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# Non-probabilistic interval analysis method for dynamic response analysis of nonlinear systems with uncertainty

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**Abstract**

Effects of uncertainties on the dynamic response of the nonlinear vibration systems with general form are investigated. Based on interval mathematics, modeling the uncertain parameters as interval numbers, a non-probabilistic interval analysis method, which estimates the range of the nonlinear dynamic response with the help of Taylor series expansion, is presented, where the partial derivatives of the dynamic response with respect to uncertain parameters are considered to be interval numbers. The sensitivity matrices of dynamic response with the uncertain parameters are derived. For the presented method, only the bounds on uncertain parameters are needed, instead of probabilistic density distribution or statistical quantities. Numerical examples are used to illustrate the validity and feasibility of the presented method.


## 1. Introduction

The dynamic response analysis plays an important role in the design and analysis of structural systems. However, the material properties and structural geometry usually exhibit uncertainties due to the manufacture errors, measurement errors and other factors. Consequently, the dynamic response of structural system is also uncertain.

In recent years, the dynamic response analysis of linear vibration system with uncertain parameters has been studied extensively, including probabilistic methods [1–3] and non-probabilistic methods [4–7]. Shinozuka presented a method by which a least favorable structural response can be estimated to the excitation of non-deterministic nature such as earthquake acceleration [1]. Chen et al. [2] proposed a probabilistic method to evaluate the effect of uncertainties in geometrical and material properties of structures on the vibration response of random excitation. The stochastic orthogonal polynomial expansion method is extended with the pseudo-excitation method by Li and Liao [3], and this extension enables the stochastic orthogonal polynomial method to be readily used in the analysis of stochastic parameter structures under non-stationary random excitation. Based on the matrix perturbation theory and the interval extension of function, the upper and lower bounds of the dynamic response are obtained by Chen et al. [4], while the sharp bounds are guaranteed

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by the interval operations. Qiu and Wang [5] developed non-probabilistic interval analysis method for the dynamical response of structures with uncertain-but-bounded parameters by combining the finite element method and interval mathematics. Subsequently, they evaluated the range of dynamic responses of structures with uncertain-but-bounded parameters by using the parameter perturbation method [6]. Based on the improved first-order Taylor interval expansion, a new interval analysis method for the static or dynamic response of the linear structural system with interval parameters is presented [7].

However, the nonlinear vibration systems with uncertain parameters are far more complicated than linear vibration systems, and the effective methods for linear system (such as the superposition principle) are not valid for nonlinear systems. As a result, the research of the nonlinear dynamic response analysis of nonlinear vibration system with uncertainty is in the developmental stage.

So far, the main methods for solving the nonlinear dynamic response with uncertain parameters are probabilistic methods. Probability perturbation finite element method was presented by Wen et al. [8] to obtain the mean value and variance of the dynamic response of random nonlinear vibration systems with 2D matrix functions. Impollonia and Muscolino [9] applied the improved perturbation method in the second-order analysis of geometrically nonlinear stochastic systems subjected to static and dynamic deterministic forces. Pradlwarter et al. [10] presented an algorithm for the computation of the stochastic non-stationary nonlinear response of large FE-model. Li and Chen [11,12] proposed the probability density evolution method for dynamic response analysis of nonlinear stochastic structures. Cai and Lin [13] proposed an improved equivalent linearization procedure for nonlinear systems under bounded random excitations. Pishkenari et al. [14] discussed the nonlinear dynamic analysis of atomic force microscopy under deterministic and random excitation. Guo et al. [15] studied the nonlinear random response of laminated composite shallow shells using finite element modal method. Kumar and Datta [16] discussed the determination of probability density function of the response for strongly nonlinear single-degree-of-freedom system subjected to both multiplicative and additive random excitations using stochastic averaging technique.

In above probabilistic models, the uncertain variables are usually dealt with random quantity or stochastic process, where extensive knowledge of the probabilistic characteristics of the uncertain information is required. However, the information about the uncertain parameters is often scanty so that the distributed characteristics are difficult to get. In these cases, the non-probabilistic interval models can be used as alternative way. Non-probabilistic interval analysis methods are less information-intensive than probabilistic models, since only the bounds on uncertain parameters are required. By far, research in non-probabilistic modeling on uncertain nonlinear dynamic response has not been widespread and is only currently gaining attention.

In this paper, an approximate solution technique for nonlinear dynamic response with uncertain-but-bounded parameters is proposed based on interval mathematics, resorting to Taylor series expansion. The sensitivity of nonlinear dynamic response with respect to uncertain parameters is analyzed. Finally, a single degree-of-freedom system with nonlinear damping and a 60-bar power pagoda with nonlinear stiffness are used to illustrate the applications of the presented method.

## 2. Problem description

Consider the $n$ degree-of-freedom nonlinear system with uncertain parameters. The general form of differential equation can be described by

$$f(B, x, \dot{x}, \ddot{x}) = F(B, t) \tag{1}$$

with the initial conditions

$$x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0 \tag{2}$$

where $f$, $x$, $F$ are the nonlinear function, displacement and external force, respectively; and the superscript "$\cdot$" represents the derivative with respect to time $t$. $B$ is the $m$-dimensional uncertain parameter vector, which is used to describe the uncertain effects. The deterministic part and the uncertain part are implicit in Eq. (1). If the probabilistic distributions of all uncertain parameters variable are known, the probabilistic statistics information of nonlinear structural response system can be solved by the method introduced in Ref. [8].

However, in some cases, the probabilistic characteristics of the uncertain parameters are unknown, and the only known parameter is the interval of the uncertain parameters, namely

$$\underline{B} \leqslant B \leqslant \overline{B} \tag{3}$$

or the component form

$$\underline{b}_i \leqslant b_i \leqslant \overline{b}_i, \quad i = 1, 2, \ldots, m \tag{4}$$

in which $\overline{B} = (\overline{b}_i)$ and $\underline{B} = b_i$ are, respectively, the upper bound vector and the lower bound vector of the uncertain-but-bounded parameter vector $B = (b_i)$. By virtue of the interval vector notation [17,18], the vector inequality constraint conditions can be written as

$$B \in B^I = (b_i^I) = [\underline{B}, \overline{B}] \text{ or } b_i^I = [\underline{b}_i, \overline{b}_i], \quad i = 1, 2, \ldots, m \tag{5}$$

where $B^I$ is the $m$-dimensional interval vector. Then the solution of Eq. (1) subject to Eq. (3) or Eq. (4) is a set, and this set is given by

$$\Gamma = \{x : f(B, x, \dot{x}, \ddot{x}) = F(B, t), B \in B^I\} \tag{6}$$

In general, the set $\Gamma$ has a complicated region. In interval mathematics [18], solving the nonlinear problem (1) subject to (3) or (4) is synonymous to finding a multidimensional rectangle or interval vector containing nonlinear response set (6). In other words, the upper and lower bounds on the nonlinear response set (6) will be sought as follows:

$$x_i \in x_i^I = [\underline{x}_i, \overline{x}_i], \quad i = 1, 2, \ldots, n \tag{7}$$

where

$$\overline{x}(t) = \max_{B \in B^I} \{x(t) : f(B, x, \dot{x}, \ddot{x}) = F(B, t), B \in B^I\} \tag{8}$$

and

$$\underline{x}(t) = \min_{B \in B^I} \{x(t) : f(B, x, \dot{x}, \ddot{x}) = F(B, t), B \in B^I\} \tag{9}$$

Obviously, the maximum and minimum values in Eqs. (8) and (9) are all global optimal solutions.

## 3. Interval analysis method based on Taylor series expansion

Based on the interval transformation, Eq. (5) can be put into the more useful form

$$B^I = B^c + [-\Delta B, \Delta B] \tag{10}$$

where $B^c$ and $\Delta B$ denote the middle vector and the uncertain vector (or the maximum error) of $B^I$, respectively. It follows that

$$B^c = (\overline{B} + \underline{B})/2 \text{ or } b_i^c = (\overline{b}_i + \underline{b}_i)/2, \quad i = 1, 2, \ldots, m \tag{11}$$

$$\Delta B = (\overline{B} - \underline{B})/2 \text{ or } \Delta b_i = (\overline{b}_i - \underline{b}_i)/2, \quad i = 1, 2, \ldots, m \tag{12}$$

Then the uncertain-but-bounded parameter vector $B$ could be denoted as the following vector form:

$$B = B^c + \delta B, \quad |\delta B| \leqslant \Delta B \tag{13}$$

or the component form

$$b_i = b_i^c + \delta b_i, \quad |\delta b_i| \leqslant \Delta b_i, \quad i = 1, 2, \ldots, m \tag{14}$$

Let $\alpha$ be a scalar, then the nonlinear dynamic response $x_k(B^c + \alpha(B - B^c), t)$ can be regarded as a function of the single variable $\alpha$. By virtue of Lagrangian mean value theorem, expanding $x_k(B^c + \alpha(B - B^c), t)$ about $\alpha = 0$ and setting $\alpha = 1$, we will have

$$x_k(B, t) = x_k(B^c, t) + (B - B^c)^{\mathrm{T}} g(B^c + \xi(B - B^c), t) \tag{15}$$

where $0 \leqslant \xi \leqslant 1$ and $g$ is the gradient of $x_k(B,t)$. Since $B \in B^I$ and $B^c \in B^I$, then $B^c + \xi(B - B^c) \in B^I$. Therefore

$$x_k(B,t) \in x_k(B^c,t) + (B - B^c)^{\mathrm{T}} g(B^I) \subset x_k(B^c,t) + (B^I - B^c)^{\mathrm{T}} g(B^I) \tag{16}$$

Since Eq. (16) holds for all $B \in B^I$, we have

$$x_k(B^I,t) \approx x_k(B^c,t) + (B^I - B^c)^{\mathrm{T}} g(B^I) \tag{17}$$

It can be seen from Eq. (17) all the variables in $g$ are intervals. In the following we can illustrate that some of the variables in $g(B^I)$ can be substituted with real quantities so that a sharper bound on $x_k(B^I,t)$ can be obtained.

Firstly, $x_k(b_1,b_2,\ldots,b_m,t)$ is considered as a function of $b_m$ only. Expanding $x_k(b_1,b_2,\ldots,b_m,t)$ about $b_m^c$ yields

$$x_k(b_1,b_2,\ldots,b_m,t) = x_k(b_1,b_2,\ldots,b_m^c,t) + (b_m - b_m^c) g_m(b_1,b_2,\ldots,\xi_m,t) \tag{18}$$

Then, expanding $x_k(b_1,b_2,\ldots,b_m{}^c,t)$ about $b_{m-1}^c$ as a function of $b_{m-1}$ yields

$$x_k(b_1,b_2,\ldots,b_m^c,t) = x_k(b_1,b_2,\ldots,b_{m-1}^c,b_m^c,t) + (b_{m-1} - b_{m-1}^c) g_{m-1}(b_1,b_2,\ldots,\xi_{m-1},b_m^c,t) \tag{19}$$

Substituting Eq. (19) into Eq. (18) yields

$$\begin{aligned} x_k(b_1,b_2,\ldots,b_m,t) = {} & x_k(b_1,b_2,\ldots,b_{m-1}^c,b_m^c,t) + (b_{m-1} - b_{m-1}^c) g_{m-1}(b_1,b_2,\ldots,\xi_{m-1},b_m^c,t) \\ & + (b_m - b_m^c) g_m(b_1,b_2,\ldots,\xi_m,t) \end{aligned} \tag{20}$$

The rest may be deduced by analogy, and we have

$$x_k(b_1,b_2,\ldots,b_m,t) = x_k(b_1^c,b_2^c,\ldots,b_{m-1}^c,b_m^c,t) + \sum_{i=1}^{m} (b_i - b_i^c) g_i(b_1,\ldots,b_{i-1},\xi_i,b_{i+1}^c\ldots,b_m^c,t) \tag{21}$$

where $\xi_i \in b_i^I$. Due to $b_i \in b_i^I$, so we can obtain

$$x_k(b_1,b_2,\ldots,b_m,t) \in x_k(b_1^c,b_2^c,\ldots,b_{m-1}^c,b_m^c,t) + \sum_{i=1}^{m} (b_i^I - b_i^c) g_i(b_1^I,\ldots,b_{i-1}^I,b_i^I,b_{i+1}^c\ldots,b_m^c,t) \tag{22}$$

Since the relation (22) holds for all $b_i \in b_i^I (i = 1,2,\ldots,m)$, the nonlinear dynamic response interval can be approximately obtained as follows:

$$x_k(b_1^I,b_2^I,\ldots,b_m^I,t) \approx x_k(b_1^c,b_2^c,\ldots,b_{m-1}^c,b_m^c,t) + \sum_{i=1}^{m} (b_i^I - b_i^c) g_i(b_1^I,\ldots,b_{i-1}^I,b_i^I,b_{i+1}^c\ldots,b_m^c,t) \tag{23}$$

By virtue of the first-order Taylor series expansion, the gradient $g_i$ of $x_k(B,t)$ about $B^c$ can be developed as

$$g_i(B,t) = \frac{\partial x_k(B,t)}{\partial b_i} \approx \frac{\partial x_k(B^c,t)}{\partial b_i} + \sum_{j=1}^{m} \frac{\partial^2 x_k(B^c,t)}{\partial b_i \partial b_j}(b_j - b_j^c), \quad i = 1,2,\ldots,m \tag{24}$$

By making use of the natural interval extension, from Eq. (24) we can obtain the interval of the gradient $g_i$ as follows:

$$g_i(B^I,t) = \frac{\partial x_k(B^I,t)}{\partial b_i} \approx \frac{\partial x_k(B^c,t)}{\partial b_i} + \sum_{j=1}^{m} \frac{\partial^2 x_k(B^c,t)}{\partial b_i \partial b_j}(b_j^I - b_j^c), \quad i = 1,2,\ldots,m \tag{25}$$

Substituting Eq. (25) into Eq. (23) leads to

$$x_k(b_1^I,b_2^I,\ldots,b_m^I,t) \approx x_k(b_1^c,b_2^c,\ldots,b_{m-1}^c,b_m^c,t) + \sum_{i=1}^{m} (b_i^I - b_i^c) \left[ \frac{\partial x_k(B^c,t)}{\partial b_i} + \sum_{j=1}^{i} \frac{\partial^2 x_k(B^c,t)}{\partial b_i \partial b_j}(b_j^I - b_j^c) \right] \tag{26}$$

By the interval operations, from the above Eq. (26), we can determine the interval region of the nonlinear dynamic response of structures with uncertain-but-bounded parameters using the interval analysis method.

## 4. Partial derivatives determination of nonlinear dynamic response

In this section, based on the assumption that the nonlinear function $f(B, x, \dot{x}, \ddot{x})$ and the external force $F(B,t)$ are continuously two-order differentiable with respect to uncertain parameters, the first- and second-order partial derivatives of nonlinear dynamic response with respect to uncertain parameters will be derived.

Differentiating both sides of Eq. (1) with respect to $b_i$ yields

$$M\frac{\partial \ddot{x}}{\partial b_i} + C\frac{\partial \dot{x}}{\partial b_i} + K\frac{\partial x}{\partial b_i} = \frac{\partial F}{\partial b_i} - \frac{\partial f}{\partial b_i} \tag{27}$$

where $M$, $C$, $K$ are the mass matrix, the damping matrix and the stiffness matrix, respectively, and they are defined by

$$M = \frac{\partial f}{\partial \ddot{x}^{\mathrm{T}}}, \quad C = \frac{\partial f}{\partial \dot{x}^{\mathrm{T}}}, \quad K = \frac{\partial f}{\partial x^{\mathrm{T}}} \tag{28}$$

Differentiating both sides of Eq. (27) with respect to $b_j$ leads to

$$M\frac{\partial^2 \ddot{x}}{\partial b_i \partial b_j} + C\frac{\partial^2 \dot{x}}{\partial b_i \partial b_j} + K\frac{\partial^2 x}{\partial b_i \partial b_j} = \frac{\partial^2 F}{\partial b_i \partial b_j} - \frac{\partial^2 f}{\partial b_i \partial b_j} - \frac{\partial M}{\partial b_j}\frac{\partial \ddot{x}}{\partial b_i} - \frac{\partial C}{\partial b_j}\frac{\partial \dot{x}}{\partial b_i} - \frac{\partial K}{\partial b_j}\frac{\partial x}{\partial b_i} \tag{29}$$

Substituting $B = B^c$ into Eq. (27) and Eq. (29) and using *Newmark-β* method, we can obtain the first-order partial derivative $\partial x(B^c, t)/\partial b_i$ and second-order partial derivative $\partial^2 x(B^c, t)/\partial b_i \partial b_j$.

Generally, in nonlinear dynamic structural analysis the response function may only be formulated implicitly so that the first- and second-order partial derivatives must be determined numerically. Under these circumstances, we often resort to the difference approximation of partial derivatives. So this will lead to a high computational effort especially in the case of high dimensional input spaces, which may be one shortcoming of the presented method but be unavoidable usually.

## 5. Probabilistic analysis method

When the uncertain parameters are considered to be random variables, the probabilistic analysis method can be used to determine the range of nonlinear dynamic response of structures. Here only the simplified introduction for this method is given. The detailed procedures can be found in Ref. [8].

If $B = (b_i)$ is random variable vector, then the structure response $x(B,t)$ is also random. The mean value of the random variable vector $B = (b_i)$ is

$$E(B) = (E(b_i)) = B^E = ((b^E)_i) \tag{30}$$

So Eq. (15) can be viewed as the Taylor series expansion of the random response $x(B^E + \delta B, t)$ about the mean value $B^E$.

The mean value or expected value of the dynamic response can be obtained by taking the expected value of both sides of Eq. (15). In so doing, it follows that

$$\begin{aligned} E(x_k(B,t)) &= E(x_k(B^E,t)) + E\left(\sum_{i=1}^{m}\frac{\partial x_k(B^E,t)}{\partial b_i}\delta b_i\right) \\ &= x_k(B^E,t) + \sum_{i=1}^{m}\frac{\partial x_k(B^E,t)}{\partial b_i}E(b_i - b_i^E), \quad k = 1,2,\ldots,n \end{aligned} \tag{31}$$

It is noted that the term $E(\delta b_i) = E(b_i - b_i^E) = 0$ is zero, then we obtain

$$E(x_k(B,t)) = x_i(B^E,t) \tag{32}$$

The variance of the nonlinear dynamic response can be obtained in a similar way as follows:

$$\mathrm{Var}(x_k(B,t)) = \sum_{i=1}^{m}\left(\frac{\partial x_k(B^E,t)}{\partial b_i}\right)^2 \mathrm{Var}(b_i) + \sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\partial x_k(B^E,t)}{\partial b_i}\frac{\partial x_k(B^E,t)}{\partial b_j}\mathrm{Cov}(b_i,b_j) \tag{33}$$

where $\mathrm{Cov}(b_i,b_j)$ is the covariance of the random structure parameter variables. When the random structural parameter variables are independent, the variance of the dynamic response can be reduced as

$$\mathrm{Var}(x_k(B,t)) = \sum_{i=1}^{m}\left(\frac{\partial x_k(B^E,t)}{\partial b_i}\right)^2 \mathrm{Var}(b_i) = \sum_{i=1}^{m}\left(\frac{\partial x_k(B^E,t)}{\partial b_i}\sigma_i\right)^2 \tag{34}$$

Obviously, the standard deviation of the nonlinear dynamic response $x_i(B,t)$ is

$$\sigma(x_k(B,t)) = \sqrt{\mathrm{Var}(x_k(B,t))} = \sqrt{\sum_{i=1}^{m}\left(\frac{\partial x_k(B^E,t)}{\partial b_i}\sigma_i\right)^2} \tag{35}$$

According to the Tchebyche's inequality [5], the probabilistic region of $l$ (a positive integer) times standard deviations from the mean value of the nonlinear dynamic response is

$$y_k^I = [\underline{y}_k(B,t), \overline{y}_k(B,t)] = [x_k(B^E,t) - l\sigma(x_k(B,t)), x_k(B^E,t) + l\sigma(x_k(B,t))], \quad k = 1,2,\ldots,n \tag{36}$$

where the probabilistic upper bound and lower bound are, respectively

$$\overline{y}_k(B,t) = x_k(B^E,t) + l\sigma(x_k(B,t)) = x_k(B^E,t) + l\sqrt{\sum_{i=1}^{m}\left(\frac{\partial x_k(B^E,t)}{\partial b_i}\sigma_i\right)^2} \tag{37}$$

and

$$\underline{y}_k(B,t) = x_k(B^c,t) - l\sigma(x_k(B,t)) = x_k(B^E,t) - l\sqrt{\sum_{i=1}^{m}\left(\frac{\partial x_k(B^E,t)}{\partial b_i}\sigma_i\right)^2} \tag{38}$$

So far, from Eqs. (37) and (38), we can obtain the range of the nonlinear dynamic response of structure with uncertain parameters using the probabilistic approach.

## 6. Numerical examples

In order to illustrate the validity of the presented method for the dynamic response analysis of nonlinear vibration system with uncertain parameters, we apply it to two numerical examples, which include a single degree-of-freedom system with nonlinear damping and a 60-bar power pagoda with nonlinear stiffness.

### 6.1. Single degree-of-freedom system with nonlinear damping

Suppose that the nonlinear vibration equation of a single degree-of-freedom system subject to external excitation can be described as

$$\ddot{u} + 2b\sin\dot{u} + u = k\left(\sin\left(2t + \frac{\pi}{2}\right) + \cos\left(10t + \frac{\pi}{3}\right) + \sin\left(20t + \frac{\pi}{4}\right)\right) \tag{39}$$

and the initial condition

$$u(0) = 0.0, \quad \dot{u}(0) = 0.0 \tag{40}$$

Here, it is assumed that the uncertain parameters vector is $B = [b,k]^{\mathrm{T}}$, and the uncertain parameters $b$ and $k$ are normally distributed with a coefficient of variation $\xi = 1\%$. The mean values of these parameters are $b^E = 5.0$, $k^E = 5$, respectively. For comparison, their interval quantities are taken as $b^I = [b^E - l\xi b^I, b^E + l\xi b^I]$ and $k^I = [k^E - l\xi k^E, k^E + l\xi k^E]$, where $l = 3$. The presented method and probabilistic analysis method are used to calculate the regions of the dynamic displacement response $u$. The exact interval solutions of them obtained by using quasi-Newton method will be taken as the benchmarks of comparison. Particularly, the difference is used to approximate the derivative computation, and six re-analysis of structure is needed for this example.

Fig. 1 shows the comparison of time history interval curves between the interval analysis method and the exact solution method, where IU and IL denote the upper bound and lower bound of the interval analysis

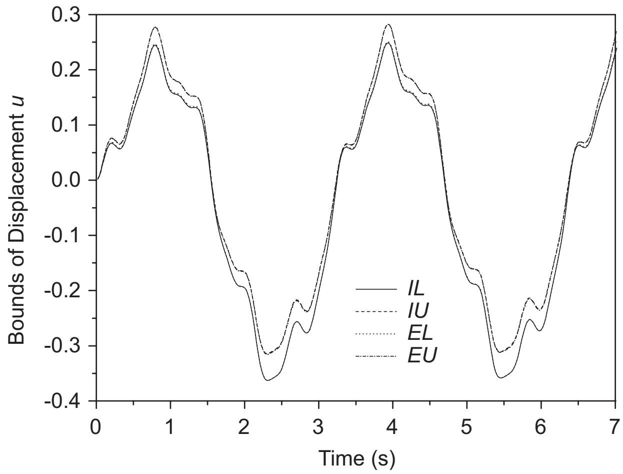


Fig. 1. Comparison of time history curves for the bounds of displacement response $u$ by the interval analysis method and the quasi-Newton method.

Table 1
Comparison of response intervals between the interval analysis method and the exact solution method

| Time | Lower bound | | | Upper bound | | |
|---|---|---|---|---|---|---|
| | Interval | Exact | Error (%) | Interval | Exact | Error (%) |
| 0.00 | 0.0000 | 0.0000 | 0.00 | 0.0000 | 0.0000 | 0.00 |
| 0.36 | 0.0618 | 0.0634 | 2.52 | 0.0720 | 0.0714 | 0.84 |
| 0.72 | 0.2253 | 0.2272 | 0.84 | 0.2546 | 0.2541 | 0.20 |
| 1.08 | 0.1553 | 0.1574 | 1.33 | 0.1803 | 0.1797 | 0.33 |
| 1.44 | 0.1026 | 0.1045 | 1.82 | 0.1206 | 0.1199 | 0.58 |
| 1.80 | −0.1578 | −0.1574 | 0.25 | −0.1356 | −0.1362 | 0.44 |
| 2.16 | −0.2924 | −0.2923 | 0.03 | −0.2543 | −0.2554 | 0.43 |
| 2.52 | −0.3349 | −0.3348 | 0.03 | −0.2864 | −0.2880 | 0.56 |
| 2.88 | −0.2704 | −0.2701 | 0.11 | −0.2311 | −0.2325 | 0.60 |
| 3.24 | −0.0063 | −0.0062 | 1.61 | 0.0027 | 0.0027 | 0.00 |
| 3.60 | 0.1076 | 0.1092 | 1.47 | 0.1220 | 0.1214 | 0.49 |
| 3.96 | 0.2457 | 0.2478 | 0.85 | 0.2802 | 0.2799 | 0.11 |
| 4.32 | 0.1479 | 0.1500 | 1.40 | 0.1721 | 0.1717 | 0.23 |
| 4.68 | 0.0154 | 0.0157 | 1.91 | 0.0233 | 0.0225 | 3.56 |
| 5.04 | −0.1848 | −0.1844 | 0.22 | −0.1585 | −0.1593 | 0.50 |
| 5.40 | −0.3515 | −0.3514 | 0.03 | −0.3061 | −0.3073 | 0.39 |
| 5.76 | −0.2776 | −0.2773 | 0.11 | −0.2348 | −0.2364 | 0.68 |
| 6.12 | −0.2130 | −0.2125 | 0.24 | −0.1790 | −0.1803 | 0.72 |
| 6.48 | 0.0603 | 0.0614 | 1.79 | 0.0649 | 0.0642 | 1.09 |
| 6.84 | 0.1536 | 0.1554 | 1.16 | 0.1745 | 0.1741 | 0.23 |
| 7.20 | 0.2015 | 0.2038 | 1.13 | 0.2341 | 0.2338 | 0.13 |

method; EU and EL denote the upper bound and lower bound of the exact solutions, respectively. It can be seen from Fig. 1 that the upper bounds and the lower bounds yielded by the interval analysis method is almost coincident with the exact solutions. The Fig. 1 only provide a qualitative insight. Errors of the approximate solutions obtained by the presented interval analysis method are given in Table 1. From Table 1, we can find the maximal error is 3.56%, and in most instances the error is 1% more or less, which generally can satisfy the accuracy demanding in practice. Moreover, the presented interval analysis method has less computational efforts than the exact solution method.

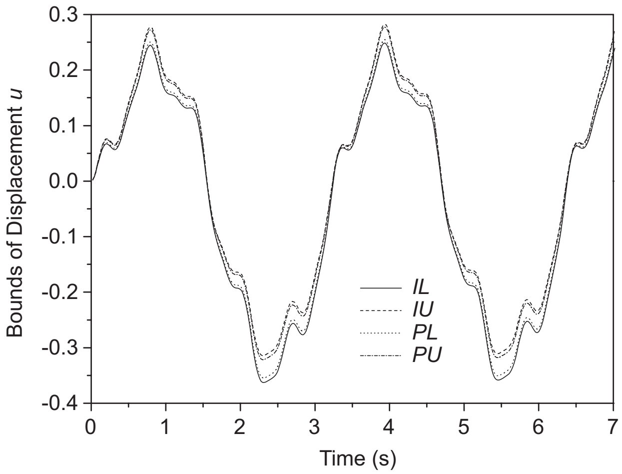


Fig. 2. Comparison of time history curves for the bounds of displacement response $u$ by the interval analysis method and the probabilistic analysis method.

Fig. 2 shows the comparison of time history interval curves between the interval analysis method and the probabilistic analysis method. where IU and IL denote the upper bound and lower bound of the interval analysis method; PU and PL denote the upper bound and lower bound of the probabilistic analysis method, respectively. It can be seen from Fig. 2 that the interval bounds yielded by the interval analysis method is basically in agreement with those calculated by probabilistic analysis method, but the former encloses the latter.

### 6.2. A 60-bar power pagoda with nonlinear stiffness

Consider a 60-bar power pagoda subject to an $x$-directional force $p(t)$ as shown in Fig. 3. Suppose that the stiffness of the system is nonlinear, and the vibration differential equation of system can be represented by

$$M\ddot{x} + Kx + Kx^2 = F(t) \tag{41}$$

with initial condition

$$x(0) = 0, \quad \dot{x}(0) = 0 \tag{42}$$

where $M$ and $K$ are, respectively, the linear mass matrix and stiffness matrix, and $F(t)$ is the external force vector. The cross-sectional areas of the bars are $A = 1.0 \times 10^{-3}\,\mathrm{m}^2$.

The density and Young's moduli of the bars are considered as uncertain parameters, and their intervals are taken as $\rho^I = [\rho^c - \beta\rho^c, \rho^c + \beta\rho^c]$ and $E^I = [E^c - \beta E^c, E^c + \beta E^c]$, where $\rho^c = 7800.0\,\mathrm{kg/m^3}$, $E^c = 2.1 \times 10^{11}\,\mathrm{N/m^2}$ and $\beta$ is a variable coefficient or uncertain degree. Particularly, the difference is used to approximate the derivative computation, and six re-analysis of structure is also needed for this example. Due to large degrees of freedom, the computational efforts are far more than the previous example. Fig. 4 show the time history interval curves of $x$-directional displacement response of Node 21 by the presented interval analysis method when $\beta$ is taken as 0.01.

## 7. Conclusions

In this paper, a non-probabilistic model—interval analysis method is presented to investigate the effects of uncertainties on dynamic response of the nonlinear vibration systems with general form. The uncertain parameters are modeled as interval numbers, and the formulations for the presented interval Taylor series method are derived to approximately estimate the nonlinear dynamic response range. The sensitivity of

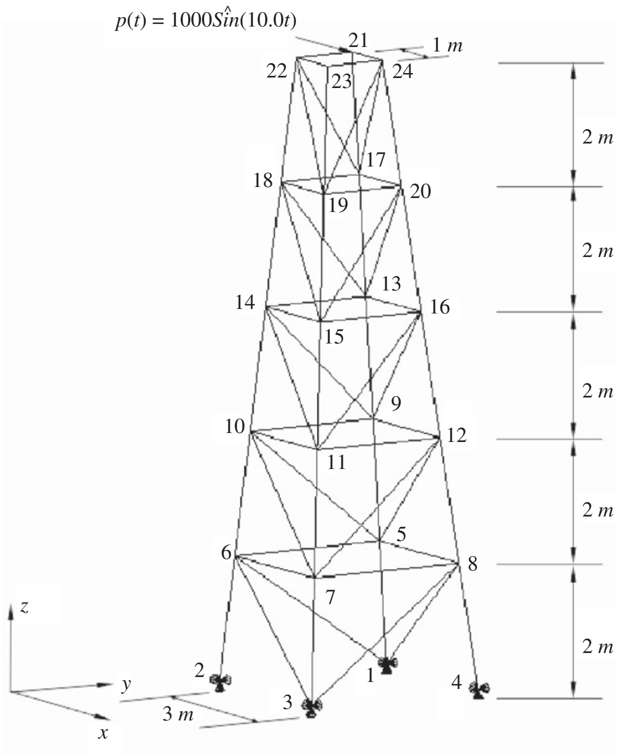


Fig. 3. A 60-bar power pagoda.

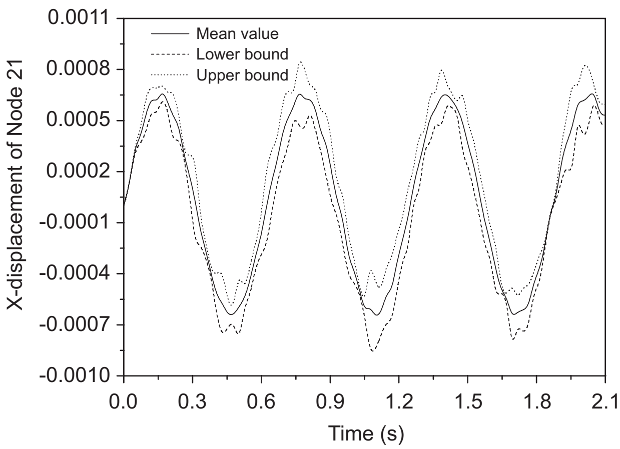


Fig. 4. Time history curves of bounds of $x$-displacement response of Node 21.

nonlinear dynamic response with respect to uncertain parameters is analyzed. Numerical examples including different kinds of nonlinearity are performed to validate the presented method by comparison with the exact solution method and the probabilistic analysis method. The results show the presented method has good accuracy and less computational efforts in comparison with the exact solution method, so the presented method is more suitable for large engineering structures. Furthermore, the numerical results also show the interval bounds obtained by the presented interval analysis method is basically in agreement with those calculated by probabilistic analysis method, but the former encloses the latter.

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