}$$

Substituted into Eq. (59), these yield (after some rearrangement)

$$\begin{Bmatrix} \mathbf{F}_{L_0} \\ \mathbf{F}_{R_0} \end{Bmatrix} = -\begin{bmatrix} {}^0\boldsymbol{\alpha}_{LL} + {}^-\boldsymbol{\alpha}_{0\text{semi}} & {}^0\boldsymbol{\alpha}_{LR} \\ {}^0\boldsymbol{\alpha}_{RL} & {}^0\boldsymbol{\alpha}_{RR} + {}^-\boldsymbol{\alpha}_{0\text{semi}} \end{bmatrix} \begin{Bmatrix} {}^0\boldsymbol{\alpha}_{Li} \\ {}^0\boldsymbol{\alpha}_{Ri} \end{Bmatrix} \tag{61}$$

from which the displacements due to $\mathbf{F}_{i_0}$ at any point in the whole system can be determined.

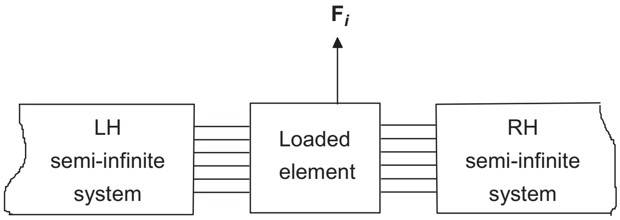

Fig. 2. Three dynamic sub-systems linked end-to-end and forced within the middle system.

3.7. Periodic systems whose elements are themselves periodic

These constitute *bi-periodic* systems, an example of which is a long periodically supported rail with a complicated section, the periodic element of which is itself modelled as a periodic system of N_{el} elements. The propagation constants of the whole periodic system are found (as before) from the eigenvalues of Eq. (6), but the dynamic stiffness matrix required is now that which relates the end forces and displacements of the finite set of N_{el} elements, i.e. the inverse of the receptance matrix of Eq. (44). Denote this by $\mathbf{D}_{N_{\mathrm{el}}}$, so

$$\mathbf{D}_{N_{\mathrm{el}}} = \begin{bmatrix} {}^{+}\mathbf{\Phi} & {}^{-}\mathbf{\Phi}{}^{-}\mathbf{E}^{\mu N_{\mathrm{el}}} \\ -{}^{+}\mathbf{\Phi}{}^{+}\mathbf{E}^{\mu N_{\mathrm{el}}} & -{}^{-}\mathbf{\Phi} \end{bmatrix} \begin{bmatrix} {}^{+}\mathbf{\Theta} & {}^{-}\mathbf{\Theta}{}^{-}\mathbf{E}^{\mu N_{\mathrm{el}}} \\ {}^{+}\mathbf{\Theta}{}^{+}\mathbf{E}^{\mu N_{\mathrm{el}}} & {}^{-}\mathbf{\Theta} \end{bmatrix}^{-1} \quad (62)$$

The eigenvalues obtained from the new eigenvalue equation can now be used to find the whole system motion generated by a set of harmonic forces at a single span-wise location. Use should be made of the appropriate analyses of Sections 3.1 to 3.6, in which the new eigenvalues and eigenvectors are incorporated.

4. Forced response calculations with a reduced set of characteristic waves

4.1. Free-wave propagation in the reduced system

Hundreds of coupling coordinates exist between periodic elements in some finite-element calculations. Twice as many characteristic waves participate (at least to some degree) in any forced motion of the system and this can lead to excessive computation times. Usually, at any given frequency only a few of the characteristic waves contribute appreciably to the total response. If these waves can be identified in a preliminary investigation, subsequent calculations can be undertaken which involve only those waves and much less time is expended. It will now be shown formally how the preceding forced vibration theory can be modified to accomplish this. It is based on observations made initially by Gry [20] who developed his own theory which is fundamentally (though not obviously) on similar lines. Brown and Byrne [23] have also used this approach.

Suppose n_{cw} waves have been identified as the main contributors to the response over a limited frequency range. In general, their eigenvectors (as determined from Eq. (6)) are complex, the real and imaginary parts of which describe a different mode of displacement at a periodic junction. These different modes will be referred to as 'sub-modes' and the reduced system will be analysed in terms of them. Each will be treated as independent and essentially real, leading to $2n_{\mathrm{cw}}$ sub-modes in all. Experience, however, has shown that fewer may be used for the following reasons:

(a) An evanescent wave may have a purely real mode and can then be represented by just one sub-wave.
(b) A good approximation to such an evanescent wave may be possible by linearly combining the real and imaginary parts of an associated propagating wave. (This is valid in the case of low-frequency flexural motion.) It is unnecessary (and can even be undesirable) to include that evanescent sub-mode separately at all.
(c) A pair of waves which have complex-conjugate propagation constants have positive and negative complex-conjugate wave modes. Their real parts are essentially identical as also are their imaginary parts. Only the one real part and the one imaginary part need be included, i.e. just two real sub-modes to represent a pair of complex modes.

In consequence, the total number of wave modes required to analyse the reduced system will usually be less than $2n_{\mathrm{cw}}$. Denote the actual number of sub-modes used by n_{red}, the value of which is the order of the reduced matrices to be generated.

The magnitude of the jth sub-mode at the left-hand end of an element will be denoted by the single generalised coordinate $q_{L\mathrm{sub}_j}$, the column vector $\mathbf{q}_{L\mathrm{sub}}$ representing the whole set of them. $\mathbf{q}_{R\mathrm{sub}}$ is the corresponding vector for the right-hand end of the element. These $\mathbf{q}_{\mathrm{sub}}$'s combine in the reduced system to form a new set of characteristic waves which approximate those of the original system and have propagation constants which should also closely approximate those of the original system. They are the new characteristic modes to be used in the forced response calculations for the reduced system.

Now the general relationship between $\mathbf{q}_{L\text{sub}}$,$\mathbf{q}_{R\text{sub}}$ and $\mathbf{q}_L$, $\mathbf{q}_R$ has the form

$$\begin{Bmatrix} \mathbf{q}_L \\ \mathbf{q}_R \end{Bmatrix} = \lfloor \mathbf{\Theta}_{\text{red}L},\ \mathbf{\Theta}_{\text{red}R} \rfloor \begin{Bmatrix} \mathbf{q}_{L\text{sub}} \\ \mathbf{q}_{R\text{sub}} \end{Bmatrix} \tag{63a}$$

$\mathbf{\Theta}_{\text{red}L}$ and $\mathbf{\Theta}_{\text{red}R}$ are reduced real forms of the original complex $\mathbf{\Theta}$ matrix of Eq. (11). Their n_{red} columns are the sub-modes of the n_{cw} selected original characteristic waves, each column consisting of n_c normalised q_L coordinates. The normalised sub-modes for the right- and left-hand ends of the element are identical, so $\mathbf{\Theta}_{\text{red}L} = \mathbf{\Theta}_{\text{red}R}$ and can both be denoted simply by $\mathbf{\Theta}_{\text{red}}$. This leaves Eq. (63a) in the simpler form

$$\begin{Bmatrix} \mathbf{q}_L \\ \mathbf{q}_R \end{Bmatrix} = \lfloor \mathbf{\Theta}_{\text{red}},\ \mathbf{\Theta}_{\text{red}} \rfloor \begin{Bmatrix} \mathbf{q}_{L\text{sub}} \\ \mathbf{q}_{R\text{sub}} \end{Bmatrix} \tag{63b}$$

One of Gry's observations [20] was that the calculated modes θ_j of low-frequency waves did not vary significantly over a useful (but limited) frequency range. He therefore assumed that the same values of $\mathbf{\Theta}_{\text{red}}$ calculated at a single frequency within that range could be used as a basis for calculating good approximate response values from a reduced set of waves over that range. Both he and Brown and Byrne [23] confirmed the validity of this from calculations on a railway rail. The same assumption will be therefore be made in this section. A new and reduced-order dynamic stiffness matrix will be derived for the periodic element, corresponding to the reduced set of wave coordinates. $\mathbf{\Theta}_{\text{red}}$ is taken to be the same for all frequencies within the acceptable restricted range, so does not have to be recalculated for each frequency.

Eq. (63b) is now used to transform Eq. (2) (which relates the actual coupling coordinates to the coupling forces) into a relationship between the new sub-wave coordinates and their generalised forces. The transformation yields

$$\lfloor \mathbf{\Theta}_{\text{red}},\ \mathbf{\Theta}_{\text{red}} \rfloor^{\text{T}} \begin{bmatrix} \mathbf{D}_{LL} & \mathbf{D}_{LR} \\ \mathbf{D}_{RL} & \mathbf{D}_{RR} \end{bmatrix} \lfloor \mathbf{\Theta}_{\text{red}},\ \mathbf{\Theta}_{\text{red}} \rfloor \begin{Bmatrix} \mathbf{q}_{L\text{sub}} \\ \mathbf{q}_{R\text{sub}} \end{Bmatrix} = \lfloor \mathbf{\Theta}_{\text{red}},\ \mathbf{\Theta}_{\text{red}} \rfloor^{\text{T}} \begin{Bmatrix} \mathbf{F}_L \\ \mathbf{F}_R \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_{L\text{gen}} \\ \mathbf{F}_{R\text{gen}} \end{Bmatrix} \tag{64}$$

$\mathbf{F}_{L\text{gen}}$ and $\mathbf{F}_{R\text{gen}}$ (as defined by this equation) are the generalised coupling forces acting on the left- and right-hand ends of an element. The dynamic stiffness matrix appropriate to the reduced set is therefore

$$\mathbf{D}_{\text{red}} = \lfloor \mathbf{\Theta}_{\text{red}},\ \mathbf{\Theta}_{\text{red}} \rfloor^{\text{T}} \begin{bmatrix} \mathbf{D}_{LL} & \mathbf{D}_{LR} \\ \mathbf{D}_{RL} & \mathbf{D}_{RR} \end{bmatrix} \lfloor \mathbf{\Theta}_{\text{red}},\ \mathbf{\Theta}_{\text{red}} \rfloor \tag{65}$$

which is of order $2n_{\text{red}} \times 2n_{\text{red}}$ and must now be used as in Section 2.3 to find a new set of propagation constants μ_{new} for the required frequency range. The values so obtained are only approximations to the original set and pertain to a new set of approximate characteristic waves, each of which is a unique combination of the sub-modes of the selected waves. Each combination is dominated by a different member, which explains Gry's observations [20] (a) that the wave modes of the original and reduced sets were very similar and (b) that over a limited frequency range his calculated μ_{new} values were very close to the accurate μ values for the original complete wave set. This latter feature justifies the use of the reduced wave set theory over the limited frequency range. Whenever the reduced wave theory is used, its range of validity should be tested by the closeness of the agreement.

To find the new values of μ_{new}, first expand Eq. (65) and express it in the form

$$\mathbf{D}_{\text{red}} = \begin{bmatrix} \mathbf{\Theta}_{\text{red}}^{\text{T}} \mathbf{D}_{LL} \mathbf{\Theta}_{\text{red}} & \mathbf{\Theta}_{\text{red}}^{\text{T}} \mathbf{D}_{LR} \mathbf{\Theta}_{\text{red}} \\ \mathbf{\Theta}_{\text{red}}^{\text{T}} \mathbf{D}_{RL} \mathbf{\Theta}_{\text{red}} & \mathbf{\Theta}_{\text{red}}^{\text{T}} \mathbf{D}_{RR} \mathbf{\Theta}_{\text{red}} \end{bmatrix} = \begin{bmatrix} \mathbf{D}_{\text{red}LL} & \mathbf{D}_{\text{red}LR} \\ \mathbf{D}_{\text{red}RL} & \mathbf{D}_{\text{red}RR} \end{bmatrix} \tag{66}$$

so the whole equation relating $\mathbf{q}_{L\text{red}}$, $\mathbf{q}_{R\text{red}}$, $\mathbf{F}_{L\text{gen}}$ and $\mathbf{F}_{R\text{gen}}$ becomes

$$\begin{bmatrix} \mathbf{D}_{\text{red}LL} & \mathbf{D}_{\text{red}LR} \\ \mathbf{D}_{\text{red}RL} & \mathbf{D}_{\text{red}RR} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_{L\text{red}} \\ \mathbf{q}_{R\text{red}} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_{L\text{gen}} \\ \mathbf{F}_{R\text{gen}} \end{Bmatrix} \tag{67}$$

The quadratic eigenvalue equation for $e^{\mu_{\text{new}}}$ is now found from this as in Section 2.2. In the same form as Eq. (4) it is

$$[\mathbf{D}_{\text{red}RL} + [\mathbf{D}_{\text{red}LL} + \mathbf{D}_{\text{red}RR}]e^{\mu_{\text{new}}} + \mathbf{D}_{LR\text{red}}e^{2\mu_{\text{new}}}]\mathbf{q}_{L\text{red}} = 0 \tag{68}$$

Its eigenvectors form a new matrix $\mathbf{\Theta}_{\text{new}}$ of order $n_{\text{red}} \times 2n_{\text{red}}$, half of which pertain to positive-going waves and half to negative-going waves. Its jth column $\mathbf{\theta}_{\text{new},j}$ contains the contributions of the $\mathbf{q}_{L\text{red}}$'s to the jth new approximate characteristic wave, normalised in some convenient way.

Denote by $\psi_{\text{new},j}$ the (scalar) complex magnitude of the new wave j at junction 0 when the system is harmonically forced, and the whole set of $\psi_{\text{new},j}$ values by the column vector $\mathbf{\Psi}_{\text{new}}$. The contributions of the new wave j to $\mathbf{q}_{L\text{sub}}$ are then simply

$$\mathbf{q}_{L\text{sub},j} = \mathbf{\Theta}_{\text{new},j}\,\psi_{\text{new},j} \tag{69}$$

Next, denote by $\mathbf{F}_{L\text{sub},j}$ the generalised coupling forces between the periodic elements associated with the jth new wave. It is related to $\mathbf{q}_{L\text{sub},j}$ through a new matrix, $\mathbf{\Phi}_{\text{new}}$, in the same way as $\mathbf{\Theta}$ and $\mathbf{\Phi}$ are related for the original system in Eq. (7) (see Sections 2.3 and 2.4). Column j of $\mathbf{\Phi}_{\text{new}}$ is therefore related to column j of $\mathbf{\Theta}_{\text{new}}$ by

$$\mathbf{\Phi}_{\text{new},j} = [\mathbf{D}_{\text{red}LL} + \mathbf{D}_{\text{red}LR}e^{\mu_{\text{new},j}}]\mathbf{\Theta}_{\text{new},j} \tag{70}$$

The generalised coupling forces corresponding to new wave j are $\mathbf{F}_{L\text{sub},j} = \mathbf{\Phi}_{\text{new},j}\psi_{\text{new},j}$ and the whole set of them is

$$\mathbf{F}_{L\text{sub}} = \mathbf{\Phi}_{\text{new}}\mathbf{\Psi}_{\text{new}} \tag{71}$$

4.2. Forced wave propagation in the reduced system

As in Section 3.1 consider first the semi-infinite periodic system loaded at its left-hand end by an arbitrary set of external forces $\mathbf{F}_{L\text{ext}}$. These generate the generalised external forces $\mathbf{F}_{L\text{ext(gen)}}$ on the sub-wave modes of the reduced system as given in Eq. (64) by

$$\mathbf{F}_{L\text{ext(gen)}} = \mathbf{\Theta}_{\text{red}}^{\text{T}}\mathbf{F}_{L\text{ext}} \tag{72}$$

which generate a new set of $2n_{\text{red}}$ characteristic waves. (There are $2n_{\text{red}}$ since there are *pairs* of positive- and negative-going waves.) Denote by $^{+}\mathbf{\Psi}_{\text{new}}$ the magnitudes of the n_{red} positive-going waves of this set. It is given as in Eq. (16) by

$$^{+}\mathbf{\Psi}_{\text{new}} = {}^{+}\mathbf{\Phi}_{\text{new}}^{-1}\mathbf{F}_{L\text{ext(gen)}} \tag{73}$$

The positive prefixes on $^{+}\mathbf{\Phi}_{\text{new}}$ and $^{+}\mathbf{\Psi}_{\text{new}}$ imply the inclusion of terms only from the positive-going waves. Combining Eqs. (69), (72) and (73) one finds the whole set of corresponding sub-mode coordinates to be

$$\mathbf{q}_{L\text{sub}} = {}^{+}\mathbf{\Theta}_{\text{new}}{}^{+}\mathbf{\Phi}_{\text{new}}^{-1}\mathbf{\Theta}_{\text{red}}^{\text{T}}\mathbf{F}_{L\text{ext}}. \tag{74}$$

Eq. (63b) gives the corresponding q_L coordinates of the reduced system as $\mathbf{q}_L = \mathbf{\Theta}_{\text{red}}\mathbf{q}_{L\text{sub}}$. Combining this with Eqs. (72)–(74) yields

$$\mathbf{q}_L = \mathbf{\Theta}_{\text{red}}{}^{+}\mathbf{\Theta}_{\text{new}}{}^{+}\mathbf{\Phi}_{\text{new}}^{-1}\mathbf{\Theta}_{\text{red}}^{\text{T}}\mathbf{F}_{L\text{ext}}. \tag{75}$$

Obviously, when analysing a semi-infinite system in the positive domain which is excited at its end, one should only include the positive-going waves in the selected reduced set. If the system is finite their negative-going counterparts must also be included (as in Section 3.3), so altogether there are $2n_{\text{red}}$ characteristic waves. Now the total number of boundary conditions to be satisfied at the two ends of the whole system is $2n_c$ but since $2n_{\text{red}} < 2n_c$, this satisfaction is clearly impossible. Another arbitrary selection process must therefore be involved, in addition to the process of selecting the waves in the first place. This involves a decision about which boundary conditions should be satisfied and requires both common sense and a clear understanding of the relative importance of the different boundary conditions in the particular system being considered. Neither Gry [20] nor Brown and Byrne [23] explicitly addressed this problem in their study of forced response levels in infinite (not finite) periodic systems. Features such as the following should be taken into account.

If the boundary conditions at the two ends of the finite periodic system are the same (e.g. both ends are solidly locked or both are quite free, or both have the same combination of prescribed and free conditions) it would clearly be sensible to satisfy the boundary conditions at the same coordinate locations at each end. If one end is locked and the other end is free, then different locations may be considered at the two ends. It is difficult to generalise on this matter and better to consider each real problem as it comes. It is conceivable that computations could be carried out by satisfying just one boundary condition at one end of the system and at ($n_{red}-1$) locations at the other, but this would need to be justified in the case being considered. Otherwise the results could be of dubious significance.

The form in which Eqs. (72)–(75) have been presented assumes that all the external forces at the left-hand end of the system are prescribed and all the left-hand-end displacements are free. If the boundary conditions at the forced end are mixed (some prescribed forces and some prescribed displacements) the same method must be adopted as in Section 3.1.

5. Validation of the theory: an application to a finite element model of a thin flat plate

5.1. The method and the periodic model

The above algebraic formulations can be validated by using them to calculate responses for a simple periodic system for which exact or near-exact solutions can also be found. If the results from both methods agree, the formulations are validated. The simplest feasible multi-coupled system for this purpose is a FE model of a uniform thin flat and parallel plate. Divided length-wise into identical (i.e. periodic) slices, each of which is subdivided identically into a single stack of simple rectangular plate elements, it constitutes a multi-coupled PS (see Fig. 3). Each slice will be referred to as 'the stack'.

Finite and infinite lengths of this plate will be considered as it undergoes pure in-plane motion $u(x,y)$, $v(x,y)$ in the length-wise (x) and width-wise (y) directions, respectively. At low to moderate frequencies, the motions are those of in-plane bending, in-plane longitudinal or in-plane shear wave motion. Simple 'engineering' theories, of course, yield near-exact responses for these motions at low frequencies, so the formulations of this paper are validated if low-frequency responses from the periodic FE plate model agree with them. The simple theories to be used are the Euler–Bernoulli theory of bending (ETB) and the Timoshenko theory of bending ('Timo').

Denote the length of each stack by δL and the number of FEs in each stack by N_{st}. (These elements need not be identical (see Fig. 3a).). The simple rectangular thin plate FE has four corner nodes and two degrees of

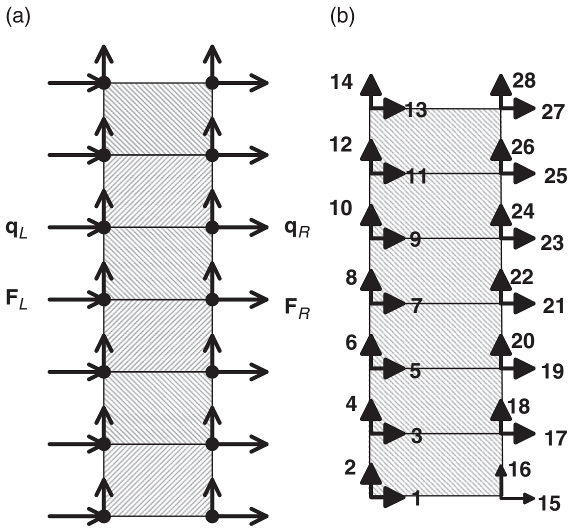

Fig. 3. (a) A periodic element consisting of a stack of rectangular finite elements, showing the coupling coordinates and forces. (b) The coordinate numbering system for a stack of six rectangular finite plate elements.

freedom per node, and all its displacements vary linearly across the element. The total number of coupling coordinates between adjacent stacks is $n_c = 2(N_{st} + 1)$ so a finite plate of length L has $L/\delta L = N_{el}$ stacks and a total of $2(N_{st} + 1)(N_{el} + 1)$ degrees of freedom. For the greatest accuracy of computation, δL should be very small (but not too small) so N_{st} should be large. This can lead to enormous numbers of displacement freedoms and vast matrices in a conventional FE analysis. Application of the PS theory embodying the Floquet principle reduces the effective number of freedoms of an infinite system to n_c, and of a finite system to $2n_c$. The combination of PS and FE analysis in this way will be referred to as the 'PS–FE' method.

The actual model studied for this investigation had only six identical elements in each stack but 2^{10} stacks per unit length. The unit length was taken as the plate width and the length/height ratio of each FE was very small at 6×2^{-10}. The overall length of the finite model considered was only 10 plate widths. Many more than six elements per stack would have been used had the author had access to commercial software such as ANSYS© but his available software made six the largest practicable number.

5.2. Computed results and comparisons with simple theories

Computer programmes were written within a software package which used precisely the same expressions as derived in this paper, using them directly for the response calculations. Now the accuracy of the forced vibration calculations depends fundamentally on the accuracy of the computed free-wave propagation constants it uses. This was therefore investigated by comparing the computed values of the wavenumbers for the model (i.e. the calculated propagation constants $\times$ the number N_E of periodic elements per unit length) with exact values derived from a solution of the exact partial differential equations of the in-plane motion of a flat plate. Details of this are not presented. Suffice it to say that very good agreement was found for the low-order propagating and attenuating waves over the non-dimensional frequency range $\Omega = 0$–2, and satisfactory agreement was found over the range $\Omega = 2$–4. ($\Omega = \omega b/c_L$ and $\omega =$ radian frequency, $b =$ plate width (taken to be unity), $c_L =$ longitudinal wave speed in the plate material. $\Omega = 2$ is a very high frequency as far as in-plane plate bending is concerned. At this frequency the half-wavelength of ETB in-plane flexural waves is only 1.2 plate widths.

One feature of the computed PS–FE wavenumbers does differ significantly from those of the exact solution, but not so significantly as far as the current investigations are concerned. The PS–FE theory predicts the correct finite number of propagating wavenumbers at low to medium frequencies, but the number of attenuating (evanescent) waves it predicts is finite. The exact theory, however, predicts an infinite number of attenuating waves at all frequencies. The low-order propagating wavenumbers from the PS–FE theory agreed closely with those of the exact theory, but this was true only for the lowest order evanescent wave. Agreement between the propagation constants of the higher-order evanescent waves was not good and, in fact, degenerated with increasing wave order. However, the detrimental effect of this on PS–FE forced vibration calculations would only be significant when local distortions close to the exciting forces were being examined. This paper is not of concerned with these.

Response calculations have been carried out for the FE plate model with the boundary conditions analysed above in Sections 3.1 to 4.2. Equal y-wise forces of non-dimensional magnitude 0.5 were considered at a single x-wise location on the plate model at coordinates 2 and 14 (see Fig. 3), the two forces together constituting a unit force. They acted at the ends of the semi-infinite and finite plates, and at a 'central' x-wise junction on a doubly infinite plate. In each case, the corresponding non-dimensional y-wise displacements (v) of the plate were computed for the same locations as the forces. As these displacements are equal at coordinates 2 and 14, either one of them can be called the direct receptance of the plate to the total unit transverse force applied at that x-wise location.

Fig. 4 shows the variation with Ω of the direct receptance of the semi-infinite FE plate model as given by Eq. (20a), and compares it with the receptances predicted by the ETB and Timo theories of bending. Timoshenko's recommended shear factor of 0.833 was used. The convergence of all three curves at frequencies below $\Omega = 1$ validates the theory (and algebra) of Section 3.1. (It is shown much more clearly at low frequencies when the curves are plotted with a logarithmic frequency scale.) Their divergence at higher frequencies simply demonstrates the well-known inadequacy of ETB and Timoshenko at higher frequencies. The agreement between the PS–FE and Timoshenko results from $\Omega = 0$ to 2 and the progressive disagreement

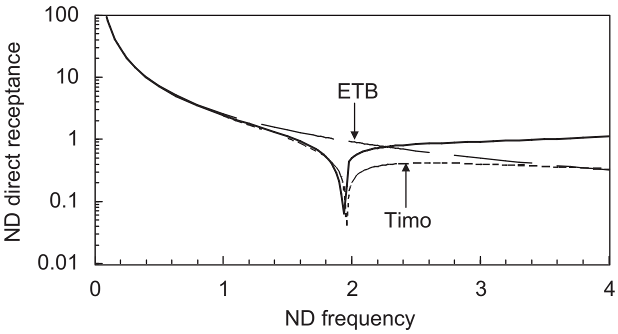

Fig. 4. The variation with frequency of the direct in-plane receptance of a semi-infinite thin plate subjected at its end to two equal transverse in-plane forces of 0.5: (——) values from the PS–FE method; (—— ——) values from the Euler–Bernoulli theory of bending (ETB) and (– – – –) values from the Timoshenko theory of bending (Timo).

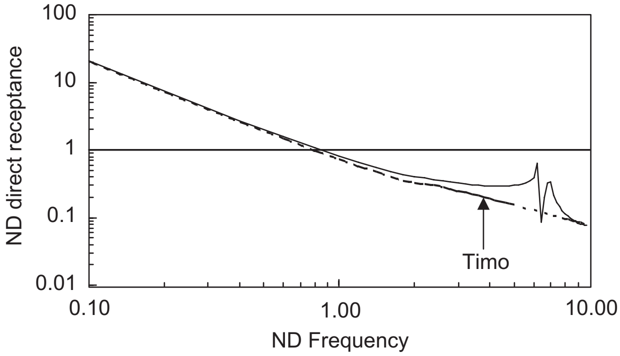

Fig. 5. The direct in-plane receptance of the doubly infinite thin plate subjected at its centre to two equal transverse in-plane forces of 0.5: (——) values from the PS-FE method and (——) Timoshenko values, shear factor = 1.

with ETB demonstrates the already well-known superiority of Timo over ETB. It gives further confidence in the forced vibration theory of this paper.

Similar conclusions can be drawn from the calculated direct receptances shown in Fig. 5 for the doubly infinite plate (see Section 3.2) computed by the method of prescribed forces and displacements described in Section 3.1. The agreement between these and the Timoshenko results is validation of the details of Section 3.2.

Shown in Fig. 6 are the direct receptances for a 'free–free' plate (i.e. free at each end) as derived in Section 3.3 and given by Eq. (39). The plate is loaded at just one end, and the PS–FE results are compared with two Timoshenko curves, one having been computed with a shear factor of 0.833 and the other with a shear factor of 1.0. The curve for the higher shear factor has the slightly higher frequencies (as expected) but it remains to be explained why the pairs of Timo peaks for this particular plate always straddle the PS–FE peaks. It does nothing to suggest that the theory of Section 3.3 is invalid. Rather, its validity is demonstrated by the convergence of all the curves at the lowest frequencies.

Receptances found from Eq. (46) for the finite plate loaded at a free end and fully fixed at the other (a fixed-free plate or short thin cantilever beam) are shown in Fig. 7. A much narrower frequency range has been considered here, simply because a wider range contains so many resonances that the figure becomes almost unintelligible. By demonstrating good agreement between the two sets of receptances, the smaller range is quite sufficient for the current validation purpose. All the resonance peaks for the undamped plate model would approach much higher values had smaller frequency increments been used in the calculations. No significance should therefore be given to the differences between the peak values of the two sets.

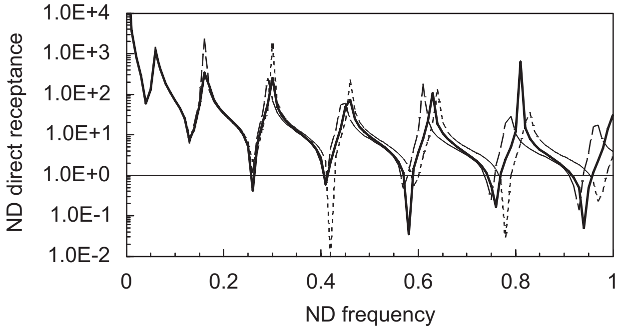

Fig. 6. The direct in-plane receptance of a finite free-free thin plate subjected at one end to two equal transverse in-plane forces of 0.5: (——) values from the PS–FE method; (– – – –) Timoshenko values, shear factor = 1; and (— —) Timoshenko values, shear factor = 0.833.

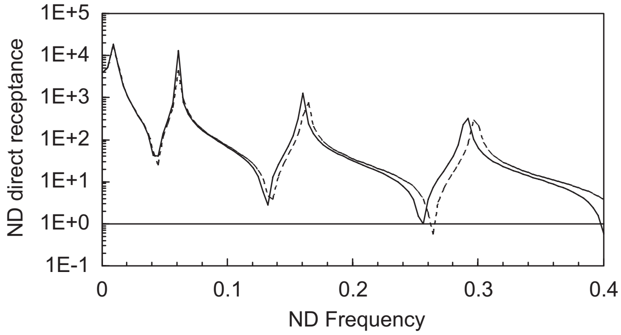

Fig. 7. The direct in-plane receptance of a finite fixed-free thin plate subjected at its free end to two equal transverse in-plane forces of 0.5: (——) values from the PS–FE method and (– – –) Timoshenko values, shear factor = 0.833.

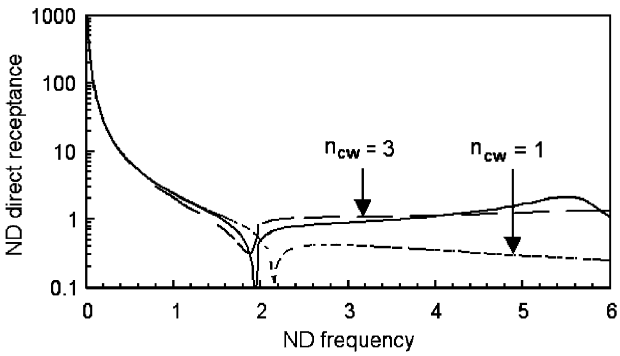

Fig. 8. The effect of reducing the number of participating modes on the direct in-plane receptance of the semi-infinite thin plate of Fig. 4, as calculated by the PS–FE method: (——) all 14 complex characteristic waves included, (– – – –) two parts of one complex wave included ($n_{cw} = 1$) and (— — —) two parts of each of three complex waves included ($n_{cw} = 3$).

Fig. 8 shows the effect on the computed receptances of reducing the number of wave coordinates in the calculations in the manner described in Section 4.2. The figure compares the computed PS–FE receptance curve of Fig. 4 for the semi-infinite plate (with all 14 positive-going waves being included) with values determined firstly by including the two sub-waves from a single characteristic wave (see Section 4.1) and then by including six sub-waves from just three characteristic waves. The two sub-wave modes were the separate real and imaginary parts of the single dominant complex wave-mode identified from the 14-wave response calculation at $\Omega = 0.0126$. This was the lowest order propagating wave, equivalent to the fundamental in-plane flexural wave in the plate. The lowest order evanescent wave was not included separately as it was automatically regenerated from the two parts of the propagating wave as mentioned in Section 4.1. The six sub-wave modes were the two parts of each of the three dominant waves identified at $\Omega = 5.2$. These were the lowest order propagating wave and two distinctly different 'complex-conjugate' evanescent waves.

The results shown in Fig. 8 are sufficient to validate the underlying theory and algebraic details of Section 4. The receptances from the two sub-wave approximation are very close to the 14-wave values over the surprisingly wide low-frequency range $\Omega = 0$–1.6, but these two sub-waves are clearly inadequate above $\Omega = 1.8$. The six sub-wave approximation is superior over the high-frequency range $\Omega = 2$–6, and is also good over the low-frequency range $\Omega = 0$–1. Further detailed comparisons are unwarranted as six elements in a stack are quite insufficient at the higher frequencies.

6. Conclusions

The former general theory of free-wave motion in multi-coupled PSs has provided the basis for a systematically developed, expanded and successfully applied general theory of forced wave motion in 1D PSs. Like the former theory it starts from the dynamic stiffness matrix of a single periodic element. The forced motion of a semi-infinite periodic system excited by forces at its finite end is then analysed in terms of the well-known characteristic waves of free motion. The forced motion of this particular system is fundamental to all the subsequent theory and is used to analyse the responses of both doubly infinite and then finite periodic systems. The computational process involved can be very long if the dynamic stiffness matrix of the periodic element is very large, leading to many waves participating in the forced motion. Full details are given whereby this matrix can be suitably transformed and reduced to allow only the most significant wave motions to be considered, Computation times are thereby greatly reduced.

The theory and its details have been validated through calculations of the in-plane vibration response of a uniform flat and parallel thin plate, modelled by a periodic array of FEs and excited by in-plane forces. Plates of both infinite and finite length with various boundary conditions were considered. The excellent agreement obtained between the computed low-frequency responses and those predicted by well-known approximate thin beam theories was sufficient validation, despite only six simple thin plate elements being considered in the periodic slices.

The combination of PS theory with FE analysis as presented is ideal for studying forced vibrations of long uniform rails, bars and complicated structural sections, especially at high frequencies when cross-sectional distortion is significant. Wavenumbers and forced responses can be computed by considering just one short periodic length-wise slice of the bar subdivided into the simplest of elements. Computational accuracy can be enhanced by making the slice extremely short, down to a point determined by frequency and computational ill-conditioning. Computation times depend only on the number of elements in the single slice and not on the total number in the whole system. The number within the slice determines the number of coupling coordinates between adjacent slices and limits the number of independent free harmonic wave motions that can exist in the model. This limit is most significant at high frequencies and leads to errors in computed responses which increase with frequency.

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