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Synchronization and anti-synchronization of chaos in an extended Bonhöffer–van der Pol oscillator using active control

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Abstract

In this paper, we investigate the complete synchronization and anti-synchronization (AS) of double-scroll chaotic attractor exhibited by an extended Bonhöffer–van der Pol (BVP) oscillator, using active control technique. In both synchronization schemes, the oscillators show good transient performance; while the AS state is further shown to correspond with complete inverse synchronization. Numerical simulations are also presented to verify the theoretical results.

1. Introduction

In the last few decades, considerable research has been done in nonlinear systems and their various properties. One of the most important aspects of nonlinear dynamical systems is the property of synchronization [1–3], which classically, represents the entrainment of frequency of oscillators due to weak interactions. Studies in this field are partly motivated by experimental realization in lasers, electronic circuits, plasma discharge and chemical reactions [2–5]. Chaos synchronization is related to the observer problem in control theory [5]. The problem may be treated as the design of control law for full chaotic observer (the slave system) using the known information of the plant (the master system) so as to ensure that the controlled receiver synchronizes with the plant. Hence, the slave chaotic system completely traces the dynamics of the master in the course of time.

On the other hand, a related phenomenon, anti-synchronization (AS), which is the vanishing of the sum of the relevant state variables of synchronized systems has been investigated both experimentally and theoretically in many physical systems [6–14]. A very recent study of the AS phenomenon in non-equilibrium

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systems suggests that AS could be exploited as a technique for particle separation in a mixture of interacting particles [14].

In general, various techniques have been proposed for achieving stable synchronization between identical and non-identical systems. Notable among these methods, the active control scheme proposed by Bai and Lonngren [15] has received considerable attention during the last decade. Applications to various systems abound, some of which include the electronic circuits which model a third-order ''jerk'' equation [16], Lorenz, Chen and Lü system [17], geophysical model [18], nonlinear equations of acoustic gravity waves [19], Qi system [19,20]; van der Pol–Duffing oscillator [21], periodically forced pendulum [22], nuclear magnetic resonance (NMR) modelled by the nonlinear Bloch equations [23], parametrically excited oscillators [24,25], the so-called Unified chaotic attractor [26]; and most recently in RCL-shunted Josephson junction [27], permanent magnet reluctance machine [13] and inertial ratchets [14,28].

Recently, there has been resurgent interest in the theoretical and experimental generation of multi-scroll chaotic attractors [29]. This is partially due to foreseen varieties of practical applications in such fields as digital and secure communications, synchronous prediction, random bit generation, information systems, to name but a few [29]. An efficient secure-communication model system should exhibit appreciable synchronization performance even in its chaotic state. Thus, exploring the synchronization behaviour of multi-scroll chaotic attractors would be an important and stimulating subject of research interest. However, this has received inadequate attention. A few reports on this can be found in Refs. [30–32].

The goal of this paper is to employ the active control technique to investigate the synchronization and AS performance of two identical double-scroll chaotic attractors generated from an extended Bonhöffer–van der Pol (BVP) oscillator circuit. This problem has not been considered to the best of our knowledge. The BVP (or Fitzhugh–Nagumo) oscillator model was derived from the van der Pol oscillator to give a more accurate and reliable description of nonlinear dynamical systems, which can show a stable state and threshold phenomena as well as stable oscillations. The BVP which is closely related to the Fitzhugh–Huxley (FH) model of the squid giant axon, the cats carotid sinus nerve [33] and the iron wire model of the nerve [34]; serves as a simplified model for electrical waves in the heart. Indeed, most studies on the BVP have shown its various applications in medicine (see for example Refs. [33–36] and references therein). In addition, Chimi et al. [37] demonstrated in a very recent study, the application of the synchronization in secure communication using two periodically forced BVP oscillators. The rest of this paper is organized as follows. In Section 2, we give a brief description of the physical circuit of the extended BVP model and the governing equations. Sections 3 and 4 are devoted to active control formulations for chaos synchronization as well as numerical simulation results. The last section contains the conclusions.

2. Circuit model

Here we consider an extended BVP oscillator, which consists of two capacitors, an inductor and a linear resistor as shown in Fig. 1 [38]. By applying Kirchoff's laws to the various branches of the circuit, the following equations are obtained:

$$C\frac{dv_1}{dt} = -i - g(v_1)$$
$$C\frac{dv_2}{dt} = i - \frac{v_2}{r}$$
$$L\frac{di}{dt} = v_1 - v_2 \tag{1}$$

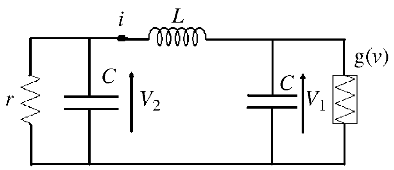

Fig. 1. The extended BVP oscillator circuit.

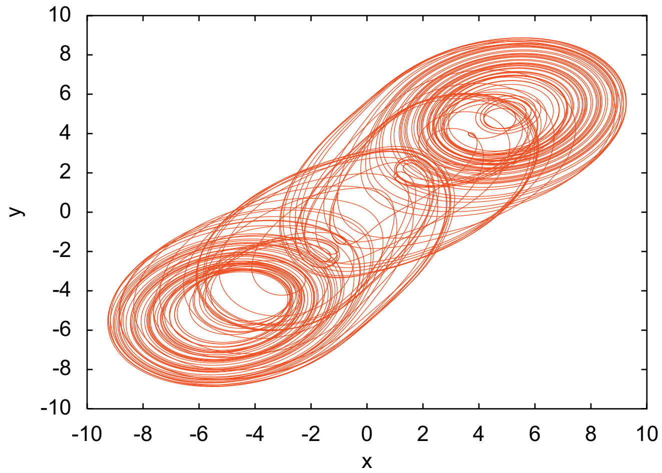

Fig. 2. Two-dimensional view of the double-scroll chaotic attractor of the extended Bonhöffer–van der Pol oscillators $A = B = 1.0$ and $\delta = 1.2$.

where $g(v_1)$ is the v–i characteristics of the nonlinear resistor and is given by

$$g(v_1) = -av_1 - b \tanh cv_1 \tag{2}$$

Using the following scaling factors:

$$\cdot = \frac{\mathrm{d}}{\mathrm{d}\tau}, \quad \tau = \frac{1}{\sqrt{LC}}t, \quad A = a\sqrt{\frac{L}{C}}, \quad B = bc\sqrt{\frac{L}{C}}, \quad \delta = \frac{1}{r}\sqrt{\frac{L}{C}}, \quad x = \frac{v_1}{b}\sqrt{\frac{C}{L}},$$
$$y = \frac{v_2}{b}\sqrt{\frac{C}{L}}, \quad z = \frac{i}{b}$$

the normalized equation for the extended BVP oscillator (1) can be written as

$$\begin{aligned} \dot{x} &= -z + Ax + \tanh Bx \\ \dot{y} &= z - \delta y \\ \dot{z} &= x - y \end{aligned} \tag{3}$$

where x and y are state variables corresponding to the voltages across the capacitors. The z variable is proportional to the current in the inductor, while δ corresponds to the value of a fixed resistor. A and B are the control parameters of the system. Extensive experimental and numerical study of the bifurcation and chaotic phenomenon of the oscillator described by system (3) was recently carried out by Nishiuchi et al. [38]. Beside various dynamical behaviours observed in the BVP oscillator, a double-scroll chaotic attractor was reported for $(A, B, \delta) = (1.0,1.0,1.2)$. This attractor is displayed in Fig. 2.

3. Complete synchronization using active control

Let the drive system of the BVP oscillator be written as

$$\begin{aligned} \dot{x}_1 &= -z_1 + Ax_1 + \tanh Bx_1 \\ \dot{y}_1 &= z_1 - \delta y_1 \\ \dot{z}_1 &= x_1 - y_1 \end{aligned} \tag{4}$$

Then the response system is

$$\begin{aligned}\dot{x}_2 &= -z_2 + Ax_2 + \tanh Bx_2 + u_x\\ \dot{y}_2 &= z_2 - \delta y_2 + u_y\\ \dot{z}_2 &= x_2 - y_2 + u_z\end{aligned} \tag{5}$$

where $u_i(t)$, $i = x, y, z$ are control functions to be determined. Subtracting Eq. (4) from Eq. (5) we obtain the error dynamics as

$$\begin{aligned}\dot{e}_x &= -e_z + Ae_x + \tanh Bx_2 - \tanh Bx_1 + u_x\\ \dot{e}_y &= e_z - \delta e_y + u_y\\ \dot{e}_z &= e_x - e_y + u_z\end{aligned} \tag{6}$$

where $e_i = i_2 - i_1$, $i = x, y, z$. In the absence of the controls, the error dynamics system (6) would have an equilibrium at (0,0,0). If the controls are chosen such that the equilibrium (0,0,0) is unchanged, then the synchronization between the driver system (4) and the response system (5) reduces to that of finding the asymptotic stability of the error system (6) at equilibrium. To achieve this, the control functions are re-defined to eliminate terms in Eq. (6), which cannot be expressed as linear terms in e_x, e_y and e_z, as follows:

$$\begin{aligned}u_x &= \tanh Bx_1 - \tanh Bx_2 + v_x(t)\\ u_y &= v_y(t)\\ u_z &= v_z(t)\end{aligned} \tag{7}$$

Substituting Eq. (7) into Eq. (6) we have

$$\begin{aligned}\dot{e}_x &= e_z + Ae_x + v_x\\ \dot{e}_y &= e_z - \delta e_y + v_y\\ \dot{e}_z &= e_x - e_y + v_z\end{aligned} \tag{8}$$

Using the active control method, a constant matrix **M** is chosen which will control the error dynamics (6) such that the feedback matrix

$$\begin{pmatrix} v_x(t)\\ v_y(t)\\ v_z(t)\end{pmatrix} = \mathbf{M}\begin{pmatrix} e_x\\ e_y\\ e_z\end{pmatrix} \tag{9}$$

with

$$\mathbf{M} = \begin{pmatrix} \lambda_1 - A & 0 & 1\\ 0 & \lambda_2 + \delta & -1\\ -1 & 1 & \lambda_3\end{pmatrix} \tag{10}$$

In Eq. (10), the three eigenvalues λ_1, λ_2 and λ_3 have been chosen as -1, -1 and -1 in order that a stable and synchronized identical BVP oscillators are achieved. With Eq. (10), the control law is

$$\begin{aligned}u_x &= \tanh Bx_1 - \tanh Bx_2 - (1 + A)e_x + e_z\\ u_y &= (\delta - 1)e_y - e_z\\ u_z &= -e_x + e_y - e_z\end{aligned} \tag{11}$$

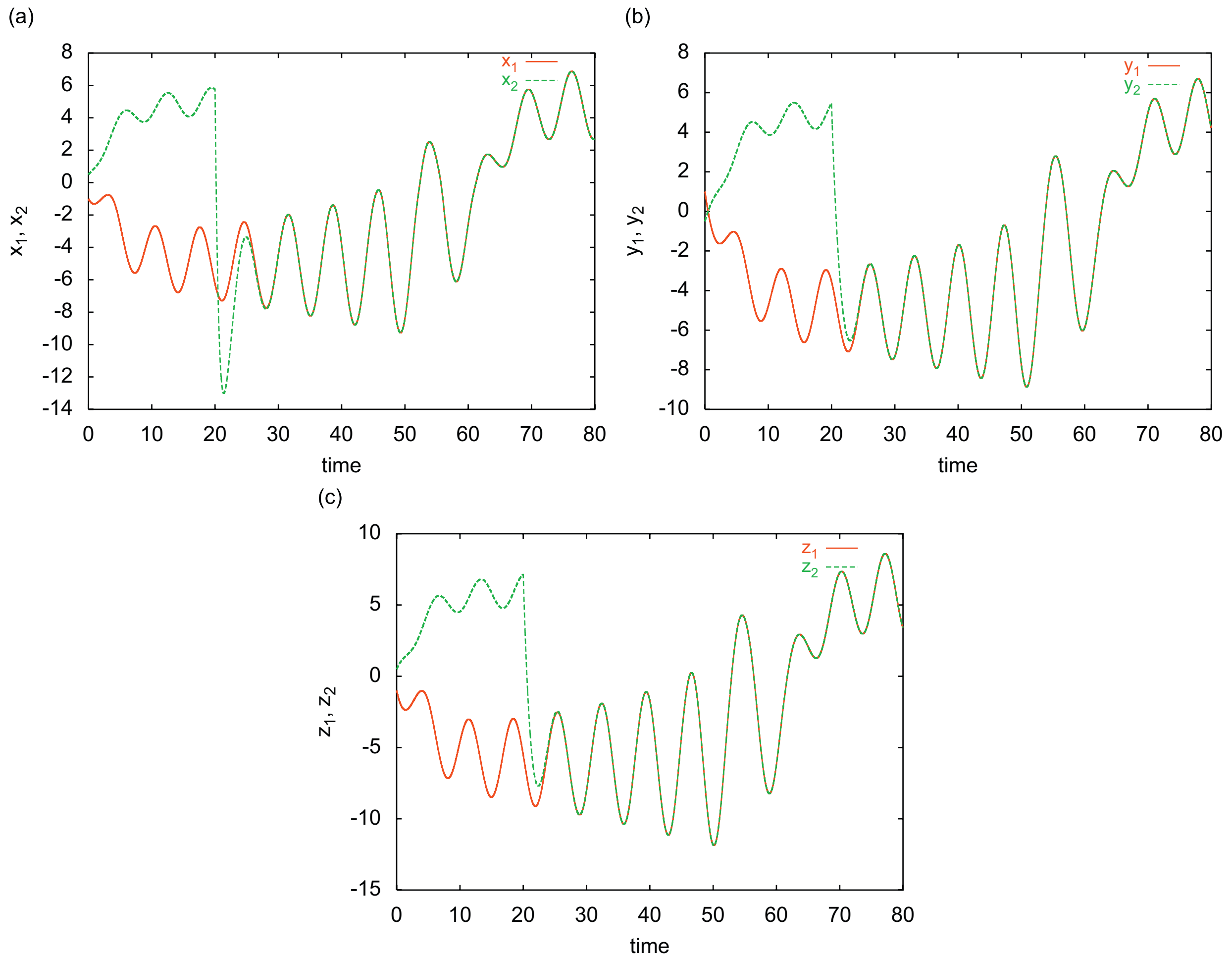

Fig. 3. Synchronization dynamics of the state variables when the control is activated at $t = 20$: (a) x_1 and x_2, (b) y_1 and y_2, and (c) z_1 and z_2.

To numerically verify the effectiveness of the designed controllers, we used the standard fourth-order Runge–Kutta algorithm to solve the drive-response systems (2) and (3), with a time step size of 0.005. The parameters of the system were selected such that the system exhibits two-scroll chaotic attractor as shown in Fig. 2. That is $A = 1.0$, $B = 1.0$ and $\delta = 1.2$. The initial conditions were taken as $x_1(0) = -1.0$, $x_2(0) = 0.5$, $y_1(0) = 1.0$, $y_2(0) = -0.5$, $z_1(0) = -1.0$ and $z_2(0) = 0.5$. The simulation results for x_1, x_2; y_1, y_2 and z_1, z_2 are illustrated in Figs. 3(a)–(c) respectively. In Fig. 3, the controls have been activated at $t = 20$. Prior to $t = 20$, the two systems exhibit different dynamics due to the difference in initial conditions. However, as soon as the controls are switched on, the response system is forced to trace the dynamics of the driver system. The synchronization is a complete one as depicted in Fig. 7(a) where a direct and linear relationship is shown to exist between x_1 and x_2 variables along the synchronization manifold defined by $x_1 = x_2$. In Fig. 4, we show the asymptotic convergence of the synchronization errors when the controls have been activated at $t = 0$. The plots show a good transience performance as synchrony is achieved for $t \geqslant 7$.

4. Anti-synchronization using active control

To investigate AS in the extended BVP oscillator, we define the AS errors for the drive-response system as

$$s_1 = x_1 + x_2, \quad s_2 = y_1 + y_2, \quad s_3 = z_1 + z_2 \tag{12}$$

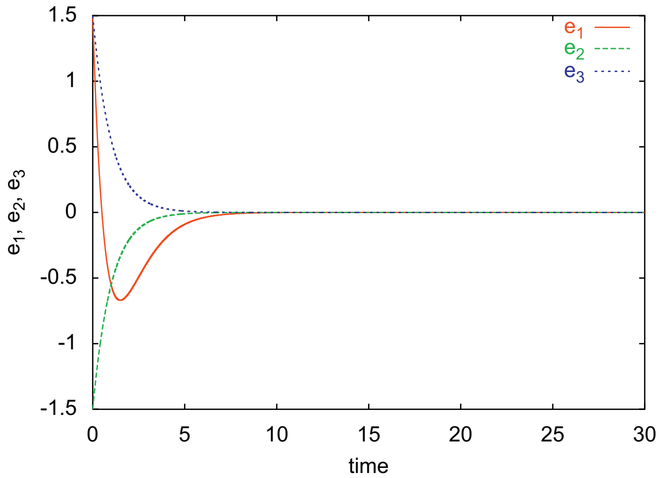

Fig. 4. Error dynamics in the synchronized state, showing the transient performance of the control when activated at $t = 0$.

Adding Eqs. (4) and (5) and using the definitions in Eq. (12), we have

$$\begin{aligned} \dot{s}_1 &= As_1 - s_3 + \tanh Bx_1 + \tanh Bx_2 + u_1 \\ \dot{s}_2 &= -\delta s_2 + s_3 + u_2 \\ \dot{s}_3 &= s_1 - s_2 + u_3 \end{aligned} \tag{13}$$

Redefining the controls ($u_i(i = 1,2,3)$) as follows:

$$\begin{aligned} u_1 &= -(\tanh Bx_1 + \tanh Bx_2) + v_1 \\ u_2 &= v_2 \\ u_3 &= v_3 \end{aligned} \tag{14}$$

the AS error system (13) becomes

$$\begin{aligned} \dot{s}_1 &= As_1 - s_3 + v_1 \\ \dot{s}_2 &= -\delta s_2 + s_3 + v_2 \\ \dot{s}_3 &= s_1 - s_2 + v_3 \end{aligned} \tag{15}$$

Proceeding as before, it is found that the same matrix as in Eq. (10) is chosen such that

$$[v_1, v_2, v_3]^{\mathrm{T}} = \mathbf{M}[s_1, s_2, s_3]^{\mathrm{T}} \tag{16}$$

and obtain the following control law:

$$\begin{aligned} u_1 &= -(\tanh Bx_1 + \tanh Bx_2) - (1 + A)s_1 + s_3 \\ u_2 &= (\delta - 1)s_2 - s_3 \\ u_3 &= -s_1 + s_2 - s_3 \end{aligned} \tag{17}$$

Again, we perform a numerical investigation for the AS scheme using the same parameters and initial conditions as in the previous section. While Fig. 6 shows the asymptotic convergence of the AS errors; Fig. 5 shows the dynamics of the state variables in the AS state. The plots in Fig. 6 also reveal a good transient performance. However, the AS synchrony is achieved when $t \geqslant 10$, implying that the CS state has better transient performance compared to the AS state. Finally, we neglect the initial transient and plot in Figs. 7(a) and (b) x_1 vs. x_2. The dynamics are obviously confined to the manifold $x_1 = x_2$ for complete synchronization (Fig. 7(a)) and $x_1 = -x_2$ for AS (Fig. 7(b)), the latter corresponding to the phenomenon of complete inverse synchronization reported by Shahverdiev et al. [39,40].

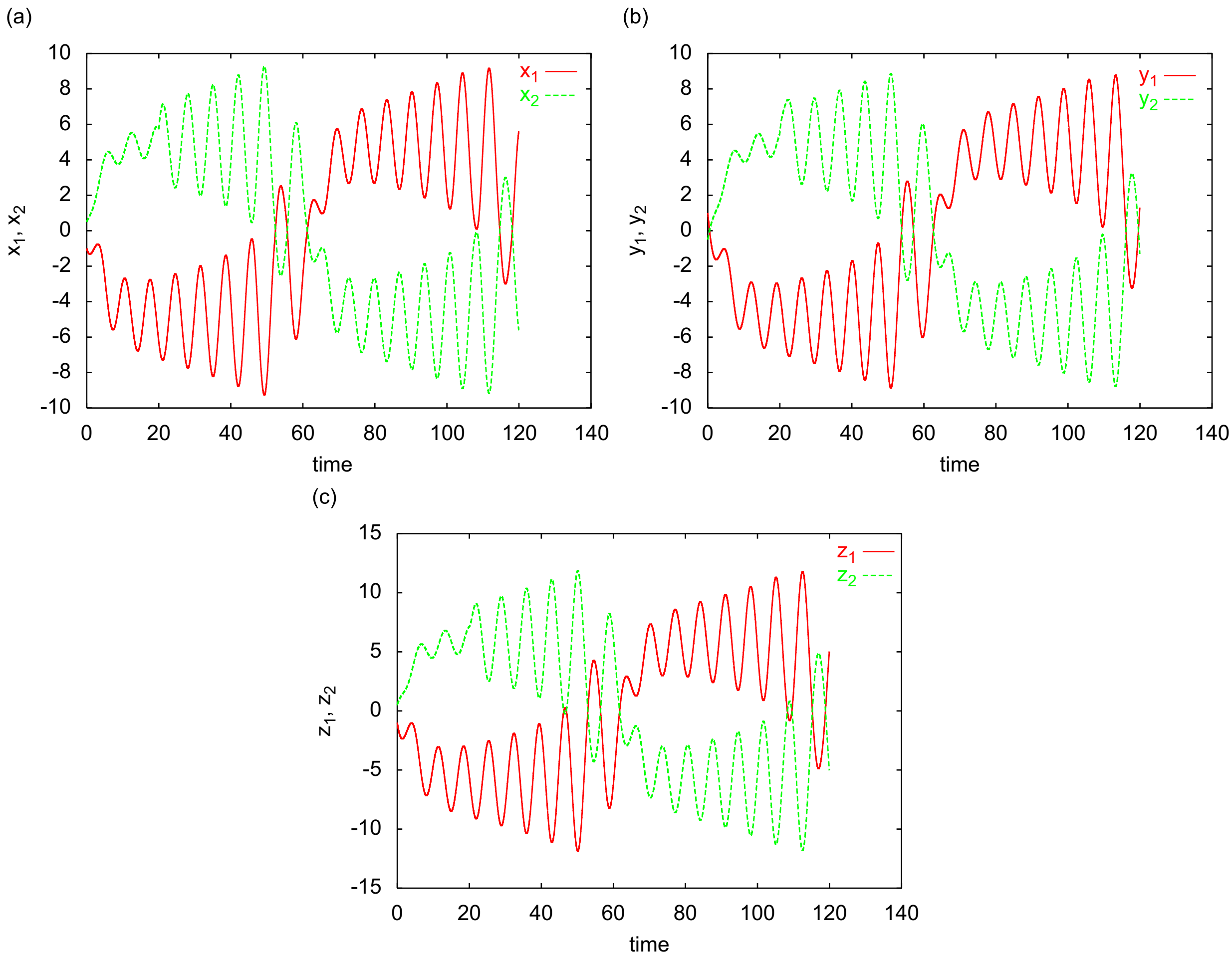

Fig. 5. AS dynamics of the state variables when the control is activated at $t = 0$: (a) x_1 and x_2, (b) y_1 and y_2, and (c) z_1 and z_2.

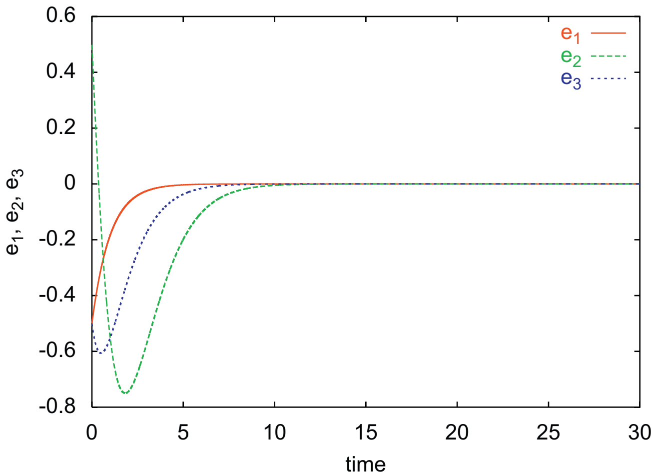

Fig. 6. AS error dynamics, showing the transient performance of the control when activated at $t = 0$.

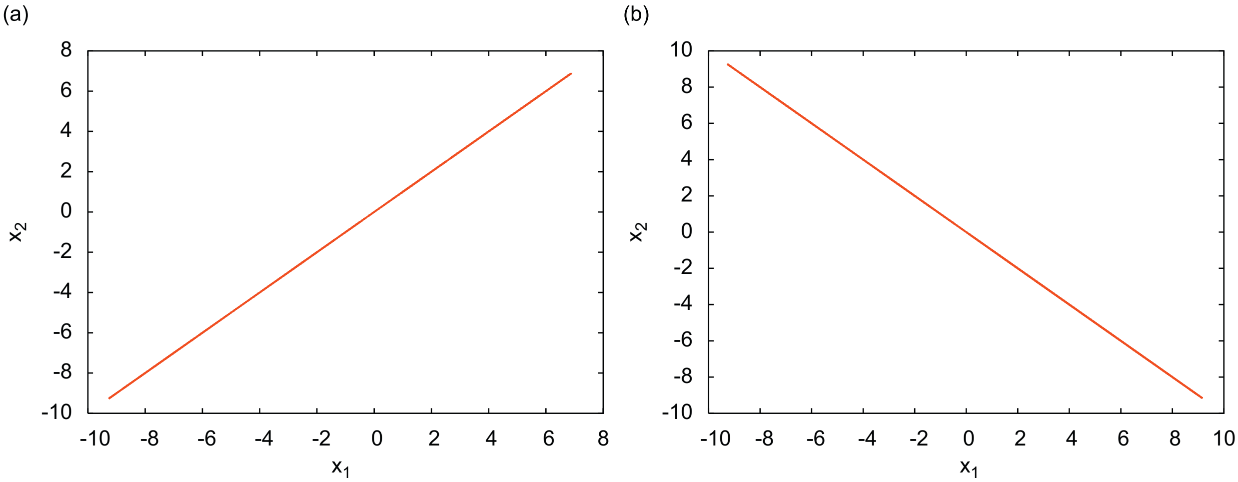

Fig. 7. x_1 vs. x_2 showing evidence of (a) complete and (b) complete-inverse synchronization.

5. Concluding remarks

Conclusively, this paper has presented active control-based synchronization and AS schemes for a simple three-dimensional autonomous circuit describing an extended BVP oscillators. The operating regime for double-scroll chaotic behaviour was used. Thus, the complexity of the systems was retained. The theoretical results have been validated with corresponding numerical simulations; and stable synchronized and anti-synchronized states were achieved. The BVP oscillator studied here could serve as a good model for chaos-based secure-communication system particularly when operated in the multi-scroll chaotic attractor regime. In this direction, the possibilities of realizing multi-scroll chaotic attractors and its synchronization from the BVP oscillator are issues that will be addressed in a future paper.

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