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Response and stability of a SDOF strongly nonlinear stochastic system with light damping modeled by a fractional derivative

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Abstract

A stochastic averaging procedure for a single-degree-of-freedom (SDOF) strongly nonlinear system with light damping modeled by a fractional derivative under Gaussian white noise excitations is developed by using the so-called generalized harmonic functions. The approximate stationary probability density and the largest Lyapunov exponent of the system are obtained from the averaged Itô stochastic differential equation of the system. It is shown that the approximate stationary solutions obtained by using the stochastic averaging procedure agree well with those from the numerical simulation of original systems. The effects of system parameters on the approxiamte stationary probability density and the largest Lyapunov exponent of the system are also discussed.

1. Introduction

The fractional calculus has a history of more than 300 years. It is a subject on derivatives and integrals of arbitrary order. The system model with fractional derivatives is established to describe the dynamic properties of the systems in many fields, e.g., electrochemistry, thermal engineering, acoustics, electromagnetics, mechanics, control and especially viscoelastic materials. Constitutive relations of viscoelastic materials based on a fractional derivative are used to accurately describe the frequency-dependent damping behavior of many materials. Gement [1] first proposed the fractional derivative constitutive model of the viscoelastic material in the 1930s. Adolfsson [2] formulated a fractional order viscoelastic model for large deformations. It was also shown by Bagley and Torvik [3–5] and Koeller [6] that a fractional derivative model can be applied to describe the characteristics of some viscoelastic materials very well. Agrawal [7] presented the application of fractional derivatives in thermal analysis of disk brakes. Deng et al. [8] used the fractional derivative model of dissipative effects to study the response of polyurethane foam in quasi-static compression tests. Depollier et al. [9] established fractional differential equations for the scattering operators deduced from the fractional telegraph equation which described the propagation of transient acoustic signals in a layered porous medium.

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Chen et al. [10] proposed to use the fractional order disturbance observer for vibration suppression applications such as hard disk drive servo control.

Several methods have been developed to study deterministic systems with damping modeled by a fractional derivative, including Laplace transforms [4,5], Fourier transforms [11], numerical methods [12–15], eigenvector expansion [16], the averaging method [17–19] and so on. Spanos and Zeldin [20] proposed a frequency-domain approach for systems with fractional derivatives. Agrawal [21–23] presented an analytical scheme for stochastic dynamic systems with fractional derivatives by using the eigenvector expansion method and the properties of Laplace transforms of convolution integrals. Ye et al. [24] studied the stochastic seismic response of structures with viscoelastic dampers by using the Fourier-transform-based technique and the Duhamel integral-type expression. Agrawal [25] presented an analytical solution for stochastic response of a fractionally damped beam by using the normal-mode approach and the Laplace transform technique. All the methods proposed for the stochastic systems with fractional derivative damping are only applicable to weak nonlinear systems. To the authors' knowledge, there is no analytical procedure for strongly nonlinear stochastic systems with a fractional derivative.

In the present paper, based on the generalized harmonic functions, a stochastic averaging procedure for SDOF strongly nonlinear stochastic dynamic systems with light damping modeled by a fractional derivative is developed. The approximate stationary probability density and the largest Lyapunov exponent of the system are obtained by using the averaged Itô stochastic differential equations. Three examples are given to illustrate the application of the proposed procedure.

2. Generalized harmonic functions

Consider the free vibration of a nonlinear conservative oscillator. The equation of motion of the oscillator is of the form [26,27]

$$\ddot{x} + g(x) = 0 \tag{1}$$

where $g(x)$ is the strongly nonlinear odd function, i.e. $g(-x) = -g(x)$.

The first (energy) integral of the oscillator is

$$\frac{1}{2}\dot{x}^2 + U(x) = H \tag{2}$$

where H is the total energy and

$$U(x) = \int_0^x g(u)\,\mathrm{d}u \tag{3}$$

is the potential energy. Assume that functions $g(x)$ and $U(x)$are such that Eq. (1) has periodic solutions surrounding the origin in domain Ω on the phase plane $(x, \dot{x})$ and the origin is an equilibrium point. The periodic solution of Eq. (1) in Ω can be written as [26]

$$\begin{aligned} x(t) &= a\cos\theta(t) \\ \dot{x}(t) &= -av(a,\theta)\sin\theta(t) \\ \theta(t) &= \phi(t) + \Gamma \end{aligned} \tag{4}$$

where

$$v(a,\theta) = \frac{\mathrm{d}\phi}{\mathrm{d}t} = \sqrt{\frac{2[U(a) - U(a\cos\theta)]}{a^2\sin^2\theta}} \tag{5}$$

in which a is related to H as follows:

$$U(a) = U(-a) = H \tag{6}$$

where $\cos\theta$ and $\sin\theta$ are the so-called generalized harmonic functions [26], a is the amplitude of displacement and $v(a, \theta)$ is the instantaneous frequency of the oscillation. Expand v^{-1} into a Fourier series as follows:

$$v^{-1}(a,\theta) = C_0(a) + \sum_{r=1}^{\infty} C_r(a)\cos r\theta \tag{7}$$

Substituting Eq. (5) into Eq. (7) and integrating Eq. (7) with respect to θ from 0 to 2π lead to the following approximate averaged frequency of the system:

$$\omega(a) = 1/C_0(a) \tag{8}$$

Thus, in averaging we can use the following approximate relation:

$$\theta(t) \approx \omega(a)t + \Gamma \tag{9}$$

3. Stochastic averaging procedure for the system with light damping modeled by a fractional derivative

Consider a strongly nonlinear conservative oscillator subject to light damping containing a fractional derivative and weak Gaussian white noise excitations. The equation of motion is of the form

$$\begin{aligned}&\frac{\mathrm{d}^2}{\mathrm{d}t^2}X(t) + \varepsilon c_1 D^\alpha X(t) + \varepsilon c_2 h\left(\frac{\mathrm{d}X(t)}{\mathrm{d}t}\right) + g(X) = \varepsilon^{1/2} f_k\left(X, \frac{\mathrm{d}X(t)}{\mathrm{d}t}\right)W_k(t)\\ &0<\alpha<1 \text{ or } 1<\alpha<2, \quad k=1,\ldots,m\end{aligned} \tag{10}$$

where ε is a small positive parameter, c_1, c_2 constant coefficients, $h(\mathrm{d}X(t)/\mathrm{d}t)$ a linear or nonlinear function of $\mathrm{d}X(t)/\mathrm{d}t$, $g(X)$ the same as that in Eq. (1), $\varepsilon^{1/2}f_k(X, \mathrm{d}X(t)/\mathrm{d}t)$ amplitudes of weakly external and (or) parametric excitations and $W_k(t)$ Gaussian white noises with correlation functions

$$E[W_k(t)W_l(t+\tau)] = 2D_{kl}\delta(\tau) \tag{11}$$

$\varepsilon c_1 D^\alpha x(t)$ denotes the fractional derivative damping. There are many definitions for a fractional derivative. In the present paper, the following Riemann–Liouville definition is adopted [28]:

$$D^\alpha X(t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\int_0^t \frac{X(t-\tau)}{\tau^\alpha}\mathrm{d}\tau, & 0<\alpha<1\\ \dfrac{1}{\Gamma(2-\alpha)}\dfrac{\mathrm{d}^2}{\mathrm{d}t^2}\displaystyle\int_0^t \frac{X(t-\tau)}{\tau^{\alpha-1}}\mathrm{d}\tau, & 1<\alpha<2\end{cases} \tag{12}$$

The sample motion of system (10) is nearly periodic and can be written as

$$X_1 = X(t) = A\cos\Theta(t) \tag{13}$$

$$X_2 = \dot{X} = \frac{\mathrm{d}X(t)}{\mathrm{d}t} = -Av(A,\Theta)\sin\Theta(t) \tag{14}$$

where the dot represents the derivative with respect to t and

$$\begin{aligned}\Theta(t) &= \Phi(t) + \Gamma(t)\\ v(A,\Theta) &= \frac{\mathrm{d}\Phi}{\mathrm{d}t} = \sqrt{\frac{2[U(A) - U(A\cos\Theta)]}{A^2\sin^2\Theta}}\\ &= b_0(A) + \sum_{r=1}^{\infty} b_r(A)\cos r\Theta\end{aligned} \tag{15}$$

in which A, Θ, Φ and Γ are random processes. The averaged frequency can be obtained in a manner similar to that in Eq. (8). Differentiating Eq. (13) with respect to t and equating the resulting equation to Eq. (14) yield

$$\dot{A}\cos\Theta - \dot{\Gamma}A\sin\Theta = 0 \tag{16}$$

Differentiating Eq. (14) with respect to t and substituting the resulting equation together with Eqs. (13) and (14) into Eq. (10) lead to

$$\begin{aligned}&\dot{A}\frac{g(A)-g(A\cos\Theta)\cos\Theta}{Av\sin\Theta}+\dot{\Gamma}\frac{g(A\cos\Theta)}{v}\\&=\varepsilon c_1D^{\alpha}(A\cos\Theta)+\varepsilon c_2h(-Av\sin\Theta)-\varepsilon^{1/2}f_k(A\cos\Theta,-Av\sin\Theta)W_k(t)\end{aligned}\tag{17}$$

Solving Eqs. (16) and (17), yields

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}t}&=\varepsilon F_1(A,\Gamma)+\varepsilon^{1/2}G_{1k}(A,\Gamma)W_k(t)\\\frac{\mathrm{d}\Gamma}{\mathrm{d}t}&=\varepsilon F_2(A,\Gamma)+\varepsilon^{1/2}G_{2k}(A,\Gamma)W_k(t)\end{aligned}\tag{18}$$

where

$$F_1=F_{11}+F_{12}\tag{19a}$$

$$F_2=\frac{v\cos\Theta}{g(A)}c_1D^{\alpha}(A\cos\Theta)+\frac{v\cos\Theta}{g(A)}c_2h(-Av\sin\Theta)\tag{19b}$$

$$G_{1k}=\frac{-Av\sin\Theta}{g(A)}f_k(A\cos\Theta,-Av\sin\Theta)\tag{19c}$$

$$G_{2k}=\frac{-v\cos\Theta}{g(A)}f_k(A\cos\Theta,-Av\sin\Theta)\tag{19d}$$

$$F_{11}=\frac{Av\sin\Theta}{g(A)}c_1D^{\alpha}(A\cos\Theta)\tag{19e}$$

$$F_{12}=\frac{Av\sin\Theta}{g(A)}c_2h(-Av\sin\Theta)\tag{19f}$$

According to the Stratonovich–Khasminskii theorem [29], $A(t)$ and $\Gamma(t)$ converge weakly into a two-dimensional diffusion Markov process as $\varepsilon\to0$, at a time interval $0\leqslant t\leqslant T$, where $T\sim\mathrm{O}(\varepsilon^{-1})$. The limiting process can be described by the averaged Itô stochastic differential equations. Because the averaged Itô equation for A is independent of Γ, the limiting process $A(t)$ is a one-dimensional diffusion process governed by

$$\mathrm{d}A=m(A)\mathrm{d}t+\sigma(A)\mathrm{d}B(t)\tag{20}$$

where the drift and diffusion coefficients are

$$\begin{aligned}m(A)&=\varepsilon\left\langle F_{11}+F_{12}+D_{kl}\frac{\partial G_{1k}}{\partial A}G_{1l}+D_{kl}\frac{\partial G_{1k}}{\partial\Gamma}G_{2l}\right\rangle_{\Theta}\\\sigma^2(A)&=\varepsilon\langle2D_{kl}G_{1k}G_{1l}\rangle_{\Theta}\end{aligned}\tag{21}$$

in which $\langle\cdot\rangle_{\Theta}$ represents the averaging with respect to Θ.

Because A and Γ vary slowly with time, the following approximate relation can be obtained by using Eq. (9):

$$\Theta(t-\tau)\approx\Theta(t)-\omega(A)\tau\tag{22}$$

By using the approximate relation (22), the deterministic averaging of the term associated with fractional derivative on the right-hand side of the first equation of Eq. (21) can be simplified as follows:

(a) $0<\alpha<1$

$$\begin{aligned}\langle F_{11}\rangle_\Theta &= \frac{c_1}{g(A)}\lim_{T\to\infty}\frac{1}{T}\int_0^T D^\alpha(A\cos\Theta)\times Av\sin\Theta\,\mathrm{d}t \\ &= \frac{c_1}{g(A)}\lim_{T\to\infty}\frac{1}{T\Gamma(1-\alpha)}\int_0^T Av\sin\Theta\times\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t\frac{X(t-\tau)}{\tau^\alpha}\,\mathrm{d}\tau\right)\mathrm{d}t \\ &= \frac{c_1}{g(A)\Gamma(1-\alpha)}\lim_{T\to\infty}\frac{1}{T}\int_0^T Av\sin\Theta\,\mathrm{d}\left(\int_0^t\frac{X(t-\tau)}{\tau^\alpha}\,\mathrm{d}\tau\right) \\ &= \frac{c_1}{g(A)\Gamma(1-\alpha)}\lim_{T\to\infty}\left\{\frac{1}{T}\left(Av\sin\Theta\int_0^t\frac{X(t-\tau)}{\tau^\alpha}\,\mathrm{d}\tau\right)\Bigg|_0^T-\frac{1}{T}\int_0^T\left(\int_0^t\frac{X(t-\tau)}{\tau^\alpha}\,\mathrm{d}\tau\right)\frac{\mathrm{d}}{\mathrm{d}t}(v\sin\Theta)\,\mathrm{d}t\right\}\end{aligned} \tag{23a}$$

(b) $1<\alpha<2$

$$\begin{aligned}\langle F_{11}\rangle_\Theta &= \frac{c_1}{g(A)}\lim_{T\to\infty}\frac{1}{T}\int_0^T D^\alpha(A\cos\Theta)\times Av\sin\Theta\,\mathrm{d}t \\ &= \frac{c_1}{g(A)}\lim_{T\to\infty}\frac{1}{T\Gamma(2-\alpha)}\int_0^T Av\sin\Theta\times\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_0^t\frac{X(t-\tau)}{\tau^{\alpha-1}}\,\mathrm{d}\tau\right)\mathrm{d}t \\ &= \frac{c_1}{g(A)}\lim_{T\to\infty}\frac{1}{T}\int_0^T Av\sin\Theta\,\mathrm{d}(D^{\alpha-1}X(t)) \\ &= \frac{c_1}{g(A)}\lim_{T\to\infty}\left\{\frac{1}{T}(Av\sin\Theta D^{\alpha-1}X(t))|_0^T-\frac{1}{T}\int_0^T D^{\alpha-1}X(t)\frac{\mathrm{d}}{\mathrm{d}t}(v\sin\Theta)\,\mathrm{d}t\right\} \\ &= \frac{c_1}{g(A)}\lim_{T\to\infty}\left[-\frac{1}{T}\int_0^T D^{\alpha-1}X(t)\frac{\mathrm{d}}{\mathrm{d}t}(v\sin\Theta)\,\mathrm{d}t\right]\end{aligned} \tag{23b}$$

In Eqs. (23), A is treated as a constant in the integration because A varies slowly.

To simplify Eqs. (23), the following asymptotic integrals are used:

$$\begin{aligned}&\int_0^t\frac{\cos(\omega\tau)}{\tau^q}\,\mathrm{d}\tau=\omega^{(q-1)}\int_0^s\frac{\cos(u)}{u^q}\,\mathrm{d}u=\omega^{(q-1)}\left(\Gamma(1-q)\sin\left(\frac{q\pi}{2}\right)+\frac{\sin(s)}{s^q}+O(s^{(-q-1)})\right)\\ &\int_0^t\frac{\sin(\omega\tau)}{\tau^q}\,\mathrm{d}\tau=\omega^{(q-1)}\int_0^s\frac{\sin(u)}{u^q}\,\mathrm{d}u=\omega^{(q-1)}\left(\Gamma(1-q)\cos\left(\frac{q\pi}{2}\right)-\frac{\cos(s)}{s^q}+O(s^{(-q-1)})\right)\\ &\qquad(u=\omega\tau,\ s=\omega t)\end{aligned} \tag{24}$$

Substituting Eq. (24) into Eq. (23a), the first term on the right-hand side of Eq. (23a) is

$$\begin{aligned}&\lim_{T\to\infty}\frac{1}{T}\left(v\sin\Theta\int_0^t\frac{X(t-\tau)}{\tau^\alpha}\,\mathrm{d}\tau\right)\Bigg|_0^T\approx\lim_{T\to\infty}\frac{1}{T}\left(Av\sin\Theta\cos\Theta\int_0^T\frac{\cos(\omega\tau)}{\tau^\alpha}\,\mathrm{d}\tau+Av\sin^2\Theta\int_0^T\frac{\sin(\omega\tau)}{\tau^\alpha}\,\mathrm{d}\tau\right)\\ &\approx\lim_{T\to\infty}\frac{Av\sin\Theta\omega^{(\alpha-1)}}{T}\left(\Gamma(1-\alpha)\sin\left(\Theta+\frac{\alpha\pi}{2}\right)+\frac{\sin(\omega T-\Theta)}{(\omega T)^\alpha}\right)=0\end{aligned} \tag{25}$$

Substituting Eq. (24) into Eq. (23a) and using Eq. (25), Eq. (23a) can be further simplified as follows:

$$\begin{aligned}\langle F_{11}\rangle_\Theta &\approx \frac{-c_1}{g(A)\Gamma(1-\alpha)}\lim_{T\to\infty}\frac{1}{T}\int_0^T Ag(A\cos\Theta)\left[\cos\Theta\int_0^t\frac{\cos\omega(A)\tau}{\tau^\alpha}\,\mathrm{d}\tau+\sin\Theta\int_0^t\frac{\sin\omega(A)\tau}{\tau^\alpha}\,\mathrm{d}\tau\right]\mathrm{d}t\\ &\approx\frac{-A}{g(A)}\times\frac{c_1}{2\pi\omega^{1-\alpha}}\int_0^{2\pi}g(A\cos\Theta)[\cos\Theta\sin(\alpha\pi/2)+\sin\Theta\cos(\alpha\pi/2)]\,\mathrm{d}\Theta\quad(0<\alpha<1)\end{aligned} \tag{26a}$$

Similar to the derivation of $\langle F_{11} \rangle_\Theta$ for $0<\alpha<1$, Eq. (23b) can be reduced to

$$\begin{aligned}\langle F_{11}\rangle_\Theta &\approx \frac{-c_1}{g(A)}\lim_{T\to\infty}\left[\frac{1}{T}\int_0^T D^{\alpha-1}X(t)g(A\cos\Theta)\,\mathrm{d}t\right]\\ &= \frac{-c_1}{g(A)\Gamma(2-\alpha)}\lim_{T\to\infty}\left\{\frac{1}{T}\left(g(A\cos\Theta)\int_0^t \frac{X(t-\tau)}{\tau^{\alpha-1}}\,\mathrm{d}\tau\right)\Bigg|_0^T + \frac{1}{T}\int_0^T\left(\int_0^t \frac{X(t-\tau)}{\tau^{\alpha-1}}\,\mathrm{d}\tau\right)\right.\\ &\quad\left.\times\frac{\mathrm{d}g(X)}{\mathrm{d}X}Av\sin\Theta\,\mathrm{d}t\right\} = \frac{-A^2}{g(A)}\times\frac{c_1}{2\pi\omega^{2-\alpha}}\int_0^{2\pi}\frac{\mathrm{d}g(X)}{\mathrm{d}X}v\sin\Theta[\cos\Theta\sin((\alpha-1)\pi/2)\\ &\quad+\sin\Theta\cos((\alpha-1)\pi/2)]\,\mathrm{d}\Theta\quad(1<\alpha<2)\end{aligned}\tag{26b}$$

To complete the deterministic averaging of the drift and diffusion coefficients in Eq. (21), G_{ik} are expanded into Fourier series with respect to Θ as follows [27]:

$$G_{ik} = G_{ik0}(A) + \sum_{n=1}^{\infty}[G_{ikn}^{(c)}(A)\cos(n\Theta) + G_{ikn}^{(s)}(A)\sin(n\Theta)]\quad i=1,2\tag{27}$$

Substituting Eq. (27) into Eq. (21) and completing the averaging with respect to Θ, one obtains the following explicit expressions for the averaged drift and diffusion coefficients

$$\begin{aligned}m(A) &= \varepsilon\langle F_{11}\rangle_\Theta + \frac{\varepsilon c_2 A}{g(A)}\int_0^{2\pi} v\sin\Theta h(-Av\sin\Theta)\,\mathrm{d}\Theta + \varepsilon D_{kl}\\ &\quad\times\left[\frac{\mathrm{d}G_{1k0}}{\mathrm{d}A}G_{1l0} + \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{\mathrm{d}G_{1kn}^{(c)}}{\mathrm{d}A}G_{1ln}^{(c)} + \frac{\mathrm{d}G_{1kn}^{(s)}}{\mathrm{d}A}G_{1ln}^{(s)} + nG_{1kn}^{(s)}G_{2ln}^{(c)} - nG_{1kn}^{(c)}G_{2ln}^{(s)}\right)\right]\\ \sigma^2(A) &= \varepsilon D_{kl}\left[2G_{1k0}G_{1l0} + \sum_{n=1}^{\infty}\left(G_{1kn}^{(c)}G_{1ln}^{(c)} + G_{1kn}^{(s)}G_{1ln}^{(s)}\right)\right]\end{aligned}\tag{28}$$

The Fokker–Planck–Kolmogorov (FPK) equation associated with Eq. (20) is

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial A}[m(A)p] + \frac{1}{2}\frac{\partial^2}{\partial A^2}[\sigma^2(A)p]\tag{29}$$

The boundary condition for Eq. (29) depends on the domain Ω. If $g(X)$ is a monotonic increasing function for $X>0$, the boundary conditions are $p=$ finite at $A=0$ and $p, \partial p/\partial A \to 0$ as $A\to\infty$. Herein, only these boundary conditions are considered and the stationary solution of Eq. (29) is of the following form:

$$p(A) = \frac{C}{\sigma^2(A)}\exp\left[\int_0^A \frac{2m(u)}{\sigma^2(u)}\,\mathrm{d}u\right]\tag{30}$$

where C is a normalization constant.

The stationary probability density of the system Hamiltonian $H=U(A)$ can be obtained from Eq. (30) as follows:

$$p(H) = p(A)\left|\frac{\mathrm{d}A}{\mathrm{d}H}\right| = \frac{p(A)}{g(A)}\bigg|_{A=U^{-1}(H)}\tag{31}$$

where U^{-1} is the inverse function of U. Then the joint stationary probability density of the generalized displacement and velocity can be obtained as follows:

$$p(x_1,x_2) = p(H)/T(H)\big|_{H=x_2^2/2+U(x_1)}\tag{32}$$

where

$$T(H) = \frac{2\pi}{\omega(A)}\bigg|_{A=U^{-1}(H)}\tag{33}$$

4. Asymptotic stability with probability one

The asymptotic stability with probability one of a stochastic system can be studied by using a Lyapunov function or a Lyapunov exponent. The largest Lyapunov exponent can be used to determine the necessary and sufficient condition for the asymptotic stability with probability one of the trivial solution of a stochastic system. To obtain the largest Lyapunov exponent of the system by using the averaged Itô equation Eq. (20), the new norm defined as the square root of the system Hamiltonian [30] is introduced

$$Y(t) = H^{1/2}(t) \tag{34}$$

The averaged Itô stochastic differential equation for $Y(t)$ can be obtained from Eq. (20) by using the Itô differential rule

$$\mathrm{d}Y = \bar{m}(Y)\,\mathrm{d}t + \bar{\sigma}(Y)\,\mathrm{d}B(t) \tag{35}$$

where

$$\begin{aligned} \bar{m}(Y) &= \frac{1}{2}Y^{-1}g(A)m(A) - \left[\frac{1}{8}Y^{-3}g^2(A) - \frac{Y^{-1}}{4}\left.\frac{\mathrm{d}g(x)}{\mathrm{d}x}\right|_{x=A}\right]\sigma^2(A) \\ \bar{\sigma}(Y) &= \tfrac{1}{2}Y^{-1}g(A)\sigma(A) \end{aligned} \tag{36}$$

in which A is a function of Y because $Y = U^{1/2}(A)$.

The approximate largest Lyapunov exponent can be derived by using Eq. (36) as follows [30]:

$$\lambda = \lim_{t\to\infty}\frac{1}{t}\ln Y(t) = \bar{m}'(0) - (\bar{\sigma}'(0))^2/2 \tag{37}$$

Based on the Oseledec multiplicative ergodic theorem [31], the trivial solution of averaged system (35) is asymptotically stable with probability one if $\lambda<0$ and unstable if $\lambda>0$. The boundary between asymptotically stable and unstable original system (10) is approximately determined by $\lambda = 0$.

5. Numerical simulation of a fractional derivative

The fractional derivative defined in Eq. (12) can be rewritten as follows:

$$\begin{aligned} D^\alpha X(t) &= \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{X(t-\tau)}{\tau^\alpha}\,\mathrm{d}\tau = \frac{1}{\Gamma(1-\alpha)}\left(\frac{X(0)}{t^\alpha} + \int_0^t \frac{\dot{X}(\tau)}{(t-\tau)^\alpha}\,\mathrm{d}\tau\right) \\ &= \frac{1}{\Gamma(1-\alpha)}\left(\frac{X(0)}{t^\alpha} + \int_0^t \frac{\dot{X}(t-\tau)}{\tau^\alpha}\,\mathrm{d}\tau\right) = \frac{1}{\Gamma(1-\alpha)}\frac{X(0)}{t^\alpha} \\ &\quad + \frac{1}{\Gamma(2-\alpha)}\int_0^{t^{1-\alpha}} \dot{X}(t - s^{1/(1-\alpha)})\,\mathrm{d}s \quad (s=\tau^\alpha, 0<\alpha<1) \\ D^\alpha X(t) &= \frac{1}{\Gamma(2-\alpha)}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_0^t \frac{X(t-\tau)}{\tau^{\alpha-1}}\,\mathrm{d}\tau \\ &= \frac{1}{\Gamma(2-\alpha)}\left[\frac{(1-\alpha)X(0)}{t^\alpha} + \frac{\dot{X}(0)}{t^{\alpha-1}} + \int_0^t \frac{\ddot{X}(t-\tau)}{\tau^{\alpha-1}}\,\mathrm{d}\tau\right] \quad (1<\alpha<2) \end{aligned} \tag{38}$$

By using the approximate initial relation $D^\alpha X(0) = \dot{X}_0$ as defined in Ref. [14], the following numerical integration of fractional derivative is adopted:

$$\begin{aligned} D^\alpha X(n\Delta t) &= \frac{1}{\Gamma(1-\alpha)}\frac{X_0}{(n\Delta t)^\alpha} + \frac{1}{\Gamma(2-\alpha)}\sum_{j=1}^n \int_{((j-1)\Delta t)^{1-\alpha}}^{(j\Delta t)^{1-\alpha}} \dot{X}(n\Delta t - s^{1/(1-\alpha)})\,\mathrm{d}s \\ &= \frac{1}{\Gamma(1-\alpha)}\frac{X(0)}{(n\Delta t)^\alpha} + \frac{1}{\Gamma(2-\alpha)}\sum_{j=1}^n \frac{(j\Delta t)^{1-\alpha} - ((j-1)\Delta t)^{1-\alpha}}{2}(\dot{X}_{n-j} + \dot{X}_{n+1-j}) \quad (0<\alpha<1) \end{aligned}$$

$$D^{\alpha}X(n\Delta t) = \frac{1}{\Gamma(1-\alpha)}\frac{X(0)}{(n\Delta t)^{\alpha}} + \frac{\dot{X}(0)}{\Gamma(2-\alpha)(n\Delta t)^{\alpha-1}} + \frac{\sum_{j=1}^{n}[(j)^{2-\alpha} - (j-1)^{2-\alpha}](\dot{X}_{n+1-j} - \dot{X}_{n-j})}{\Gamma(3-\alpha)(\Delta t)^{\alpha-1}} \quad (1<\alpha<2) \tag{39}$$

The samples of Gaussian white noises are generated by using a random number between zero and one [32]. The responses of the original system (10) are calculated by using the fourth-order Runge–Kutta algorithm and Eq. (39).

6. Examples

6.1. Example 1

Consider a linear oscillator subject to nonlinear dampings containing a fractional derivative and Gaussian white noise excitation. The equation of motion is of the form

$$\frac{d^2}{dt^2}X(t) + a_1 D^{\alpha}X(t) + a_2\left(\frac{dX(t)}{dt}\right)^3 + \omega^2 X(t) = W(t) \tag{40}$$

where $W(t)$ is a Gaussian white noise with intensity $2D_1$, ω constant natural frequency and a_1, a_2 light damping coefficients.

The conservative oscillator associated with system (40) is a linear oscillator, the following transformations are adopted:

$$\begin{aligned} g(X) &= \omega^2 X \\ X_1 &= X(t) = A\cos\theta \\ X_2 &= \dot{X}(t) = -A\omega\sin\theta \\ \theta &= \omega t + \phi \end{aligned} \tag{41}$$

Based on the stochastic averaging method described in Section 3, the averaged Itô equation for A is of the form of Eq. (20) with the following averaged drift and diffusion coefficients of the form:

$$\begin{aligned} m(A) &= -\frac{a_1 A\sin(\alpha\pi/2)}{2\omega^{1-\alpha}} - \frac{3a_2\omega^2 A^3}{8} + \frac{D_1}{2\omega^2 A} \\ \sigma^2(A) &= \frac{D_1}{\omega^2} \end{aligned} \tag{42}$$

The stationary solution of the system can be obtained as follows:

$$p(A) = \frac{C\omega^2 A}{D_1}\exp\left[-\frac{a_1\omega^{1+\alpha}\sin(\alpha\pi/2)}{2D_1}A^2 - \frac{3a_2\omega^4}{16D_1}A^4\right] \tag{43}$$

where C is a normalization constant.

According to Eqs. (31) and (32), the joint stationary probability density of displacement and velocity can be derived from Eq. (43) as follows:

$$p(x_1, x_2) = \frac{C\omega}{2\pi D_1}\exp\left[-\frac{a_1\omega^{1+\alpha}\sin(\alpha\pi/2)}{2D_1}(x_1^2 + x_2^2/\omega^2) - \frac{3a_2\omega^4}{16D_1}(x_1^2 + x_2^2/\omega^2)^2\right] \tag{44}$$

The stationary probability density of displacement and the stationary probability density of velocity can be derived from Eq. (44).

The stationary probability density of amplitude, the stationary probability density of displacement and the stationary probability density of velocity are shown in Fig. 1, where the results for different α values are compared. The joint stationary probability density of displacement and velocity obtained by using a numerical

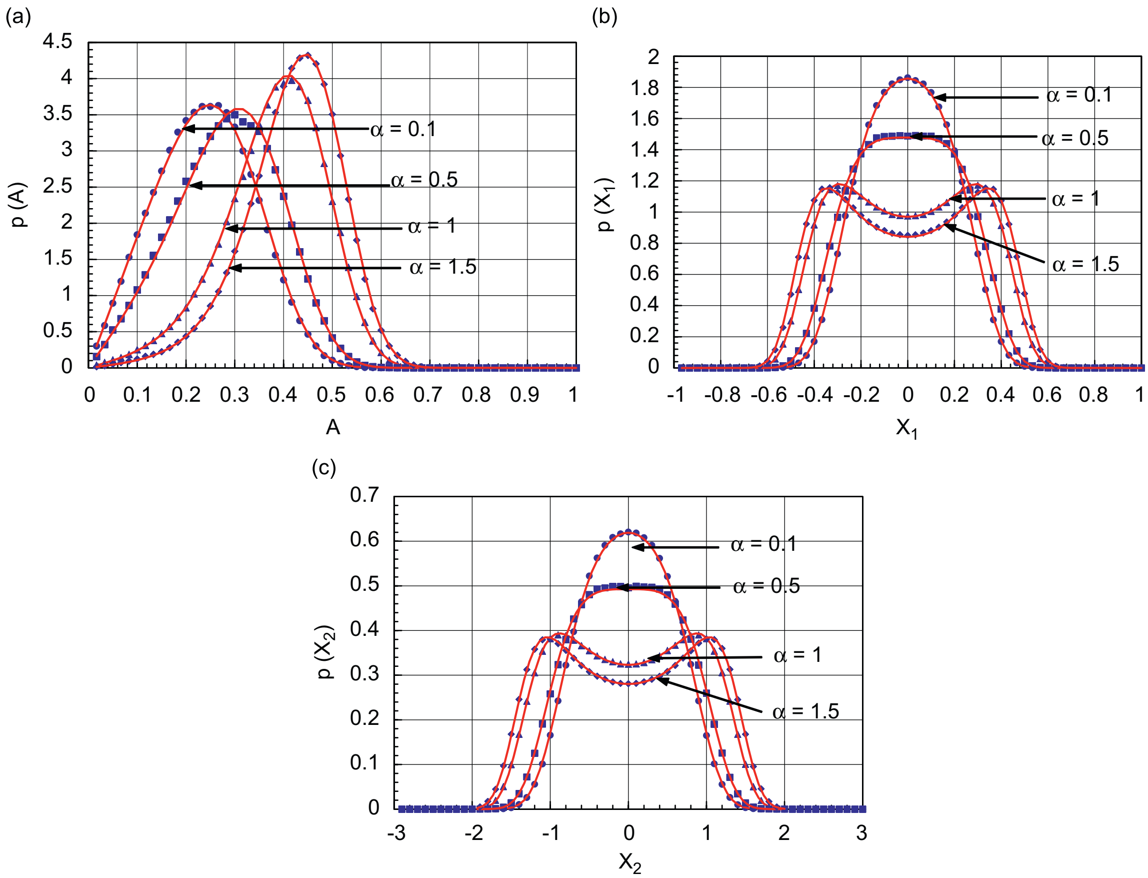

Fig. 1. The stationary probability densities of amplitude A, displacement $X_1 = X$ and velocity $X_2 = \dot{X}$ of system (40) with $a_1 = -0.05$, $a_2 = 0.05$, $\omega = 3.0$, $D_1 = 0.01$. ———, From stochastic averaging method; ● ■ ▲ ♦, from numerical simulation of original system (40).

simulation of original system (40) and by using the averaging method for the case $\alpha = 3/2$ are shown in Fig. 2. It can be seen from Figs. 1(b and c) that the stationary response of the system is close to a Gaussian distribution when the fractional order value is small, while it is a diffusive limit cycle when the value of α is large enough. It can also be observed from Figs. 1 and 2 that the results from analytical solutions by using the proposed stochastic averaging procedure agree well with those from the numerical simulation of original system (40), and the order of fractional derivative of the damping plays an important role in the stationary response of the system. In the simulation of original system (40), the time step is 0.01 s, the number of samples is 200 and the duration of time is 1600s.

6.2. Example 2

Consider a stochastic system with strongly nonlinear stiffness and damping modeled by a fractional derivative. The equation of motion is of the form

$$\frac{d^2}{dt^2}X(t) + cD^{\alpha}X(t) + \omega_0^2 X + kX^3 = W(t) \tag{45}$$

where $W(t)$ is a Gaussian white noise with intensity $2D_1$, ω_0 a linear stiffness coefficient and k a positive nonlinear stiffness coefficient. The damping coefficient c and the excitation intensity $2D_1$ are small parameters.

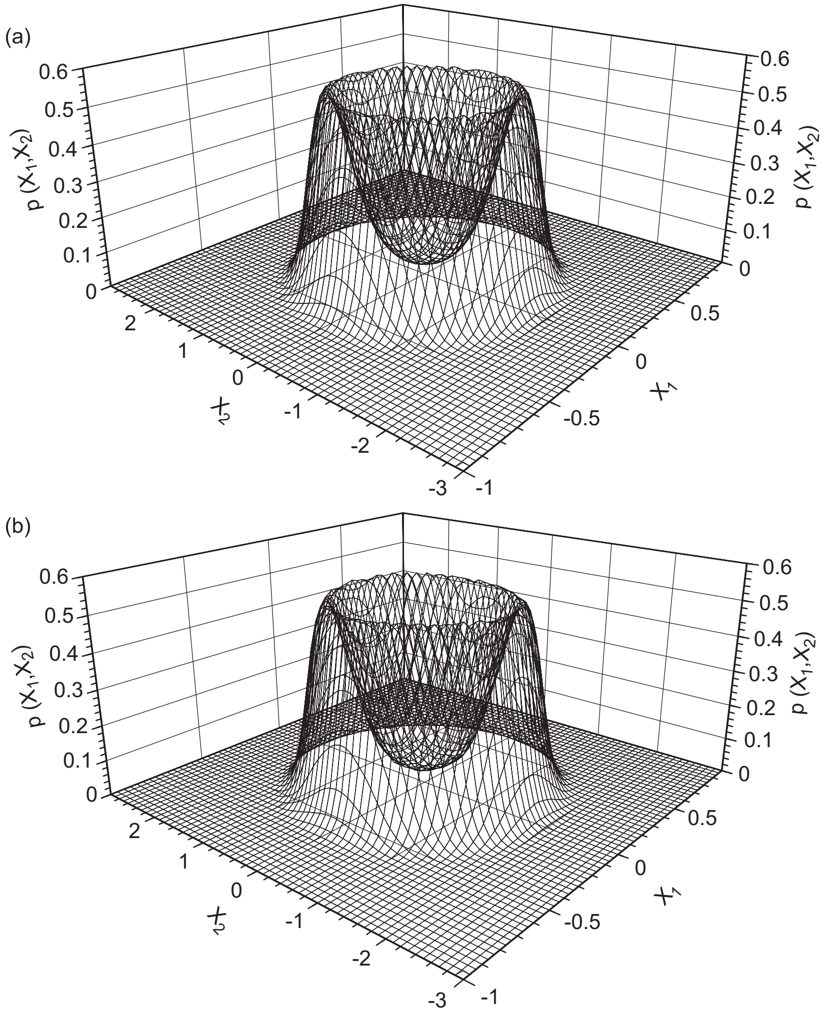

Fig. 2. The joint stationary probability density of displacement and velocity of system (40) with $a_1 = -0.05$, $a_2 = 0.05$, $\omega = 3.0$, $D_1 = 0.01$, $\alpha = 1.5$.

The following transformations are adopted:

$$
\begin{aligned}
X_1 &= X(t) = A\cos\Theta(t) \\
X_2 &= \dot{X}(t) = -Av(A,\Theta)\sin\Theta(t) \\
\Theta(t) &= \Phi(t) + \Gamma(t) \\
U(X_1) &= \omega_0^2 X_1^2/2 + kX_1^4/4
\end{aligned}
\tag{46}
$$

where

$$
\begin{aligned}
v(A,\Theta) &= [(\omega_0^2 + 3kA^2/4)(1+\eta\cos 2\Theta)]^{1/2} \approx b_0(A) + b_2(A)\cos 2\Theta + b_4(A)\cos 4\Theta + b_6(A)\cos 6\Theta \\
\eta &= kA^2/(4\omega_0^2 + 3kA^2)
\end{aligned}
\tag{47}
$$

in which

$$
\begin{aligned}
b_0(A) &= (\omega_0^2 + 3kA^2/4)^{1/2}(1-\eta^2/16) \\
b_2(A) &= (\omega_0^2 + 3kA^2/4)^{1/2}(\eta/2 + 3\eta^3/64)
\end{aligned}
$$

$$b_4(A) = (\omega_0^2 + 3kA^2/4)^{1/2}(-\eta^2/16)$$
$$b_6(A) = (\omega_0^2 + 3kA^2/4)^{1/2}(\eta^3/64) \tag{48}$$

According to the stochastic averaging method described in Section 3, the averaged Itô equation for the amplitude A has the form of Eq. (20). The averaged drift and diffusion coefficients are expressed as follows:

$$m(A) = \begin{cases} D_1\dfrac{8\omega_0^4 + 3\omega_0^2 kA^2 + (kA^2)^2}{16(\omega_0^2 + kA^2)^3 A} - \dfrac{c(\omega_0^2 A + 3kA^3/4)\sin(\alpha\pi/2)}{2(\omega_0^2 + kA^2)\omega^{1-\alpha}}, & 0<\alpha<1 \\ D_1\dfrac{8\omega_0^4 + 3\omega_0^2 kA^2 + (kA^2)^2}{16(\omega_0^2 + kA^2)^3 A} - \dfrac{cA\sin(\alpha\pi/2)}{2(\omega_0^2 + kA^2)\omega^{2-\alpha}}\left[\left(\omega_0^2 + \dfrac{3kA^2}{4}\right)b_0 - \dfrac{\omega_0^2}{2}b_2 - \dfrac{3kA^2}{8}b_4\right], & 1<\alpha<2 \end{cases}$$

$$\sigma^2(A) = D_1\frac{\omega_0^2 + 5kA^2/8}{(\omega_0^2 + kA^2)^2} \tag{49}$$

where the averaged frequency $\omega(A) = b_0(A)$. Then the stationary solutions of the system can be obtained by using Eqs. (30)–(32).

The stationary probability density of amplitude, the stationary probability density of displacement and the stationary probability density of velocity are shown in Fig. 3, where different orders of fractional derivative are considered. The joint stationary probability densities of displacement and velocity from an analytical solution and from a numerical simulation of original system (45) with $\alpha = 2/3$ are shown in Fig. 4. It can be

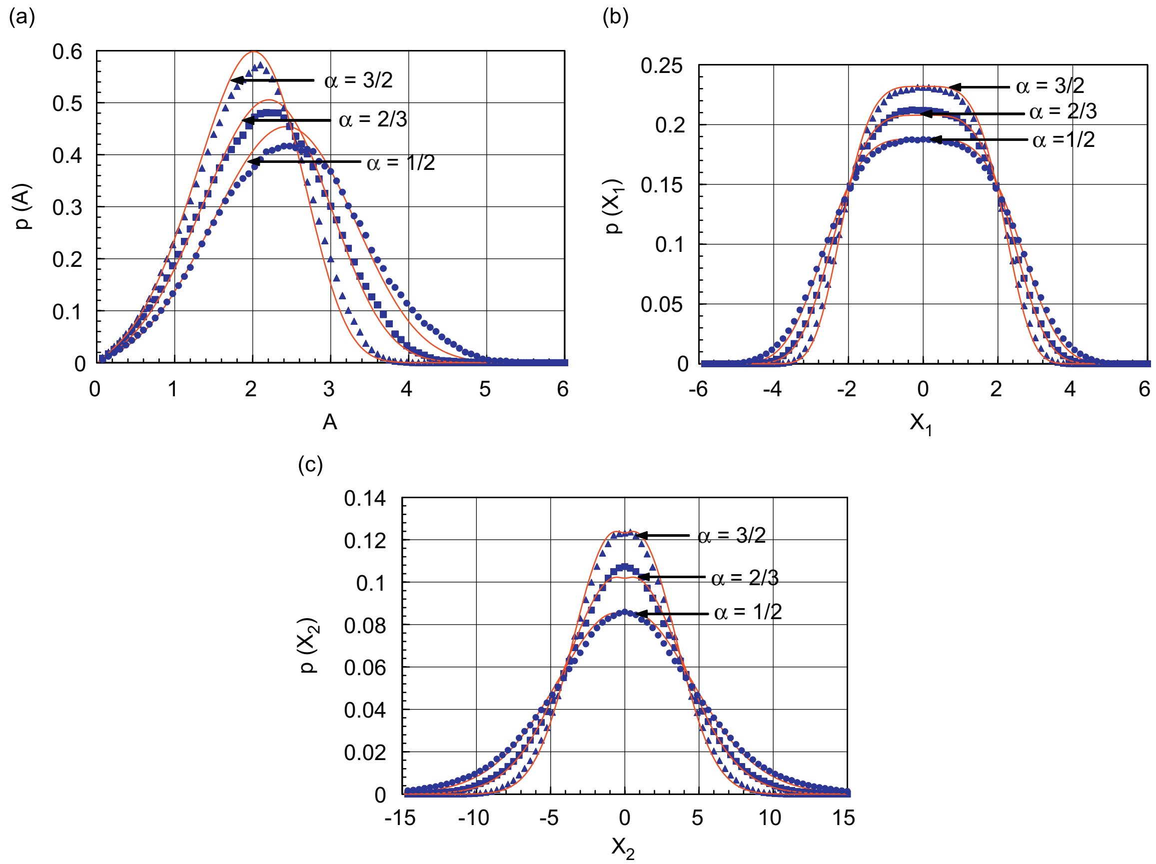

Fig. 3. The stationary probability densities of amplitude A, displacement $X_1 = X$ and velocity $X_2 = \dot{X}$ of system (45) with $c = 0.01$, $\omega_0 = 0.75$, $k = 1$, $D_1 = 0.1$. ———, From the stochastic averaging method; ● ■ ▲, from numerical simulation of original system Eq. (45).

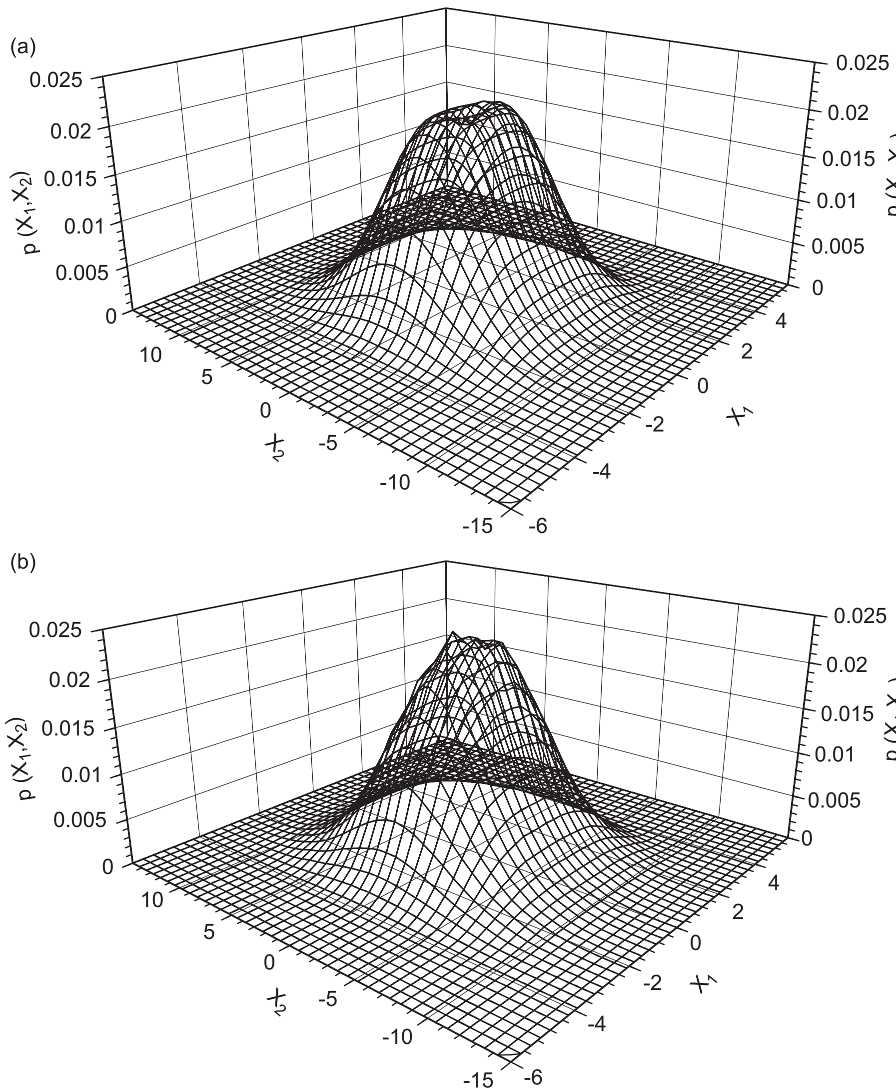

Fig. 4. The joint stationary probability density of displacement and velocity of system (45) with $c = 0.01$, $\omega_0 = 0.75$, $k = 1$, $D_1 = 0.1$, $\alpha = 2/3$.

observed from Figs. 3 and 4 that the results obtained by using the stochastic averaging method agree with those from the numerical simulation of original system (45). In the simulation of original system (45), the time step is 0.01 s, the number of samples is 300 and the duration of time is 2400 s.

6.3. Example 3

Consider the asymptotic stability with probability one of a SDOF system subject to the parametric excitation of Gaussian white noise and damping modeled by a fractional derivative. The equation of motion is of the form

$$\frac{d^2}{dt^2}x(t) + cD^{1/2}x(t) + \omega_0^2 x(t) + kx^3(t) = \dot{x}W(t) \tag{50}$$

where $W(t)$ is a Gaussian white noise with intensity $2D_1$, c the small damping coefficient, ω_0 a linear stiffness coefficient and k a positive nonlinear stiffness coefficient.

According to the stochastic averaging procedure described in Section 3, the averaged Itô equation for the amplitude A is of the form of Eq. (20) with the following averaged drift and diffusion coefficients:

$$\begin{aligned} m(A) &= -\frac{c(\omega_0^2 A + 3/4kA^3)}{2\sqrt{2}\omega(\omega_0^2 + kA^2)} + \frac{D_1 A(5\omega_0^6/8 + 7\omega_0^4 kA^2/16 + 413\omega_0^2(kA^2)^2/512 + 119(kA^2)^3/512)}{(\omega_0^2 + kA^2)^3} \\ \sigma^2(A) &= \frac{D_1 A^2(3\omega_0^4/4 + 7\omega_0^2 kA^2/8 + 67(kA^2)^2/256)}{(\omega_0^2 + kA^2)^2} \end{aligned} \tag{51}$$

where $\omega(A)$ is the same as that in Eq. (49).

Introducing the new norm in Eq. (34), the drift and diffusion coefficients for $Y(t)$ in Eq. (36) can be obtained from Eq. (51) as follows:

$$\begin{aligned} \bar{m}(Y) &= \left\{ \frac{(\omega_0^2 A + kA^3)}{[2(\omega_0^2 A^2 + kA^4/2)]^{1/2}} m(A) + \frac{\sigma^2(A)}{[2(\omega_0^2 A^2 + kA^4/2)]^{3/2}} \right. \\ &\quad \left. \times [(\omega_0^2 + 3kA^2)(\omega_0^2 A^2 + kA^4/2) - (\omega_0^2 A + kA^3)^2] \right\} \Big|_{A=A(Y)} \\ \bar{\sigma}^2(Y) &= \frac{(\omega_0^2 A + kA^3)^2}{2(\omega_0^2 A^2 + kA^4/2)} \sigma^2(A) \Bigg|_{A=A(Y)} \end{aligned} \tag{52}$$

where $A(Y)$ is the inverse function of $Y = (\omega_0^2/2A^2 + k/4A^4)^{1/2}$.

Case 1: $\omega_0 > 0$.

In this case, the asymptotic expressions for drift and diffusion coefficients as $Y \to 0$ can be obtained from Eq. (52) as follows:

$$\begin{aligned} \bar{m}'(0) &= -\frac{c}{2\sqrt{2}\omega_0} + \frac{5D_1}{8} \\ \bar{\sigma}'(0) &= \frac{\sqrt{3D_1}}{2} \end{aligned} \tag{53}$$

Thus, the largest Lyapunov exponent can be obtained by using Eq. (37) as follows:

$$\lambda = \bar{m}'(0) - (\bar{\sigma}'(0))^2/2 = -\frac{c}{2\sqrt{2}\omega_0} + \frac{D_1}{4} \tag{54}$$

The approximate necessary and sufficient condition for the asymptotic stability with probability one of the trivial solution (0,0) of system (50) is

$$c > \sqrt{2}\omega_0 D_1/2 \tag{55}$$

Case 2: $\omega_0 = 0$.

In this case, the asymptotic expressions for drift and diffusion coefficients as $Y \to 0$ can be obtained from Eq. (52) as follows:

$$\begin{aligned} \bar{m}(Y) &= -\frac{9cY^{3/4}}{6^{1/4}\sqrt{143}k^{1/8}} + \frac{253D_1}{256} Y \\ \bar{\sigma}^2(Y) &= \frac{67}{64} D_1 Y^2 \end{aligned} \tag{56}$$

The largest Lyapunov exponent can be obtained by using Eq. (37) as follows:

$$\lambda = \bar{m}'(0) - (\bar{\sigma}'(0))^2/2 = \begin{cases} -\infty, & c > 0 \\ +\infty, & c < 0 \end{cases} \tag{57}$$

Thus, the system is asymptotically stable with probability one when $c > 0$ and unstable when $c < 0$.

7. Concluding remarks

The stochastic averaging procedure for a SDOF strongly nonlinear oscillator subject to dampings modeled by a fractional derivative and Gaussian white noise excitations has been developed by using generalized harmonic functions. The original system was reduced to a one-dimensional Markov process for amplitude A by using the proposed stochastic averaging method. Then, the approximate stationary response and the largest Lyapunov exponent of the system are obtained by using the averaged Itô stochastic differential equation. The stationary probability densities obtained by using the stochastic averaging method agree well with those from the numerical simulation of the original systems for two examples. The results showed that the order of fractional derivative plays an important role in the responses of the systems. The largest Lyapunov exponent and stochastic stability with probability one of a nonlinear conservative oscillator subject to light damping modeled by a fractional derivative and stochastic parametric excitation depend on the value of the stiffness coefficient. It should be pointed out that the response and stability of a multi-degree-of-freedom stochastic system with dampings modeled by a fractional derivative can also be studied in a similar way.

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