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Vibration analysis of a continuous system subject to generic forms of actuation forces and sensing devices

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Abstract

This work provides a general formulation to solve vibration problems for continuous systems with damping effects, including modal, transient, harmonic and spectrum response analyses. In modal analysis, the system eigenvalues and corresponding eigenfunctions can be determined. The orthogonal relations of eigenfunctions are shown. For transient, harmonic and spectrum analyses, the generic force/actuator functions and response/sensing operators are introduced, respectively, and used to derive the system response. The time domain response is obtained for transient analysis, the frequency response function is derived for harmonic analysis and statistical quantities of response variables due to random excitation are determined in spectrum analysis. The solution for each type of analysis can be formulated and expressed in a concise format in terms of generic force/actuator and response/sensor mode shape functions. In particular, one-dimensional beam and two-dimensional plate vibration analyses are illustrated by following the developed generic formulation. This work provides the complete analytical solutions of four types of vibration analyses for continuous systems and can be applied to other engineering structures as well.

1. Introduction

Mechanical or structural vibration problems are of great interest and importance in engineering design. Many literatures and vibration textbooks [1–10] address the analytical and numerical approximation methods to deal with both discrete (lumped parameter) systems and continuous (distributed parameter) systems, respectively. Discrete systems include both single-degree-of-freedom (sdof) and multi-degree-of-freedom (mdof) systems whose equations of motion are an ordinary differential equation (ODE) and a coupled set of ODEs, respectively. Continuous systems deal with boundary-value problems, such as strings, bars, shafts, beams, membranes and plates, whose equations of motion are partial differential equations (PDEs).

In studying vibration problems for engineering applications, four types of vibration analyses can be categorized as follows:

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1. Modal analysis or the so-called free vibration analysis is to determine the structural natural modes, including natural frequencies and their corresponding mode shapes. Most engineering problems may only require knowledge of structural modes to avoid resonance and for structural design. Many literatures deal with analytical, semi-analytical or numerical solutions on various kinds of structures, such as beams [11–17] and plates [18–25]. The orthogonal properties of structural mode shapes are also of interest for the following types of vibration analyses.
2. Transient response analysis deals with the solution of a time domain transient response due to initial excitation and external excitation. In analytical solutions, the modal domain approach is generally adopted [26–28]. The original equation of motion for a continuous system expressed as a PDE in the physical domain can be transformed into an infinite set of ODEs in the modal domain by the theoretical modal analysis (TMA) procedure with the employment of orthogonal properties. The modal coordinate responses can then be solved for different types of damping conditions. If the excitations are deterministic, the analytical solution can be determined and is of interest.
3. Harmonic response analysis or steady-state response analysis [29–32] is used to solve system frequency response functions (FRFs) that can be defined as the output response divided by the input excitation due to harmonic or sinusoidal excitation. The steady-state response is of interest and the frequency-dependent FRF is useful to evaluate the system response for sinusoidal excitation as well as periodic excitation. In experimental modal analysis or modal testing, the FRFs can be measured and processed to determine structural modal parameters [33].
4. Spectrum response analysis is mainly for systems subject to random excitations. Transient response analysis in the time domain is generally infeasible, and frequency domain analysis techniques are employed. The random inputs of external excitation are generally represented by power spectral density (PSD) functions, and therefore the PSD functions of structural random response can then be obtained so as to determine the system response in terms of statistical quantities [34–41].

There are rigorous literatures and textbooks dealing with the four types of vibration analyses for continuous systems. Meirovitch [1,6] formulated a general approach for the differential eigenvalue problems so as to apply to different types of continuous systems. The normal mode analysis for solving structural natural modes and the transient solution analysis by the TMA approach were treated. However, the harmonic and spectrum response analyses were not provided. Soedel [10] provided a presentation in the transient and harmonic response analyses by the modal approach, in particular for plates and shells. Gardonio and Brennan [42] showed a similar formulation for different mobility equations. Wang [43] is the main reference for which improvements are given and presents the theoretical formulation of generic FRFs for continuous systems associated with various forms of actuation and sensing methods, particularly for harmonic response analysis. The generic actuator and sensor eigenfunctions (mode shape functions) are identified and interpreted physically. Although the spectrum response analysis for continuous system has been widely discussed [44–46], the solution can be dependent. This work develops a systematic approach in spectrum response analysis to obtain the statistical quantities, i.e. the mean and standard deviation of the system response.

This work basically expands Wang's work [43], regarding the harmonic response analysis only, to the transient and spectrum response analyses. In this work, the generic solutions of transient, harmonic and spectrum response analyses by the modal approach are completely shown by a consistently systematic formulation and are applicable to an arbitrary continuous structure. The solutions are shown as a concise format by generic force/actuator and response/sensor mode shape functions, respectively. This work enhances the mathematical formulation in analytically solving the vibration problems for the four types of vibration analyses.

For modal analysis, the solutions of natural modes for many structures with different boundaries can be found in handbooks [47–49]. This work defines the orthonormal eigenfunctions or displacement mode shapes with respect to the mass distribution function and summarizes the orthogonal properties for later use in transient, harmonic and spectrum analyses. The generic force/actuator mode shape functions and response/sensor mode shape functions are introduced, respectively. The system response for each type of analysis can be derived and expressed in a general format in terms of generic force/actuator mode shape functions and response/sensor mode shape functions. Section 2 presents the four types of vibration analyses for differential eigenvalue problems of self-adjoint systems. The analytical approaches can account for different structures, boundary conditions, excitation forms and sensing devices or response variables of interest. Sections 3 and 4 show the lateral vibration of one-dimensional beam and two-dimensional plate problems, respectively, by following the generic formulation in Section 2 to analytically solve for the four types of vibration analyses.

The developed systematic formulation can provide a simple approach to deal with the mathematical derivation of presented system response for complex combinations of different types of structures, boundary conditions and forcing functions. The formulation is straightforward and advantageous in numerical programming for solution. The presented general formulation of analytical approaches can also be applied to other structures, such as strings, bars, shafts and membranes as well.

2. Theoretical analysis

The PDE describing the motion of a continuous system over domain D can be expressed as follows [6]:

$$L[w(P,t)] + \frac{\partial}{\partial t} C[w(P,t)] + M(P)\frac{\partial^2 w(P,t)}{\partial t^2} = f(P,t) \tag{1}$$

where L and C are linear homogeneous self-adjoint differential operators consisting of derivatives through order $2p$ with respect to the spatial coordinates P but not with respect to time t, containing the information concerning the stiffness and damping functions. $M(P)$ is the mass distribution function of the system. $f(P, t)$ is the general force function. For simplicity, the boundary conditions are assumed to be homogeneous so that at every point on the boundaries of domain D the boundary conditions must be satisfied:

$$B_i[w(P,t)] = 0, \quad i = 1, 2, \ldots, p \tag{2}$$

B_i is the linear homogeneous differential operator containing derivatives normal to the boundary and along the boundary of order through $2p-1$.

The system initial conditions can be specified as follows:

$$w(P,0) = w_0(P) \tag{3}$$

$$\dot{w}(P,0) = \dot{w}_0(P) \tag{4}$$

where $w_0(P)$ and $\dot{w}_0(P)$ are the initial displacement and velocity, respectively.

2.1. Modal analysis

2.1.1. Eigenproblem analysis

With normal modes analysis, the eigenvalue problem associated with the homogeneous undamped system can be shown to be [6]

$$L[w] = \lambda M w = \omega^2 M w \tag{5}$$

The above equation must be satisfied over domain D, and w is subject to the boundary conditions as shown in Eq. (2). The eigenvalue problem can be solved. An infinite set of natural frequencies ω_r and their corresponding eigenfunctions $w_r(P)$ can then be obtained. And, the orthonormal eigenfunction or displacement mode shape function with respect to the mass distribution function $M(P)$ can be determined and is unique as follows:

$$\phi_r(P) = \frac{w_r(P)}{\sqrt{\int_D M(P)[w_r(P)]^2 \, \mathrm{d}D(P)}} \tag{6}$$

2.1.2. Orthogonality of eigenfunctions

If the eigenfunctions are orthonormal as defined in Eq. (6), then

$$\int_D M(P)\phi_r(P)\phi_s(P)\,\mathrm{d}D(P) = \delta_{rs} \tag{7}$$

$$\int_D \phi_r(P)L[\phi_s(P)]\,\mathrm{d}D(P) = \omega_r^2\delta_{rs} \tag{8}$$

where δ_{rs} is the Kronecker delta.

If the Rayleigh proportional damping is assumed and maintains the following relationship:

$$C = \alpha M + \beta L \tag{9}$$

where α and β are some constants, by recalling Eqs. (7) and (8), the orthogonal properties of eigenfunctions with respect to the damping can be shown to be as follows:

$$\begin{aligned}\int_D \phi_r(P)C[\phi_s(P)]\,\mathrm{d}D(P) &= \int_D \phi_r(P)(\alpha M + \beta L)[\phi_s(P)]\,\mathrm{d}D(P)\\ &= (\alpha + \beta\omega_r^2)\delta_{rs}\\ &= 2\zeta_r\omega_r\delta_{rs}\end{aligned} \tag{10}$$

where ζ_r is the rth modal damping ratio and dependent on the natural frequency as follows:

$$\zeta_r = \frac{\alpha}{2\omega_r} + \frac{\beta\omega_r}{2} \tag{11}$$

2.2. Transient response analysis

The force function for the generic form of actuation force applied at P_j can be expressed by

$$f(P_j, t) = f_j(t)\Gamma(P_j) \tag{12}$$

where $f_j(t)$ and $\Gamma(P_j)$ are the temporal function and the spatial function of the jth generic actuation force, respectively. From the expansion theorem, the displacement response can be assumed to be as follows:

$$w(P,t) = \sum_{r=1}^{\infty} \phi_r(P)q_r(t) \tag{13}$$

By the substitution of Eqs. (12) and (13) into Eq. (1), Eq. (1) is then multiplied by $\phi_s(P)$ and integrated over domain D as follows:

$$\begin{aligned}q_r(t)\Bigg\{&\sum_{r=1}^{\infty}\int_D \phi_s(P)L[\phi_r(P)]\,\mathrm{d}D(P) + \dot{q}_r(t)\sum_{r=1}^{\infty}\int_D \phi_s(P)C[\phi_r(P)]\,\mathrm{d}D(P)\\ &+\ddot{q}_r(t)\sum_{r=1}^{\infty}\int_D \phi_s(P)M(P)\phi_r(P)\,\mathrm{d}D(P)\Bigg\} = f_j(t)\int_D \phi_s(P)\Gamma(P_j)\,\mathrm{d}D(P)\end{aligned} \tag{14}$$

By the substitution of the orthogonal properties of eigenfunctions as shown in Eqs. (7), (8) and (10), the above equation can be reduced to obtain an infinite set of modal domain equations as follows:

$$\ddot{q}_r + 2\zeta_r\omega_r\dot{q}_r + \omega_r^2 q_r = N_r(t), \quad r = 1, 2, \ldots \tag{15}$$

where

$$\begin{aligned}N_r(t) &= f_j(t)\int_D \phi_r(P)\Gamma(P_j)\,\mathrm{d}D(P)\\ &= f_j(t)\phi_r^F(P_j) = f_j(t)\phi_{r,j}^F\end{aligned} \tag{16}$$

where $\phi_r^F(P_j)$ is the generic force mode shape function related to the displacement mode shape function and force spatial function as follows:

$$\phi_r^F(P_j) = \phi_{r,j}^F = \int_D \phi_r(P)\Gamma(P_j)\,\mathrm{d}D(P) \tag{17}$$

It should be noted that the original system equation in the physical domain as shown in Eq. (1) is a PDE expressed by physical parameters. Through the TMA procedure as shown above, the PDE, i.e. the physical domain equation, can be reduced to an infinite set of ODEs, i.e. the modal domain equations, as shown in Eq. (15) expressed by modal parameters. It should also be noted that q_r is termed the modal coordinate [8] in this work. However, there are some other terminologies, such as the generalized coordinate [1], modal participant factor [10] and normal coordinate [9], with the same physical contents.

By imposing the initial condition Eq. (3) in Eq. (13) as follows:

$$w(P,0) = \sum_{r=1}^{\infty} \phi_r(P)q_r(0) = w_0(0) \tag{18}$$

By multiplication of $M(P)\phi_s(P)$ with the above equation and integration over domain D as follows:

$$\int_D M(P)\phi_s(P)\left\{\sum_{r=1}^{\infty} \phi_r(P)q_r(0)\right\}\mathrm{d}D(P) = \int_D M(P)\phi_s(P)\{w_0(0)\}\,\mathrm{d}D(P) \tag{19}$$

By substituting the orthogonal properties of the eigenfunctions into the left-hand side of the above equation, one can obtain the modal coordinate initial condition as follows:

$$q_{r0} = q_r(0) = \int_D M(P)\phi_r(P)w_0(P)\,\mathrm{d}D(P) \tag{20}$$

Similarly,

$$\dot{q}_{r0} = \dot{q}_r(0) = \int_D M(P)\phi_r(P)\dot{w}_0(P)\,\mathrm{d}D(P) \tag{21}$$

Therefore, the modal coordinate response $q_r(t)$ in Eq. (15) can be solved as follows:

$$q_r(t) = q_{r,\mathrm{IC}}(t) + q_{r,\mathrm{IRF}}(t) \tag{22}$$

Soedel [10] showed the solution of the modal coordinate, and Table 1 summarizes the solution of $q_r(t)$ for different damping conditions for completion. The under-damped system, i.e. $0<\xi_r<1$, is shown as follows:

$$q_{r,\mathrm{IC}}(t) = \mathrm{e}^{-\zeta_r\omega_r t}\left[q_{r0}\cos\,\omega_{dr}t + \frac{\dot{q}_{r0}+\zeta_r\omega_r q_{r0}}{\omega_{dr}}\sin\,\omega_{dr}t\right] \tag{23}$$

$$q_{r,\mathrm{IRF}}(t) = \int_0^t N_r(\tau)h_r(t-\tau)\,\mathrm{d}\tau \tag{24}$$

$$\omega_{dr} = \omega_r\sqrt{1-\zeta_r^2} \tag{25}$$

$$h_r(t) = \frac{1}{\omega_{dr}}\mathrm{e}^{-\zeta_r\omega_r t}\sin\,\omega_{dr}t \tag{26}$$

By the substitution of $q_r(t)$ into Eq. (13), the system response can be determined.

$$w(P,t) = \sum_{r=1}^{\infty} \phi_r(P)q_r(t) \tag{27}$$

Table 1
Modal coordinate solution for different damping conditions

$$\ddot{q}_r + 2\zeta_r\omega_r\dot{q}_r + \omega_r^2 q_r = N_r(t),$$
$$q_r(0) = q_{r0},\ \dot{q}_r(0) = \dot{q}_{r0}$$
$$q_r(t) = q_{r,\mathrm{IC}}(t) + q_{r,\mathrm{IRF}}(t)$$

Damping ratio ζ_r	Free vibration response due to initial condition $q_{r,\mathrm{IC}}(t)$	Forced response due to modal force by IRF solution $q_{r,\mathrm{IRF}}(t) = \int_0^t N_r(\tau)h_r(t-\tau)\mathrm{d}\tau$
$\zeta_r = 0$ undamped	$q_{r,\mathrm{IC}}(t) = \left(q_{r0}\cos\omega_r t + \dfrac{\dot{q}_{r0}}{\omega_r}\sin\omega_r t\right)$	$h_r(t) = \dfrac{1}{\omega_r}\sin\omega_r t$
$0<\zeta_r<1$ Under-damped	$q_{r,\mathrm{IC}}(t) = \mathrm{e}^{-\xi_r\omega_r t}\left(q_{r0}\cos\omega_{dr}t + \dfrac{\dot{q}_{r0}+\zeta_r\omega_r q_{r0}}{\omega_{dr}}\sin\omega_{dr}t\right)$ $\omega_{dr} = \omega_r\sqrt{1-\zeta_r^2}$	$h_r(t) = \dfrac{1}{\omega_{dr}}\mathrm{e}^{-\zeta_r\omega_r t}\sin\omega_{dr}t$
$\zeta_r = 1$ Critical-damped	$q_{r,\mathrm{IC}}(t) = [q_{r0} + (\dot{q}_{r0} + \omega_r q_{r0})t]\mathrm{e}^{-\omega_r t}$	$h_r(t) = t\,\mathrm{e}^{-\omega_r t}$
$\zeta_r>1$ Over-damped	$q_{r,\mathrm{IC}}(t) = \mathrm{e}^{-\xi_r\omega_r t}\left(q_{r0}\cosh\varpi_{dr}t + \dfrac{\dot{q}_{r0}+\zeta_r\omega_r q_{r0}}{\bar{\omega}_{dr}}\sinh\varpi_{dr}t\right)$ $\varpi_{dr} = \omega_r\sqrt{\zeta_r^2-1}$	$h_r(t) = \dfrac{1}{\bar{\omega}_{dr}}\mathrm{e}^{-\zeta_r\omega_r t}\sinh\varpi_{dr}t$
$\omega_r = 0$	$\ddot{q}_r = N_r(t)$ if $\omega_r = 0$ $q_{r,\mathrm{IC}}(t) = q_{r0} + \dot{q}_{r0}t$	$h_r(t) = t$

Now, introduce a sensing or a response operator R on $w(P, t)$ to obtain the response variable $s(P, t)$ as follows:

$$s(P,t) = \sum_{r=1}^{\infty} R[\phi_r(P)q_r(t)] \tag{28}$$

The response operator can be related to the spatial and temporal variables. However, R is normally dependent on the spatial variable only. The response variable can then be rewritten as follows:

$$s(P,t) = \sum_{r=1}^{\infty} R[\phi_r(P)]q_r(t) = \sum_{r=1}^{\infty} \phi_r^S(P)q_r(t) \tag{29}$$

where

$$\phi_r^S(P) = R[\phi_r(P)] \tag{30}$$

$\phi_r^S(P)$ is the generic response or the sensor mode shape function. One will see the advantage of the response operator in later analysis as well as for the case studies.

2.3. Harmonic response analysis

Wang [43] has theoretically derived the formulation of generic FRFs for continuous systems. The following derivation is partly taken from Wang [43] and modified to comply with this work's formulation.

2.3.1. Harmonic excitation for generic actuation force

In harmonic response analysis, the major work is to determine the FRFs of the system. For harmonic excitation, the general force function for the generic form of actuation force applied at P_j with magnitude A_j

can be assumed and to be expressed by

$$f(P_j,t) = f_j(t)\Gamma(P_j) = A_j\,\mathrm{e}^{\mathrm{i}\omega t}\Gamma(P_j) \tag{31}$$

where $\Gamma(P_j)$ is the spatial function of the jth generic actuation force and ω is the excitation frequency. The steady-state response will also be harmonic. From the expansion theorem, the displacement response can be assumed to be as follows:

$$\begin{aligned} w(P,t) &= \sum_{r=1}^{\infty} \phi_r(P) q_r(t) \\ &= \sum_{r=1}^{\infty} \phi_r(P) Q_r(\omega) \mathrm{e}^{\mathrm{i}\omega t} \end{aligned} \tag{32}$$

where $Q_r(\omega)$ is the frequency-dependent modal amplitude of the rth mode depending on the form of actuation force. By the substitution of Eqs. (31) and (32) into the system equation in Eq. (1), Eq. (1) is then multiplied by $\phi_s(P)$ and integrated over domain D as follows:

$$\begin{aligned} &Q_r(\omega)\left\{\sum_{r=1}^{\infty}\int_D \phi_s(P)L[\phi_r(P)]\,\mathrm{d}D(P) + \mathrm{i}\omega\sum_{r=1}^{\infty}\int_D \phi_s(P)C[\phi_r(P)]\,\mathrm{d}D(P)\right. \\ &\left.-\omega^2\sum_{r=1}^{\infty}\int_D \phi_s(P)M(P)\phi_r(P)\,\mathrm{d}D(P)\right\} = A_j\int_D \phi_s(P)\Gamma(P_j)\,\mathrm{d}D(P) \end{aligned} \tag{33}$$

Notice that the $\mathrm{e}^{\mathrm{i}\omega t}$ term is canceled out. By the substitution of the orthogonal properties of eigenfunctions as shown in Eqs. (7)–(10), the above equation can be reduced to

$$Q_r(\omega)[\omega_r^2 - \omega^2 + \mathrm{i}2\zeta_r\omega_r\omega] = A_j\int_D \phi_r(P)\Gamma(P_j)\,\mathrm{d}D(P) \tag{34}$$

such that

$$Q_r(\omega) = \frac{A_j\phi_r^F(P_j)}{(\omega_r^2-\omega^2)+\mathrm{i}(2\zeta_r\omega_r\omega)} \tag{35}$$

It should be noted that $\phi_r^F(P_j)$ is the generic force mode shape function as defined in Eq. (17). In experimental modal testing, the force input is applied by actuators, and therefore, $\phi_r^F(P_j)$ can also be termed the generic actuator eigenfunction [43] that is named the generic actuator mode shape function in this work.

2.3.2. Harmonic response for generic sensing device

In practical implementation of sensing devices, accelerometers or other sensors can be applied to measure the structural response at location P_i. For harmonic response, the measured quantity $s(P_i, t)$ can then be defined by a sensing or a response operator R operating on the structural displacement response as follows:

$$s(P_i,t) = R[w(P_i,t)] = R[w(P_i)]\mathrm{e}^{\mathrm{i}\omega t} \tag{36}$$

By the substitution of the displacement response as shown in Eq. (32) into the above equation, the measured quantity can be rewritten as follows:

$$s(P_i,t) = \mathrm{e}^{\mathrm{i}\omega t}\sum_{r=1}^{\infty} Q_r(\omega)R[\phi_r(P_i)] \tag{37}$$

By substituting $Q_r(\omega)$ into Eq. (35) into the above equation, the measured quantity from the sensor at location P_i can be derived as follows:

$$s(P_i,t) = \mathrm{e}^{\mathrm{i}\omega t}\sum_{r=1}^{\infty}\frac{A_j\{R[\phi_r(P_i)]\}\{\int_D \phi_r(P)\Gamma(P_j)\,\mathrm{d}D(P)\}}{(\omega_r^2-\omega^2)+\mathrm{i}(2\zeta_r\omega_r\omega)} = S(P_i,\omega)\mathrm{e}^{\mathrm{i}\omega t} = S_i\,\mathrm{e}^{\mathrm{i}\omega t} \tag{38}$$

therefore,

$$S_i = \sum_{r=1}^{\infty} \frac{A_j\{R[\phi_r(P_i)]\}\{\int_D \phi_r(P)\Gamma(P_j)\mathrm{d}D(P)\}}{(\omega_r^2-\omega^2)+\mathrm{i}(2\zeta_r\omega_r\omega)} \tag{39}$$

2.3.3. FRF between generic sensing device and generic actuation force

The FRF between the response of the ith generic sensing device at location P_i and the magnitude of the jth generic actuation force applied at location P_j can be derived from Eq. (39) as follows:

$$\begin{aligned} H_{ij} = \frac{S_i}{A_j} &= \sum_{r=1}^{\infty} \frac{\{R[\phi_r(P_i)]\}\{\int_D \phi_r(P)\Gamma(P_j)\,\mathrm{d}D(P)\}}{(\omega_r^2-\omega^2)+\mathrm{i}(2\zeta_r\omega_r\omega)} \\ &= \sum_{r=1}^{\infty} \frac{\phi_{r,i}^S\phi_{r,j}^F}{(\omega_r^2-\omega^2)+\mathrm{i}(2\zeta_r\omega_r\omega)} \end{aligned} \tag{40}$$

where

$$\phi_{r,i}^S = R[\phi_r(P_i)] \tag{41}$$

$$\phi_{r,j}^F = \int_D \phi_r(P)\Gamma(P_j)\,\mathrm{d}D(P) \tag{42}$$

where $\phi_r^F(P)$ and $\phi_r^S(P)$ can be defined as the generic actuator and sensor mode shape functions, respectively. $\phi_{r,j}^F$ and $\phi_{r,i}^S$ can then be identified as the scalar values of the generic actuator and sensor mode shape functions at locations P_j and P_i, respectively. It should be noted that the FRF is expressed in the conventional modal format analogous to the discrete mdof system and reveals convenience in programming as well as for physical interpretation of $\phi_r^F(P)$ and $\phi_r^S(P)$ [43].

2.4. Spectrum response analysis

For a system subject to random excitation, transient response analysis in the time domain is not feasible. Frequency domain analysis is generally adopted [44–46]. With the operation of the Fourier transform on the force function as shown in Eq. (12), the Fourier spectrum of the jth generic force $f(P_j, t)$ can be obtained:

$$F(P_j,\omega) = \Im[f(P_j,t)] = \Im[f_j(t)]\Gamma(P_j) = F_j(\omega)\Gamma(P_j) \tag{43}$$

where $\Im$ denotes the Fourier transform operator and $F_j(\omega)$ is the Fourier spectrum of $f_j(t)$. From the definition of the PSD function [50], the PSD function for $f(P_j, t)$ can be defined as follows:

$$\begin{aligned} S_{ff}(P_j,\omega) &= \lim_{T\to\infty}\frac{1}{T}E[F^*(P_j,\omega)F(P_j,\omega)] \\ &= \lim_{T\to\infty}\frac{1}{T}E[F_j^*(\omega)\Gamma(P_j)F_j(\omega)\Gamma(P_j)] \\ &= \Gamma(P_j)\Gamma(P_j)\lim_{T\to\infty}\frac{1}{T}E[F_j^*(\omega)F_j(\omega)] \\ &= \Gamma(P_j)\Gamma(P_j)S_{f_jf_j}(\omega) \end{aligned} \tag{44}$$

where

$$S_{f_jf_j}(\omega) = \lim_{T\to\infty}\frac{1}{T}E[F_j^*(\omega)F_j(\omega)] \tag{45}$$

From the expansion theorem shown in Eq. (27), the system response is as follows:

$$w(P,t) = \sum_{r=1}^{\infty}\phi_r(P)q_r(t) \tag{46}$$

By employing the system response variable $s(P,\ t)$ in Eq. (28) for the response operator R related to the spatial function only, the measured quantity can be rewritten as follows:

$$\begin{aligned} s(P,t) &= \sum_{r=1}^{\infty} R[\phi_r(P)]q_r(t) \\ &= \sum_{r=1}^{\infty} \phi_r^S(P)q_r(t) \end{aligned} \tag{47}$$

Perform the Fourier transform on $s(P,\ t)$ to obtain its Fourier spectrum as follows:

$$\begin{aligned} S(P,\omega) &= \Im[s(P,t)] \\ &= \Im\left[\sum_{r=1}^{\infty} \phi_r^S(P)q_r(t)\right] \\ &= \sum_{r=1}^{\infty} \phi_r^S(P)\Im[q_r(t)] \\ &= \sum_{r=1}^{\infty} \phi_r^S(P)Q_r(\omega) \end{aligned} \tag{48}$$

The PSD function of sensing response $s(P,\ t)$ can also be obtained:

$$\begin{aligned} S_{ss}(P,\omega) &= \lim_{T\to\infty} \frac{1}{T} E[S^*(P,\omega)S(P,\omega)] \\ &= \lim_{T\to\infty} \frac{1}{T} E\left[\sum_{r=1}^{\infty} \phi_r^S(P)Q_r^*(\omega) \sum_{s=1}^{\infty} \phi_s^S(P)Q_s(\omega)\right] \\ &= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \phi_r^S(P)\phi_s^S(P) \lim_{T\to\infty} \frac{1}{T} E[Q_r^*(\omega)Q_s(\omega)] \\ &= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \phi_r^S(P)\phi_s^S(P)S_{q_rq_s}(\omega) \end{aligned} \tag{49}$$

where

$$S_{q_rq_s}(\omega) = \lim_{T\to\infty} \frac{1}{T} E[Q_r^*(\omega)Q_s(\omega)] \tag{50}$$

For the determination of $S_{q_rq_s}(\omega)$, modal equations as shown in Eqs. (15) and (16) are rewritten as follows:

$$\ddot{q}_r + 2\zeta_r\omega_r\dot{q}_r + \omega_r^2 q_r = N_r(t), \quad r = 1, 2, \ldots \tag{51}$$

where

$$N_r(t) = f_j(t)\int_D \phi_r(P)\Gamma(P_j)\,\mathrm{d}D(P) = f_j(t)\phi_r^F(P_j) = f_j(t)\phi_{r,j}^F \tag{52}$$

Let $N_r(t) = N_r\mathrm{e}^{\mathrm{i}\omega t}$ and $q_r(t) = Q_r\,\mathrm{e}^{\mathrm{i}\omega t}$ be substituted into Eq. (51), the FRF between Q_r and N_r can be determined:

$$H_r(\omega) = \frac{Q_r(\omega)}{N_r(\omega)} = \frac{1}{(\omega_r^2 - \omega^2) + \mathrm{i}(2\zeta_r\omega_r\omega)} \tag{53}$$

such that

$$Q_r(\omega) = H_r(\omega)N_r(\omega) \tag{54}$$

By the operation of the Fourier transform on $N_r(t)$ in Eq. (52) to obtain its Fourier spectrum:

$$\begin{aligned} N_r(\omega) &= \Im[N_r(t)] \\ &= \Im[f_j(t)] \int_D \phi_r(P)\Gamma(P_j)\,\mathrm{d}D(P) \\ &= F_j(\omega)\phi_r^F(P_j) = F_j(\omega)\phi_{r,j}^F \end{aligned} \tag{55}$$

The PSD function of $N_r(t)$ will be as follows:

$$\begin{aligned} S_{N_rN_s}(\omega) &= \lim_{T\to\infty} \frac{1}{T} E[N_r^*(\omega)N_s(\omega)] \\ &= \lim_{T\to\infty} \frac{1}{T} E[F_j^*(\omega)\phi_r^F(P_j)F_j(\omega)\phi_s^F(P_j)] \\ &= \phi_r^F(P_j)\phi_s^F(P_j) \lim_{T\to\infty} \frac{1}{T} E[F_j^*(\omega)F_j(\omega)] \\ &= \phi_r^F(P_j)\phi_s^F(P_j)S_{f_jf_j}(\omega) \end{aligned} \tag{56}$$

By the substitution of $Q_r(\omega)$ in Eq. (54) into $S_{q_rq_s}(\omega)$ in Eq. (50),

$$\begin{aligned} S_{q_rq_s}(\omega) &= \lim_{T\to\infty} \frac{1}{T} E[Q_r^*(\omega)Q_s(\omega)] \\ &= \lim_{T\to\infty} \frac{1}{T} E[H_r^*(\omega)N_r^*(\omega)H_s(\omega)N_s(\omega)] \\ &= H_r^*(\omega)H_s(\omega) \lim_{T\to\infty} \frac{1}{T} E[N_r^*(\omega)N_s(\omega)] \\ &= H_r^*(\omega)H_s(\omega)S_{N_rN_s}(\omega) \end{aligned} \tag{57}$$

The PSD function of response variable $s(P, t)$ from Eq. (49) can be finally expressed:

$$\begin{aligned} S_{ss}(P,\omega) &= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \phi_r^S(P)\phi_s^S(P)\{S_{q_rq_s}(\omega)\} \\ &= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \phi_r^S(P)\phi_s^S(P)\{H_r^*(\omega)H_s(\omega)[S_{N_rN_s}(\omega)]\} \\ &= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \phi_r^S(P)\phi_s^S(P)H_r^*(\omega)H_s(\omega)[\phi_r^F(P_j)\phi_s^F(P_j)S_{f_jf_j}(\omega)] \end{aligned} \tag{58}$$

The PSD function of response variable $s(P, t)$ at P_i location can then be obtained:

$$\begin{aligned} S_{s_is_i}(\omega) &= S_{ss}(P_i,\omega) \\ &= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \phi_r^S(P_i)\phi_s^S(P_i)\phi_r^F(P_j)\phi_s^F(P_j)H_r^*(\omega)H_s(\omega)S_{f_jf_j}(\omega) \end{aligned} \tag{59}$$

The root mean square (rms) value of sensing response $s(P_i, t)$ can then be obtained:

$$\begin{aligned} s_{i,rms}^2 = \overline{s_i^2} &= \int_{-\infty}^{\infty} S_{s_is_i}(\omega)\,\mathrm{d}\omega \\ &= \int_{-\infty}^{\infty} \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \phi_r^S(P_i)\phi_s^S(P_i)\phi_r^F(P_j)\phi_s^F(P_j)H_r^*(\omega)H_s(\omega)S_{f_jf_j}(\omega)\,\mathrm{d}\omega \\ &= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \phi_r^S(P_i)\phi_s^S(P_i)\phi_r^F(P_j)\phi_s^F(P_j) \int_{-\infty}^{\infty} H_r^*(\omega)H_s(\omega)S_{f_jf_j}(\omega)\,\mathrm{d}\omega \end{aligned} \tag{60}$$

and

$$\begin{aligned}\sigma_{s_i}^2 &= \overline{s_i^2} - (\bar{s}_i)^2 \\ &= (s_{i,\text{rms}})^2 - (\bar{s}_i)^2\end{aligned} \tag{61}$$

It can be noted that the rms of sensing response $s(P_i, t)$ in Eq. (60) is valid for generic random force excitation in which $H_r(\omega)$ is shown in Eq. (53). For the assumption of white noise excitation $S_{f_jf_j}(\omega) = S_0$ with zero mean $\bar{f}_j = E[f_j(t)] = 0$ and the system with light damping and well-separated modes, Eq. (60) can be simplified as follows:

$$\begin{aligned}\sigma_{s_i}^2 = s_{i,\text{rms}}^2 &= S_0 \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \phi_r^S(P_i)\phi_s^S(P_i)\phi_r^F(P_j)\phi_s^F(P_j) \int_{-\infty}^{\infty} H_r^*(\omega)H_s(\omega)\,\mathrm{d}\omega \\ &\approx S_0 \sum_{r=1}^{\infty} [\phi_r^S(P_i)\phi_r^F(P_j)]^2 \int_{-\infty}^{\infty} |H_r(\omega)|^2\,\mathrm{d}\omega\end{aligned} \tag{62}$$

where [44]

$$\int_{-\infty}^{\infty} |H_r(\omega)|^2\,\mathrm{d}\omega = \int_{-\infty}^{\infty} \left|\frac{1}{(\omega_r^2-\omega^2)+\mathrm{i}(2\zeta_r\omega_r\omega)}\right|^2 \mathrm{d}\omega = \frac{\pi}{2\zeta_r\omega_r^3} \tag{63}$$

Therefore,

$$\sigma_{s_i}^2 = s_{i,\text{rms}}^2 \approx \sum_{r=1}^{\infty} [\phi_r^S(P_i)\phi_r^F(P_j)]^2 \frac{\pi S_0}{2\zeta_r\omega_r^3} \tag{64}$$

The following derivation is to find the mean of $s(P, t)$ at P_i location, $\bar{s}_i$. Recall Eq. (12) for a general force function to obtain the mean of $f(P_j, t)$ as follows:

$$\bar{f}(P_j, t) = E[f(P_j, t)] = E[f_j(t)]\Gamma(P_j) = \bar{f}_j\Gamma(P_j) \tag{65}$$

where $\bar{f}_j = E[f_j(t)]$ is the mean of $f_j(t)$. Recall Eq. (16) to obtain the mean of modal force $N_r(t)$:

$$\bar{N}_r(t) = E[N_r(t)] = E[f_j(t)]\phi_r^F(P_j) = \bar{f}_j\phi_r^F(P_j) \tag{66}$$

Recall Eq. (22), where the $q_{r,\text{IC}}(t)$ term can be neglected for random excitation and, therefore, $q_r(t) = q_{r,\text{IRF}}(t)$, to obtain the mean of modal coordinate $q_r(t)$ as follows:

$$\begin{aligned}\bar{q}_r &= E[q_r(t)] = E[q_{r,\text{IRF}}(t)] \\ &= E\left[\int_0^t N_r(\tau)h_r(t-\tau)\,\mathrm{d}\tau\right] \\ &= E\left[\int_0^t N_r(t-\tau)h_r(\tau)\,\mathrm{d}\tau\right] \\ &= \int_0^t E[N_r(t-\tau)]h_r(\tau)\,\mathrm{d}\tau \\ &= E[N_r(t-\tau)]\int_0^t h_r(\tau)\,\mathrm{d}\tau \\ &= \bar{N}_r H_r(0)\end{aligned} \tag{67}$$

Also, recall Eq. (47) to obtain the mean of $s(P, t)$ as follows:

$$
\begin{aligned}
\bar{s} = E[s(P,t)] &= E\left[\sum_{r=1}^{\infty} \phi_r^S(P) q_r(t)\right] \\
&= \sum_{r=1}^{\infty} \phi_r^S(P)\{E[q_r(t)]\} = \sum_{r=1}^{\infty} \phi_r^S(P)\{\bar{q}_r\} \\
&= \sum_{r=1}^{\infty} \phi_r^S(P)\{[\bar{N}_r] H_r(0)\} \\
&= \sum_{r=1}^{\infty} \phi_r^S(P)[\bar{f}_j \phi_r^F(P_j)] H_r(0) \\
&= \sum_{r=1}^{\infty} \frac{\phi_r^S(P)\phi_r^F(P_j)}{\omega_r^2} \bar{f}_j
\end{aligned} \tag{68}
$$

Finally, the mean of $s(P, t)$ at P_i location, $\bar{s}_i$, can be obtained as follows:

$$
\bar{s}_i = \bar{s}(P_i, t) = \sum_{r=1}^{\infty} \frac{\phi_r^S(P_i)\phi_r^F(P_j)}{\omega_r^2} \bar{f}_j \tag{69}
$$

With the knowledge of the mean and standard deviation of $s_i(t) = s(P_i, t)$ at P_i location, i.e. $\bar{s}_i$ and σ_{si}, the Gaussian distribution function can be obtained as follows:

$$
p(s_i) = \frac{1}{\sigma_{s_i}\sqrt{2\pi}} \mathrm{e}^{[-(1/2)((s_i - \bar{s}_i)/\sigma_{s_i})^2]} \tag{70}
$$

In summary for spectrum response analysis, the PSD function and the mean of temporal function $f_j(t)$ are known as the input variables and given in Eq. (45) for $S_{f_i f_i}(\omega)$ and $\bar{f}_j = E[f_j(t)]$, respectively. The system response can be obtained as follows:

(1) $S_{s_i s_i}(\omega)$ is the PSD function of $s(P, t)$ at P_i location as shown in Eq. (59).
(2) $\bar{s}_i$ is the mean of $s(P, t)$ at P_i location as shown in Eq. (69).
(3) $s_{i,\mathrm{rms}}$ is the rms value of system response $s(P, t)$ at P_i location as shown in Eq. (60) as well as in Eq. (64), in particular for white noise random excitation with zero mean $\bar{f}_j = 0$.
(4) σ_{si} is the standard deviation of $s_i(t) = s(P_i, t)$ at P_i location as shown in Eq. (61) as well as in Eq. (64) for white noise excitation with $\bar{f}_j = 0$, respectively.
(5) $p(s_i)$ is the Gaussian distribution function of $s_i(t) = s(P_i, t)$ as shown in Eq. (70).

The above statistical quantities can be used to evaluate the structural response due to a generic sensing device for engineering design regarding generic random force input.

3. Case study: one-dimensional beam problem

Section 2 has completely derived four types of vibration analyses for continuous systems with the solutions expressed in terms of generic force/actuator and response/sensor mode shape functions, respectively. This section will follow the developed formulation shown in Section 2 and demonstrate the application to the lateral vibration of a one-dimensional uniform beam problem. First, the equation of motion for the uniform Euler–Bernoulli beam including a damping effect can be derived as follows [1]:

$$
EI\frac{\partial^4 w(x,t)}{\partial x^4} + C\frac{\partial w(x,t)}{\partial t} + \rho A\frac{\partial^2 w(x,t)}{\partial t^2} = f(x,t) \tag{71}
$$

where E is Young's modulus, I the moment of inertia, ρ the density, A the cross-sectional area and C the damping coefficient. In comparison to Eq. (1), the following variables can be identified.

$$\begin{aligned} L &= EI\frac{\partial^4}{\partial x^4} \\ C &= C \\ M &= \rho A \\ P &= x \end{aligned} \tag{72}$$

Some common boundary conditions at each of both ends in beam can be any two of the following, including deflection, slope, bending moment and shear force:

$$w(x,t) = 0 \tag{73}$$

$$\theta(x,t) = \frac{\partial w(x,t)}{\partial x} = 0 \tag{74}$$

$$M(x,t) = EI\frac{\partial^2 w(x,t)}{\partial x^2} = 0 \tag{75}$$

$$V(x,t) = \frac{\partial}{\partial x}\left(EI\frac{\partial^2 w(x,t)}{\partial x^2}\right) = 0 \tag{76}$$

The beam initial conditions can be specified as follows:

$$w(x,0) = w_0(x) \tag{77}$$

$$\dot{w}(x,0) = \dot{w}_0(x) \tag{78}$$

where $w_0(x)$ and $\dot{w}_0(x)$ indicate the initial displacement and velocity of the beam, respectively.

3.1. Modal analysis

Through eigenproblem analysis, an infinite set of natural frequencies ω_r and their corresponding eigenfunctions $w_r(x)$ for various end conditions of beams can be obtained [2]. It should be noted that $w_r(x)$ can be properly redefined by using Eq. (6) and is unique as follows:

$$\phi_r(x) = \frac{w_r(x)}{\sqrt{\int_0^L \rho A[w_r(x)]^2 \mathrm{d}x}} \tag{79}$$

The orthogonal relations of displacement mode shape functions $\phi_r(x)$ can be written as follows according to Eqs. (7)–(10):

$$\int_0^L \rho A\phi_r(x)\phi_s(x)\,\mathrm{d}x = \delta_{rs} \tag{80}$$

$$\int_0^L \phi_r(x)EI\frac{\partial^4\phi_s(x)}{\partial x^4}\,\mathrm{d}x = \omega_r^2\delta_{rs} \tag{81}$$

$$\int_0^L C\phi_r(x)\phi_s(x)\,\mathrm{d}x = 2\zeta_r\omega_r\delta_{rs} \tag{82}$$

The free vibration analysis of various kinds of beams has been widely studied [11–17] to solve for natural frequencies and corresponding mode shapes. This work does not intend to solve such a problem but provides the results of natural modes for the general approach to the solution of transient, harmonic and spectrum response analyses as follows: once the natural modes of vibration can be properly determined and their orthogonal properties demonstrated.

It should be noted that in this work those simple boundary conditions as shown in Eqs. (73)–(76) are considered so that the displacement mode shape $\phi_r(x)$ can maintain the orthogonal relations shown in Eqs. (80)–(82). Ginsberg [8] showed the mass and spring ends' orthogonal conditions as well but are omitted here for brevity. Here, two common boundary conditions of a beam are illustrated as follows:

(1) Simply supported beam:

$$\omega_r = \alpha_r^2\sqrt{\frac{EI}{\rho A}}, \quad \alpha_r = \frac{r\pi}{L} \tag{83}$$

$$\phi_r(x) = \sqrt{\frac{2}{\rho AL}}\sin \alpha_r x \tag{84}$$

(2) Cantilever beam:

$$\omega_r = \alpha_r^2\sqrt{\frac{EI}{\rho A}} \tag{85}$$

$$\begin{aligned} \phi_r(x) &= \sqrt{\frac{1}{\rho AL}}[(\sin \alpha_r x - \sinh \alpha_r x) + \sigma_r(\cos \alpha_r x - \cosh \alpha_r x)] \\ \sigma_r &= \frac{\sin \alpha_r L + \sin \alpha_r L}{\cos \alpha_r L + \cosh \alpha_r L} \end{aligned} \tag{86}$$

where for a cantilever beam the value of $\alpha_r L$ can be found in many vibration textbooks [47].

3.2. Transient response analysis

Consider the generic force acting on the beam with the temporal and spatial functions as follows:

$$f(x,t) = f_j(t)\Gamma(x) \tag{87}$$

Some examples of spatial functions for several types of forces are given in Table 2 and depicted in Fig. 1.

Table 2
Examples for different types of forces for beam transient dynamic analysis

Type of force	Ideal impact point force	Step point force	Ideal impact point moment	Ideal impact PZT actuator for bending [51]
Force function $f(x,t) = f_j(t)\Gamma(x)$	$F_j\delta(t-t_0)\delta(x-x_{f_j})$	$F_j u(t-t_0)\delta(x-x_{f_j})$	$M_j\delta(t-t_0)\delta'(x-x_{m_j})$	$M_{C_j}\delta(t-t_0)[\delta'(x-x_{c1j})-\delta'(x-x_{c2j})]$
Temporal function $f_j(t)$	$F_j\delta(t-t_0)$	$F_j u(t-t_0)$	$M_j\delta(t-t_0)$	$M_{C_j}\delta(t-t_0)$
Spatial function $\Gamma(x)$	$\delta(x-x_{f_j})$	$\delta(x-x_{f_j})$	$\delta'(x-x_{m_j})$	$[\delta'(x-x_{c1j})-\delta'(x-x_{c2j})]$
Magnitude F_j	F_j	F_j	M_j	M_{C_j}
Location x_j	x_{f_j}	x_{f_j}	x_{m_j}	x_{c1j}, x_{c2j}
Generic force mode shape function $\phi_r^F(x_j) = \int_0^L \phi_r(x)\Gamma(x_j)\mathrm{d}x$	$\phi_r^F(x_j) = \phi_r(x_{f_j})$	$\phi_r^F(x_j) = \phi_r(x_{f_j})$	$\phi_r^F(x_j) = \phi_r'(x_{m_j})$	$\phi_r^F(x_j) = \phi_r'(x_{c2j}) - \phi_r'(x_{c1j})$

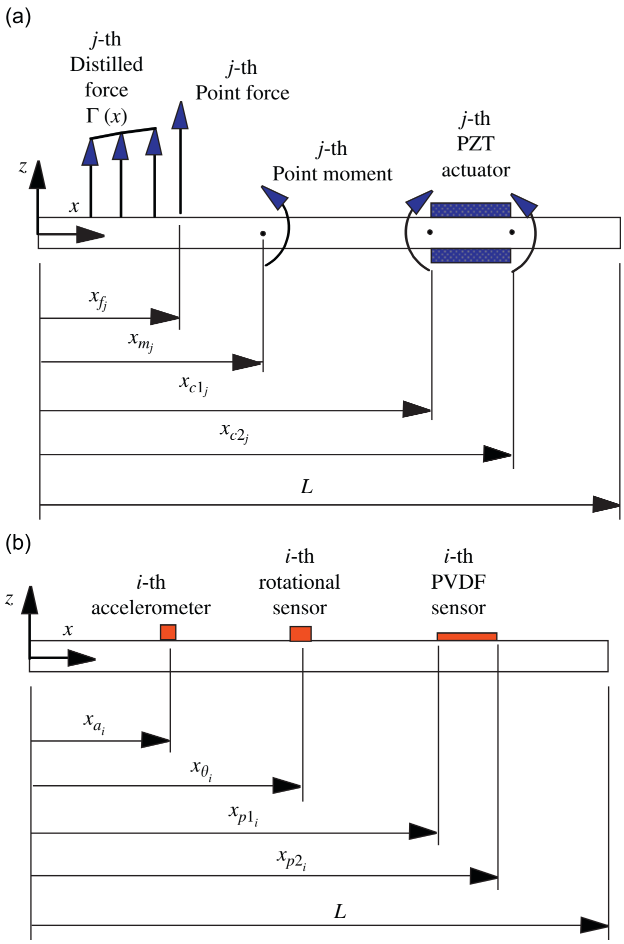


Fig. 1. The arrangement and coordinates of actuators and sensors in a beam: (a) force/actuator and (b) response/sensor.

From the expansion theorem as shown in Eq. (13), the beam lateral displacement response can be assumed to be as follows:

$$w(x,t) = \sum_{r=1}^{\infty} \phi_r(x) q_r(t) \tag{88}$$

By following the TMA procedure as shown in Section 2 and with the substitution of Eqs. (87) and (88) into Eq. (71), Eq. (71) is then multiplied by $\phi_s(x)$ and integrated over beam length L as follows:

$$\begin{aligned} q_r(t)\Bigg\{ \sum_{r=1}^{\infty} \int_0^L \phi_s(x)\left[EI\frac{\partial^4 \phi_r(x)}{\partial x^4}\right] \mathrm{d}x + \dot{q}_r(t) \sum_{r=1}^{\infty} \int_0^L C\phi_s(x)[\phi_r(x)]\,\mathrm{d}x \\ +\ddot{q}_r(t) \sum_{r=1}^{\infty} \int_0^L \rho A \phi_s(x)\phi_r(x)\,\mathrm{d}x \Bigg\} = f_j(t) \int_0^L \phi_s(x)\Gamma(x)\,\mathrm{d}x \end{aligned} \tag{89}$$

By the substitution of the orthogonal properties of the eigenfunctions as shown in Eqs. (80)–(82), the PDE in the physical domain can be reduced to an infinite set of ODEs in the modal domain as follows:

$$\ddot{q}_r + 2\zeta_r \omega_r \dot{q}_r + \omega_r^2 q_r = N_r(t), \quad r = 1, 2, \ldots \tag{90}$$

where

$$N_r(t) = f_j(t)\int_0^L \phi_r(x)\Gamma(x)\,\mathrm{d}x = f_j(t)\phi_r^F(x_j) \tag{91}$$

The modal coordinate initial conditions can be obtained from Eqs. (20) and (21) as follows:

$$q_{r0} = q_r(0) = \int_0^L \rho A\phi_r(x)w_0(x)\,\mathrm{d}x \tag{92}$$

Similarly,

$$\dot{q}_{r0} = \dot{q}_r(0) = \int_0^L \rho A\phi_r(x)\dot{w}_0(x)\,\mathrm{d}x \tag{93}$$

Finally, the modal coordinate response $q_r(t)$ in Eq. (90) can be solved and gives the same expressions as shown in Eqs. (22)–(26). Therefore, the transient displacement response due to the generic force as shown in Eq. (87) can be solved. It should be noted that the formulation is generic and can be easily adapted for different boundary conditions, such as simply supported and cantilever beams as shown, and force conditions as illustrated in Table 2.

By introducing the response operator R on $w(x, t)$ to obtain the response variable $s(x, t)$ as follows:

$$s(x,t) = R[w(x,t)] = \sum_{r=1}^{\infty} R[\phi_r(x)]q_r(t) = \sum_{r=1}^{\infty}\phi_r^S(x)q_r(t) \tag{94}$$

where

$$\phi_r^S(x) = R[\phi_r(x)] \tag{95}$$

$\phi_r^S(x)$ is the generic sensor or the response mode shape function. Table 3(a) and (b) show several examples of sensing and response operators for typical sensing devices and structural responses of interest of the beam, respectively. The advantage of the formulation can be summarized as follows:

(1) The spatial function $\Gamma(x)$ for the generic force acting on the beam is introduced and results in the generic force mode shape function $\phi_r^F(x_j)$ that is employed to characterize the modal force $N_r(t)$ as shown in Eq. (91) and suitable for arbitrary force application. Then, the modal coordinate response $q_r(t)$ can be obtained from Eqs. (22)–(26). Table 2 summarizes several examples of typical forces and their corresponding $\phi_r^F(x_j)$.
(2) The response operator R is also defined to obtain the response variable $s(x, t)$ as shown in Eq. (94) that can be determined from the generic sensor/response mode shape function $\phi_r^S(x)$. Table 3(a) shows some typical sensing devices and their corresponding generic sensor/response mode shape functions, $\phi_r^S(x)$. Displacement sensors, accelerometers and rotational sensors are point-type sensors, while the PVDF sensor is a distributed-type sensor [51–53]. Table 3(b) shows typical responses of interest, including the slope (strain), bending moment, shear force and the maximum bending stress in the beam.
(3) Both $\phi_r^F(x_j)$ and $\phi_r^S(x_i)$ are functions of displacement mode shape function $\phi_r^S(x)$. The developed formulation is valid for arbitrary boundaries when $\phi_r(x)$ satisfies the orthogonal properties as shown in Eqs. (80)–(82). From the viewpoint of numerical programming for solution, the formulation is of great convenience in application, since only both the generic force and the response mode shape functions need to be rearranged accordingly.
(4) It should be noted that the solution form revealed in Eq. (91) is not only valid for the illustrated simply supported or cantilever boundary condition but also for complex beams, such as with non-uniform thickness [14], mass-loaded [11–13,15], composite [17] or the various end boundaries [13,16], as long as the displacement mode shape $\phi_r(x)$ can maintain the corresponding orthogonal relations similar to those shown in Eqs. (80)–(82).

Table 3
Examples of response operators for typical sensing devices or structural responses of interest of beam structures
(a) *Typical sensing devices*

Sensor	Displacement sensor	Accelerometer	Rotational (slope) sensor	PVDF sensor [51,52,53]
Location x_i	x_{d_i}	x_{a_i}	x_{θ_i}	x_{p1i}, x_{p2i}
Measured quantity $s(x_j) = R[w(x_i, t)]$	$w(x_{d_i})$	$\left.\dfrac{\partial^2 w(x,t)}{\partial t^2}\right\|_{x=x_{a_i}}$	$\left.\dfrac{\partial w(x,t)}{\partial x}\right\|_{x=x_{\theta_i}}$	$K_p\left[\left.\dfrac{\partial w(x,t)}{\partial x}\right\|_{x=x_{p1i}} - \left.\dfrac{\partial w(x,t)}{\partial x}\right\|_{x=x_{p2i}}\right]$
Sensing operator R	$1\|_{x=x_{d_i}}$	$\left.\dfrac{\partial^2}{\partial t^2}\right\|_{x=x_{a_i}}$	$\left.\dfrac{\partial}{\partial x}\right\|_{x=x_{\theta_i}}$	$K_p\left[\left.\dfrac{\partial}{\partial x}\right\|_{x=x_{p1i}} - \left.\dfrac{\partial}{\partial x}\right\|_{x=x_{p2i}}\right]$
Generic sensor mode shape function $\phi_r^S(x_i) = R[\phi_r(x_i)]$	$\phi_r^S(x_i) = \phi_r(x_{d_i})$	$\phi_r^S(x_i) = \phi_r(x_{a_i})$	$\phi_r^S(x_i) = \phi_r'(x_{\theta_i})$	$\phi_r^S(x_i) = K_p[\phi_r'(x_{p1i}) - \phi_r'(x_{p2i})]$

(b) *Typical structural responses of interest*

Response	Slope (Strain)	Moment	Shear force	Max. bending stress
Location x_i	x_i	x_i	x_i	x_i
Measured quantity $s(x_j) = R[w(x_i, t)]$	$\theta(x_i, t) = \left.\dfrac{\partial w(x,t)}{\partial x}\right\|_{x=x_{\theta_i}}$	$M(x_i, t) = EI\left.\dfrac{\partial^2 w(x,t)}{\partial x^2}\right\|_{x=x_i}$	$V(x_i, t) = EI\left.\dfrac{\partial^3 w(x,t)}{\partial x^3}\right\|_{x=x_i}$	$\sigma(x_i, t) = EZ\left.\dfrac{\partial^2 w(x,t)}{\partial x^2}\right\|_{x=x_i}$
Sensing operator R	$\left.\dfrac{\partial}{\partial x}\right\|_{x=x_i}$	$EI\left.\dfrac{\partial^2}{\partial x^2}\right\|_{x=x_i}$	$EI\left.\dfrac{\partial^3}{\partial x^3}\right\|_{x=x_i}$	$EZ\left.\dfrac{\partial^2}{\partial x^2}\right\|_{x=x_i}$
Generic sensor mode shape function $\phi_r^S(x_i) = R[\phi_r(x_i)]$	$\phi_r^S(x_i) = \phi_r'(x_i)$	$\phi_r^S(x_i) = EI\phi_r''(x_i)$	$\phi_r^S(x_i) = EI\phi_r'''(x_i)$	$\phi_r^S(x_i) = EZ\phi_r''(x_i)$

Note: Z = max. distance from the neutral axis of the beam.

3.3. Harmonic response analysis

Wang [43] has demonstrated the harmonic response analysis for determining the FRFs for different combinations of actuators and sensors. Different modal testing procedures resulting in different mode shape functions were also illustrated. Here, a brief summary is provided with a slight modification in accordance with this paper.

Consider a generic harmonic force with amplitude A_j and excitation frequency ω applied at some location defined by the spatial function $\Gamma(x)$. The harmonic force function can be expressed as follows:

$$f(x,t) = f_j(t)\Gamma(x) = A_j e^{i\omega t}\Gamma(x) \tag{96}$$

Table 2 shows several types of forces with their spatial functions.

From the expansion theorem as shown in Eq. (88), the beam lateral displacement response can also be assumed to be harmonic as follows:

$$w(x,t) = \sum_{r=1}^{\infty} \phi_r(x) q_r(t) = \sum_{r=1}^{\infty} \phi_r(x) Q_r(\omega) e^{i\omega t} = W(x,\omega) e^{i\omega t} \tag{97}$$

With the employment of the sensing or the response operator R, the response variable $s(x, t)$ can be expressed as follows:

$$s(x,t) = \sum_{r=1}^{\infty} R[\phi_r(x)] q_r(t) = \sum_{r=1}^{\infty} \phi_r^S(x) Q_r(\omega) e^{i\omega t} = S(x,\omega) e^{i\omega t} \tag{98}$$

where

$$S(x,\omega)=\sum_{r=1}^{\infty}\phi_r^S(x)Q_r(\omega) \tag{99}$$

By following the TMA procedure as shown in the previous section, one can obtain $Q_r(\omega)$ directly from Eq. (35). Or with the substitution of Eqs. (96) and (98) into the beam equation in Eq. (71), Eq. (71) is then multiplied by $\phi_s(x)$ and integrated over beam length L, the amplitude of modal coordinate $Q_r(\omega)$ can be obtained as follows:

$$Q_r(\omega)=\frac{A_j\int_0^L\phi_r(x)\Gamma(x)\,\mathrm{d}x}{(\omega_r^2-\omega^2)+\mathrm{i}(2\zeta_r\omega_r\omega)}=\frac{A_j\phi_r^F(x_j)}{(\omega_r^2-\omega^2)+\mathrm{i}(2\zeta_r\omega_r\omega)} \tag{100}$$

The beam response variable $s(x, t)$ at $x = x_i$ can be obtained from Eq. (98) as follows:

$$s_i=s(x_i,t)=\sum_{r=1}^{\infty}\phi_r^S(x_i)Q_r(\omega)\mathrm{e}^{\mathrm{i}\omega t}=S(x_i,\omega)\mathrm{e}^{\mathrm{i}\omega t}=S_i\mathrm{e}^{\mathrm{i}\omega t} \tag{101}$$

where S_i is the harmonic amplitude of response variable $s(x, t)$ at $x = x_i$ as follows:

$$S_i=\sum_{r=1}^{\infty}\phi_r^S(x_i)Q_r(\omega) \tag{102}$$

Finally, from Eq. (40) the FRF between the generic force amplitude and the response variable amplitude can be obtained as follows:

$$H_{ij}=\frac{S_i}{A_j}=\sum_{r=1}^{\infty}\frac{\phi_r^S(x_i)\phi_r^F(x_j)}{(\omega_r^2-\omega^2)+\mathrm{i}(2\zeta_r\omega_r\omega)} \tag{103}$$

Tables 2 and 3 show various types of actuator/force and sensor/response for both $\phi_r^F(x_j)$ and $\phi_r^S(x_i)$ of beams. The FRF for any combination of actuator/force and sensor/response can then be derived from Eq. (103). The advantage of the formulation is that it not only shows the physical meaning for both $\phi_r^F(x_j)$ and $\phi_r^S(x_i)$ in the modal testing procedure [43] but also provides a great convenience for numerical programming for the solution. In particular, the FRF, such as the maximum bending stress in a beam due to different forms of forces, can be easily obtained. Various types of steady-state responses of the beam due to generic forms of harmonic excitation can be easily determined from Eq. (103) and are applicable to different kinds of complex beams as well.

3.4. Spectrum response analysis

The general approach for the beam subject to generic random force excitation is considered in this section. The generic force function can be expressed as follows:

$$f(x,t)=f_j(t)\Gamma(x) \tag{104}$$

Then, the Fourier spectrum of the jth generic force $f(x, t)$ can be obtained:

$$F(x,\omega)=\Im[f(x,t)]=\Im[f_j(t)]\Gamma(x)=F_j(\omega)\Gamma(x) \tag{105}$$

where $F_j(\omega)$ is the Fourier spectrum of $f_j(t)$. From Eq. (44), the PSD function for $f(x, t)$ can be obtained as follows:

$$S_{ff}(x_j,\omega)=[\Gamma(x)]^2S_{f_jf_j}(\omega) \tag{106}$$

where

$$S_{f_jf_j}(\omega)=\lim_{T\to\infty}\frac{1}{T}E[F_j^*(\omega)F_j(\omega)] \tag{107}$$

From transient response analysis as shown in Eq. (94), the beam response variable $s(x, t)$ can be rewritten as follows:

$$s(x,t) = \sum_{r=1}^{\infty} \phi_r^S(x) q_r(t) \tag{108}$$

The Fourier spectrum and PSD function of $s(x, t)$ can also be obtained from Eqs. (48) and (49), respectively, as follows:

$$S(x,\omega) = \Im[s(x,t)] = \sum_{r=1}^{\infty} \phi_r^S(P) Q_r(\omega) \tag{109}$$

$$S_{ss}(x,\omega) = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \phi_r^S(x)\phi_s^S(x) S_{q_r q_s}(\omega) \tag{110}$$

where $S_{q_r q_s}(\omega)$ is the PSD function of $q_r(t)$ and can be determined from Eq. (57)

$$S_{q_r q_s}(\omega) = H_r^*(\omega) H_s(\omega) S_{N_r N_s}(\omega) \tag{111}$$

where $S_{N_r N_s}(\omega)$ is the PSD function of $N_r(t)$ and can be determined from Eq. (56):

$$S_{N_r N_r}(\omega) = \phi_r^F(x_j)\phi_s^F(x_j) S_{f_j f_j}(\omega) \tag{112}$$

where $\phi_r^F(x_j)$ is the generic force mode shape function. Table 2 shows some typical force expressions.

Finally, from Eq. (59) and with the imposition of the above equations the PSD of the system response variable $s(x, t)$ can be finally expressed:

$$S_{ss}(x,\omega) = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \phi_r^S(x)\phi_s^S(x)\phi_r^F(x_j)\phi_s^F(x_j) H_r^*(\omega) H_s(\omega) S_{f_j f_j}(\omega) \tag{113}$$

The PSD function of $s_i(t) = s(x_i, t)$ at $x = x_i$ location can be obtained as follows:

$$S_{s_i s_i}(\omega) = S_{ss}(x_i,\omega) = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \phi_r^S(x_i)\phi_s^S(x_i)\phi_r^F(x_j)\phi_s^F(x_j) H_r^*(\omega) H_s(\omega) S_{f_j f_j}(\omega) \tag{114}$$

From Eq. (60), the rms value of $s_i(t) = s(x_i, t)$ can then be obtained:

$$\begin{aligned} s_{i,\mathrm{rms}}^2 = \overline{s_i^2} &= \int_{-\infty}^{\infty} S_{s_i s_i}(\omega)\,\mathrm{d}\omega \\ &= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \phi_r^S(x_i)\phi_s^S(x_i)\phi_r^F(x_j)\phi_s^F(x_j) \int_{-\infty}^{\infty} H_r^*(\omega) H_s(\omega) S_{f_j f_j}(\omega)\,\mathrm{d}\omega \end{aligned} \tag{115}$$

and

$$\begin{aligned} \sigma_{s_i}^2 &= \overline{s_i^2} - (\bar{s}_i)^2 \\ &= (s_{i,\mathrm{rms}})^2 - (\bar{s}_i)^2 \end{aligned} \tag{116}$$

For the assumption of white noise excitation $S_{s_i s_i}(\omega) = S_0$ with zero mean $\overline{f}_j = E[f_j(t)] = 0$ and beam with light damping and well-separated modes, Eq. (115) can be simplified as follows according to Eq. (64):

$$\sigma_{s_i}^2 = s_{i,\mathrm{rms}}^2 \approx \sum_{r=1}^{\infty} [\phi_r^S(x_i)\phi_r^F(x_j)]^2 \frac{\pi S_0}{2\zeta_r \omega_r^3} \tag{117}$$

If $\overline{f}_j \neq 0$, the mean of $s_i(t) = s(x_i, t)$ can be derived from Eq. (69) as follows:

$$\bar{s}_i = \sum_{r=1}^{\infty} \frac{\phi_r^S(x_i)\phi_r^F(x_j)}{\omega_r^2} \overline{f}_j \tag{118}$$

In summary, the statistical quantities of $s_i(t) = s(x_i, t)$, in terms of mean $\bar{s}_i$ and standard deviation σ_{si}, can be determined and thus the Gaussian distribution function as shown in Eq. (70) can be obtained and used to evaluate the structural response.

This section follows the procedure developed in Section 2 to obtain the solutions for transient, harmonic and spectrum analyses in terms of both generic force/actuator mode shape function $\phi_r^F(x_j)$ and generic response/sensor mode shape function $\phi_r^S(x_i)$, respectively. As long as the force function is identified in terms of spatial function $\Gamma(x)$ as illustrated in Table 2 and the typical responses of interest are specified as shown in Table 3, both $\phi_r^F(x_j)$ and $\phi_r^S(x_i)$ can be determined, respectively, as shown in Tables 2 and 3. The analytical approach is concise and suitable for various boundary conditions of beams, for which mode shape functions can be defined to maintain the orthogonal properties as shown in Eqs. (80)–(82), subject to different forms of forces by quantifying different types of response in beams. Although only two common boundary conditions of uniform beams are illustrated, the presented formulation can be easily made and adapted for complex beams as well, if the displacement mode shape $\phi_r(x)$ can maintain the corresponding orthogonal relations similar to those shown in Eqs. (80)–(82).

4. Case study: two-dimensional plate problem

In addition to the beam vibration problem, this section will further demonstrate the generality and application of the theoretical formulation in Section 2 to lateral vibration of a two-dimensional rectangular plate problem. The equation of motion for the uniform, thin plate including damping effects can be derived as follows [54]:

$$D\nabla^2\nabla^2 w(x,y,t) + C\frac{\partial w(x,y,t)}{\partial t} + \rho t\frac{\partial^2 w(x,y,t)}{\partial t^2} = f(x,y,t) \tag{119}$$

where

$$D = \frac{Eh^3}{12(1-v^2)} \tag{120}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{121}$$

D is the bending or flexural rigidity of the plate, h the plate thickness and E and v plate's Young's modulus and Poisson's ratio. In comparison to Eq. (1), the following variables can be identified:

$$\begin{aligned} L &= D\nabla^2\nabla^2 \\ C &= C \\ M &= \rho h \\ P &= x, y \end{aligned} \tag{122}$$

Some common boundary conditions at each of four edges of the rectangular plate can be any two of the following, including deflection, slope, bending moment and shear force [54]:

$$w(x,y,t) = 0 \tag{123}$$

$$\theta_x = \frac{\partial w}{\partial y} = 0, \quad \theta_y = \frac{\partial w}{\partial x} = 0 \tag{124}$$

$$m_x = D\left(\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2}\right) = 0, \quad m_y = D\left(\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2}\right) = 0 \tag{125}$$

$$Q_x = D\left[\frac{\partial^3 w}{\partial x^3} + (2-v)\frac{\partial^2 w}{\partial x\,\partial y^2}\right] = 0, \quad Q_y = D\left[\frac{\partial^3 w}{\partial y^3} + (2-v)\frac{\partial^3 w}{\partial x^2\,\partial y}\right] = 0 \tag{126}$$

The plate initial conditions can be specified as follows:

$$w(x, y, 0) = w_0(x, y) \tag{127}$$

$$\dot{w}(x, y, 0) = \dot{w}_0(x, y) \tag{128}$$

where $w_0(x,\ y)$ and $\dot{w}_0(x, y)$ are the initial displacement and velocity of the plate, respectively.

4.1. Modal analysis

Through eigenproblem analysis, an infinite set of natural frequencies ω_{rs} and their corresponding eigenfunctions $w_{rs}(x)$ for various boundary conditions of plates can be obtained [47,49]. $w_{rs}(x)$ can be properly redefined by invoking Eq. (6) and is unique as follows:

$$\phi_{rs}(x) = \frac{w_{rs}(x)}{\sqrt{\int_0^{L_y}\int_0^{L_x} \rho h [w_{rs}(x, y)]^2 \, \mathrm{d}x \, \mathrm{d}y}} \tag{129}$$

The orthogonal relations of displacement mode shape functions $\phi_{rs}(x, y)$ can be written as follows according to Eqs. (7)–(10):

$$\int_0^{L_y}\int_0^{L_x} \rho h \phi_{rs}(x, y)\phi_{mn}(x, y) \, \mathrm{d}x \, \mathrm{d}y = \delta_{rm}\delta_{sn} \tag{130}$$

$$\int_0^{L_y}\int_0^{L_x} \phi_{rs}(x, y)[D\nabla^2\nabla^2\phi_{mn}(x, y)] \, \mathrm{d}x \, \mathrm{d}y = \omega_{rs}^2\delta_{rm}\delta_{sn} \tag{131}$$

$$\int_0^{L_y}\int_0^{L_x} C\phi_{rs}(x, y)\phi_{mn}(x, y) \, \mathrm{d}x \, \mathrm{d}y = 2\zeta_{rs}\omega_{rs}\delta_{rm}\delta_{sn} \tag{132}$$

For a simply supported plate, the natural frequencies and mode shape functions that can satisfy the orthogonal properties are shown as follows, respectively:

$$\omega_{rs} = (\alpha_r^2 + \alpha_s^2)\sqrt{\frac{D}{\rho h}}, \quad \alpha_r = \frac{r\pi}{L_x},\ \alpha_s = \frac{s\pi}{L_y} \tag{133}$$

$$\phi_{rs}(x, y) = \phi_r(x)\phi_s(y) = \sqrt{\frac{2}{\sqrt{\rho h}L_x}}\sin\alpha_r x\sqrt{\frac{2}{\sqrt{\rho h}L_y}}\sin\alpha_s y \tag{134}$$

where

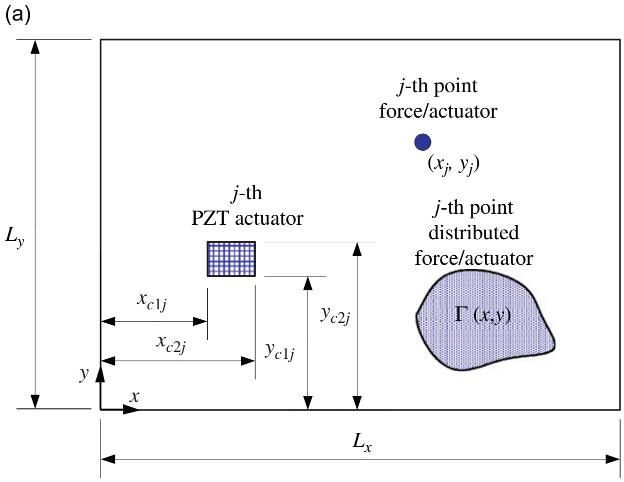
$$\phi_r(x) = \sqrt{\frac{2}{\sqrt{\rho h}L_x}}\sin\alpha_r x \tag{135}$$

$$\phi_s(y) = \sqrt{\frac{2}{\sqrt{\rho h}L_y}}\sin\alpha_s y \tag{136}$$

where L_x and L_y are the length and width of the rectangular plate. The subscripts r and s account for both x- and y-direction in plate length and width, respectively. The free vibration analysis for complex plates can be found in many literatures, such as plates with non-uniform thickness [24], mass-loaded [20,21,23], or different boundary conditions [18,19,22,25]. Again, this work does not intend to solve such a free vibration problem. The following will show the systematic approach for solving the transient, harmonic and spectrum response analyses, if the natural modes of vibration for the plates can be determined and reveals their orthogonal properties similar to Eqs. (130)–(132).

Table 4
Examples for different types of forces for plate transient dynamic analysis

Type of force	Ideal impact point force	Step point force	Ideal impact point moment	Ideal impact PZT for bending [55]
Force function $f(x,y,t)=f_j(t)\Gamma(x,y)$	$F_j\delta(t-t_0)\delta(x-x_{f_j})\delta(y-y_{f_j})$	$F_j u(t-t_0)\delta(x-x_{f_j})\delta(y-y_{f_j})$	$M_{xj}\delta(t-t_0)\delta'(x-x_{m_j})\delta(y-y_{m_j})$	$M_{C_j}\delta(t-t_0)[\delta'(x-x_{c1j})-\delta'(x-x_{c2j})][\delta'(y-y_{c1j})-\delta'(y-y_{c2j})]$
Temporal function $f_j(t)$	$F_j\delta(t-t_0)$	$F_j u(t-t_0)$	$M_{xj}\delta(t-t_0)$	$M_{C_j}\delta(t-t_0)$
Spatial function $\Gamma(x,y)$	$\delta(x-x_{f_j})\delta(y-y_{f_j})$	$\delta(x-x_{f_j})\delta(y-y_{f_j})$	$\delta'(x-x_{m_j})\delta(y-y_{m_j})$	$[\delta'(x-x_{c1j})-\delta'(x-x_{c2j})][\delta'(y-y_{c1j})-\delta'(y-y_{c2j})]$
Magnitude F_j	F_j	F_j	M_{xj}	M_{C_j}
Location x_j,y_j	$x_{f_j},\ y_{f_j}$	$x_{f_j},\ y_{f_j}$	$x_{m_j},\ y_{m_j}$	$x_{c1j},\ x_{c2j},\ y_{c1j},\ y_{c2j}$
Generic force mode shape function $\phi^F_{rs}(x_j,y_j)=\int_0^{L_y}\int_0^{L_x}\phi_{rs}(x,y)\Gamma(x,y)\,\mathrm{d}x\,\mathrm{d}y$	$\phi_{rs}(x_{f_j},\ y_{f_j})$	$\phi_{rs}(x_{f_j},\ y_{f_j})$	$\phi'_r(x_{m_j})\phi_s(y_{m_j})$	$[\phi'_r(x_{c2j})-\phi'_r(x_{c1j})][\phi'_s(y_{c2j})-\phi'_s(y_{c1j})]$



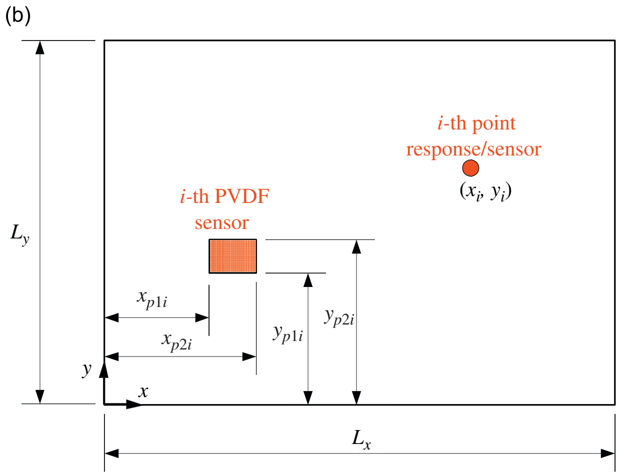


Fig. 2. The arrangement and coordinates of actuators and sensors in a rectangular plate: (a) force/actuator and (b) response/sensor.

4.2. Transient response analysis

Consider the generic force acting on the plate with the temporal and spatial functions as follows:

$$f(x,y,t) = f_j(t)\Gamma(x,y) \tag{137}$$

Some examples of spatial functions for several types of forces are given in Table 4 and depicted in Fig. 2.

From the expansion theorem as shown in Eq. (13), the plate lateral displacement response can be assumed to be as follows:

$$w(x,y,t) = \sum_{s=1}^{\infty}\sum_{r=1}^{\infty}\phi_{rs}(x,y)q_{rs}(t) \tag{138}$$

By following the TMA procedure as shown in Section 2 and with the substitution of Eqs. (137) and (138) into Eq. (119), Eq. (119) is then multiplied by $\phi_{mn}(x)$ and integrated over plate length L_x and width L_y as follows:

$$q_{rs}(t)\left\{\sum_{s=1}^{\infty}\sum_{r=1}^{\infty}\int_0^{L_y}\int_0^{L_x}\phi_{rs}(x,y)[D\nabla^2\nabla^2\phi_{mn}(x,y)]\,\mathrm{d}x\,\mathrm{d}y\right.$$
$$+\,\dot{q}_{rs}(t)\sum_{s=1}^{\infty}\sum_{r=1}^{\infty}\int_0^{L_y}\int_0^{L_x}C\phi_{rs}(x,y)[\phi_{mn}(x,y)]\,\mathrm{d}x\,\mathrm{d}y$$

$$+\ddot{q}_{rs}(t)\sum_{s=1}^{\infty}\sum_{r=1}^{\infty}\int_0^{L_y}\int_0^{L_x}\rho h\phi_{rs}(x,y)\phi_{mn}(x,y)\,\mathrm{d}x\,\mathrm{d}y\Bigg\}$$
$$=f_j(t)\int_0^{L_y}\int_0^{L_x}\phi_{mn}(x,y)\Gamma(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{139}$$

By the substitution of the orthogonal properties of eigenfunctions as shown in Eqs. (130)–(132), the PDE in the physical domain can be reduced to an infinite set of ODEs in the modal domain as follows:

$$\ddot{q}_{rs}+2\zeta_{rs}\omega_{rs}\dot{q}_{rs}+\omega_{rs}^2 q_{rs}=N_{rs}(t)\quad\begin{cases} r=1,2,\ldots \\ s=1,2,\ldots\end{cases} \tag{140}$$

where

$$N_{rs}(t)=f_j(t)\int_0^{L_y}\int_0^{L_x}\phi_{rs}(x,y)\Gamma(x,y)\,\mathrm{d}x\,\mathrm{d}y$$
$$=f_j(t)\phi_{rs}^F(x,y) \tag{141}$$

The modal coordinate initial conditions can be obtained from Eqs. (20) and (21) as follows:

$$q_{rs0}=q_{rs}(0)=\int_0^{L_y}\int_0^{L_x}\rho h\phi_{rs}(x,y)w_0(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{142}$$

Similarly,

$$\dot{q}_{rs0}=\dot{q}_{rs}(0)=\int_0^{L_y}\int_0^{L_x}\rho h\phi_{rs}(x,y)\dot{w}_0(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{143}$$

Finally, the modal coordinate response $q_{rs}(t)$ in Eq. (140) can be solved and maintains the same expressions as shown in Eqs. (22)–(26) except for the subscript r replaced by rs. Therefore, the transient displacement response due to the generic force shown in Eq. (137) can be solved. It should be noted that the formulation is generic and can be easily adapted for different boundary and force conditions.

By introducing the response operator R on $w(x, y, t)$, we obtain the response variable $s(x, y, t)$ as follows:

$$s(x,y,t)=R[w(x,y,t)]=\sum_{s=1}^{\infty}\sum_{r=1}^{\infty}R[\phi_{rs}(x,y)]q_{rs}(t)=\sum_{s=1}^{\infty}\sum_{r=1}^{\infty}\phi_{rs}^S(x)q_{rs}(t) \tag{144}$$

where $\phi_{rs}^S(x,y)$ is the generic sensor/response mode shape function. Table 5 shows several examples of sensing/response operators for typical sensing devices or structural responses of interest of the plate. The advantage of the formulation can be summarized as follows:

(1) The spatial function $\Gamma(x, y)$ for the generic force acting on the plate is introduced and results in the generic force mode shape function $\phi_{rs}^F(x_j,y_j)$ that is employed to characterize the modal force $N_{rs}(t)$ as shown in Eq. (141) and is suitable for arbitrary force application. Then, modal coordinate response $q_{rs}(t)$ can be obtained from Eqs. (22)–(26). Table 4 summarizes several examples of typical forces and their corresponding $\phi_{rs}^F(x_j,y_j)$.

(2) The response operator R is also defined to obtain the response variable $s(x, y, t)$ as shown in Eq. (144) that can be determined from the generic sensor/response mode shape function $\phi_{rs}^S(x,y)$. Table 5(a) shows some typical sensors and their $\phi_{rs}^S(x_i,y_i)$ expressions. Table 5(b) shows typical responses of interest, including the slope θ_y (strain ε_x), bending moment M_x, shear force Q_x and the maximum bending stress σ_x in a plate, and their $\phi_{rs}^S(x_i,y_i)$ expressions.

(3) Both $\phi_{rs}^F(x_j,y_j)$ and $\phi_{rs}^S(x_i,y_i)$ are functions of displacement mode shape function $\phi_{rs}(x, y)$. It should be noted that the presented formulation for the plate is valid for arbitrary boundaries when $\phi_{rs}(x, y)$ satisfies the orthogonal properties as shown in Eqs. (130)–(132). From the viewpoint of numerical programming for the solution, the formulation is of great convenience to apply, since only the generic force and response mode shape functions need to be rearranged accordingly.

Table 5
Examples of response operators for typical sensing devices or structural responses of interest of plate structures
(a) *Typical sensing devices*

Sensor	Displacement sensor	Accelerometer	Rotational sensor (slope θ_y)	PVDF sensor [52,55]
Location x_j,y_j	$x_{d_i},\ y_{d_i}$	$x_{a_i},\ y_{a_i}$	$x_{\theta_i},\ y_{\theta_i}$	$x_{p1i},\ x_{p2i},\ y_{p1i},\ y_{p2i}$
Measured quantity $s(x_i, y_i) = R[w(x_i, y_i, t)]$	$w(x_{d_i},\ y_{d_i})$	$\left.\frac{\partial^2 w(x,y,t)}{\partial t^2}\right\vert_{\substack{x = x_{a_i}\\ y = y_{a_i}}}$	$\left.\frac{\partial w(x,y,t)}{\partial x}\right\vert_{\substack{x = x_{\theta_i}\\ y = y_{\theta_i}}}$	$K_p\left[\left.\frac{\partial w(x,y,t)}{\partial x}\right\vert_{x=x_{p1i}} - \left.\frac{\partial w(x,y,t)}{\partial x}\right\vert_{x=x_{p2i}}\right]\left[\left.\frac{\partial w(x,y,t)}{\partial y}\right\vert_{y=y_{p1i}} - \left.\frac{\partial w(x,y,t)}{\partial y}\right\vert_{y=y_{p2i}}\right]$
Sensing operator R	$1\vert_{\substack{x = x_{d_i}\\ y = y_{d_i}}}$	$\left.\frac{\partial^2}{\partial t^2}\right\vert_{\substack{x = x_{a_i}\\ y = y_{a_i}}}$	$\left.\frac{\partial}{\partial x}\right\vert_{\substack{x = x_{\theta_i}\\ y = y_{\theta_i}}}$	$K_p\left[\left.\frac{\partial}{\partial x}\right\vert_{x=x_{p1i}} - \left.\frac{\partial}{\partial x}\right\vert_{x=x_{p2i}}\right]\left[\left.\frac{\partial}{\partial y}\right\vert_{y=y_{p1i}} - \left.\frac{\partial}{\partial y}\right\vert_{y=y_{p2i}}\right]$
Generic sensor function $\phi^S_{rs}(x_i, y_i) = R[\phi_r(x_i, y_i)]$	$\phi_{rs}(x_{d_i},\ y_{d_i})$	$\phi_{rs}(x_{a_i},\ y_{a_i})$	$\phi'_r(x_{\theta_i})\phi_s(y_{\theta_i})$	$K_p[\phi'_r(x_{p1i}) - \phi'_r(x_{p2i})][\phi'_s(y_{p1i}) - \phi'_s(y_{p2i})]$

(b) *Typical structural responses of interest*

Response	Slope θ_y (Strain ε_x)	Moment	Shear force	Max. bending stress
Location x_j,y_j	x_j,y_j	x_j,y_j	x_j,y_j	x_j,y_j
Measured quantity $s(x_i) = R[w(x_i, t)]$	$\theta_y(x_i, y_i, t) = \left.\frac{\partial w(x,y,t)}{\partial x}\right\vert_{\substack{x = x_i\\ y = y_i}}$	$M_x(x_i, y_i, t) = D\left.\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)\right\vert_{\substack{x = x_i\\ y = y_i}}$	$Q_x(x_i, y_i, t) = D\left.\left[\frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^2 w}{\partial x\,\partial y^2}\right]\right\vert_{\substack{x = x_i\\ y = y_i}}$	$\sigma_x(x_i, y_i, t) = \frac{EZ}{(1-\nu^2)}\left.\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)\right\vert_{\substack{x = x_i\\ y = y_i}}$
Response operator R	$\left.\frac{\partial}{\partial x}\right\vert_{\substack{x = x_i\\ y = y_i}}$	$D\left.\left(\frac{\partial^2}{\partial x^2} + \nu\frac{\partial^2}{\partial y^2}\right)\right\vert_{\substack{x = x_i\\ y = y_i}}$	$D\left.\left[\frac{\partial^3}{\partial x^3} + (2-\nu)\frac{\partial^2}{\partial x\,\partial y^2}\right]\right\vert_{\substack{x = x_i\\ y = y_i}}$	$\frac{EZ}{(1-\nu^2)}\left.\left(\frac{\partial^2}{\partial x^2} + \nu\frac{\partial^2}{\partial y^2}\right)\right\vert_{\substack{x = x_i\\ y = y_i}}$
Generic response mode shape function $\phi^S_{rs}(x_i, y_i) = R[\phi_{rs}(x_i, y_i)]$	$\phi'_r(x_i)\phi_s(y_i)$	$D[\phi''_r(x_i)\phi_s(y_i) + \nu\phi_r(x_i)\phi''_s(y_i)]$	$D[\phi'''_r(x_i)\phi_s(y_i) + (2-\nu)\phi'_r(x_i)\phi''_s(y_i)]$	$\frac{EZ}{(1-\nu^2)}[\phi''_r(x_i)\phi_s(y_i) + \nu\phi'_r(x_i)\phi''_s(y_i)]$

Note: $Z = h/2$ = max. distance from the neutral surface of the plate.

4.3. Harmonic response analysis

Consider a generic harmonic force with amplitude A_j and excitation frequency ω applied at some location defined by the spatial function $\Gamma(x)$. The harmonic force function can be expressed as follows:

$$f(x,y,t) = f_j(t)\Gamma(x,y) = A_j e^{i\omega t}\Gamma(x,y) \tag{145}$$

Table 4 shows several types of forces for their spatial functions.

From the expansion theorem as shown in Eq. (138), the plate lateral displacement response can also be assumed to be harmonic as follows:

$$\begin{aligned} w(x,y,t) &= \sum_{s=1}^{\infty}\sum_{r=1}^{\infty}\phi_{rs}(x,y)q_{rs}(t) \\ &= \sum_{s=1}^{\infty}\sum_{r=1}^{\infty}\phi_{rs}(x,y)Q_r(\omega)e^{i\omega t} \\ &= W(x,y,\omega)e^{i\omega t} \end{aligned} \tag{146}$$

With the employment of the sensing/response operator R, the response variable $s(x, y, t)$ can be expressed as follows:

$$\begin{aligned} s(x,y,t) &= \sum_{s=1}^{\infty}\sum_{r=1}^{\infty}R[\phi_{rs}(x,y)]q_{rs}(t) \\ &= \sum_{s=1}^{\infty}\sum_{r=1}^{\infty}\phi_{rs}^S(x,y)Q_{rs}(\omega)e^{i\omega t} \\ &= S(x,y,\omega)e^{i\omega t} \end{aligned} \tag{147}$$

where

$$S(x,y,\omega) = \sum_{s=1}^{\infty}\sum_{r=1}^{\infty}\phi_{rs}^S(x,y)Q_{rs}(\omega) \tag{148}$$

By following the TMA procedure as shown in the previous section, one can obtain $Q_{rs}(\omega)$ directly from Eq. (35). Or, with the substitution of Eqs. (145) and (147) into the plate equation in Eq. (119), Eq. (119) is then multiplied by $\phi_{rs}(x, y)$ and integrated over plate length L_x and width L_y, and the amplitude of modal coordinate $Q_{rs}(\omega)$ can be obtained as follows:

$$Q_{rs}(\omega) = \frac{A_j\int_0^{L_y}\int_0^{L_x}\phi_{rs}(x,y)\Gamma(x,y)\,dx\,dy}{(\omega_{rs}^2-\omega^2)+i(2\zeta_{rs}\omega_{rs}\omega)} = \frac{A_j\phi_{rs}^F(x_j,y_j)}{(\omega_{rs}^2-\omega^2)+i(2\zeta_{rs}\omega_{rs}\omega)} \tag{149}$$

The plate response variable $s(x, y, t)$ at $x = x_i$, $y = y_i$ can be obtained from Eq. (147) as follows:

$$s_i = s(x_i,y_i,t) = \sum_{s=1}^{\infty}\sum_{r=1}^{\infty}\phi_{rs}^S(x_i,y_i)Q_r(\omega)e^{i\omega t} = S(x_i,y_i,\omega)e^{i\omega t} = S_i\,e^{i\omega t} \tag{150}$$

where

$$S_i = S(x_i,y_i,\omega) = \sum_{s=1}^{\infty}\sum_{r=1}^{\infty}\phi_{rs}^S(x_i,y_i)Q_r(\omega) \tag{151}$$

Finally, from Eq. (40) the FRF between the generic force amplitude and the response variable amplitude can be obtained as follows:

$$H_{ij} = \frac{S_i}{A_j} = \sum_{s=1}^{\infty}\sum_{r=1}^{\infty}\frac{\phi_{rs}^S(x_i,y_i)\phi_{rs}^F(x_j,y_j)}{(\omega_{rs}^2-\omega^2)+i(2\zeta_{rs}\omega_{rs}\omega)} \tag{152}$$

Tables 4 and 5 show various types of actuator/force and sensor/response for both $\phi^F_{rs}(x_j, y_j)$ and $\phi^S_{rs}(x_i, y_i)$. In particular, as shown in Table 5(b), the resultant bending moment $M_x(x_i, y_i, t)$, shear force $Q_x(x_i, y_i, t)$ and of maximum bending stress $\sigma_x(x_i, y_i, t)$ at location (x_j, y_j) can be conveniently obtained [54]. Wu et al. [56] have revealed a similar concept for modal resultants. The FRF for any combination of actuator/force and sensor/response can be derived from Eq. (152). Chen and Wang [52] presented the same form of equation as shown in Eq. (152) for the use of a PZT actuator and a PVDF sensor in experimental modal testing. Koh and White [32] also presented a similar general form of driving point mobility expressions as the above approach by considering a structural damping effect and suitable for several boundary conditions. In the present work, the advantage of the formulation is that it not only shows the physical meaning for both $\phi^F_{rs}(x_j, y_j)$ and $\phi^S_{rs}(x_i, y_i)$ but is also proof a great convenience for the numerical programming and solution for different complex combinations of actuation, sensing and boundary conditions of plates.

4.4. Spectrum response analysis

The general approach for the plate subject to generic random force excitation is considered in this section. The generic force function can be expressed as follows:

$$f(x, y, t) = f_j(t)\Gamma(x, y) \tag{153}$$

Then, the Fourier spectrum of the jth point random force $f(x, y, t)$ can be obtained:

$$F(x, y, \omega) = \Im[f(x, y, t)] = \Im[f_j(t)]\Gamma(x, y) = F_j(\omega)\Gamma(x, y) \tag{154}$$

where $F_j(\omega)$ is the Fourier spectrum of $f_j(t)$. From Eq. (44), the PSD function for $f(x, y, t)$ can be found as follows:

$$S_{ff}(x_j, y_j, \omega) = [\Gamma(x, y)]^2 S_{f_j f_j}(\omega) \tag{155}$$

From transient response analysis as shown in Eq. (144), the plate response variable $s(x, y, t)$ can be written as follows:

$$s(x, y, t) = \sum_{s=1}^{\infty}\sum_{r=1}^{\infty} \phi^S_{rs}(x, y) q_{rs}(t) \tag{156}$$

The Fourier spectrum and PSD function of $s(x, y, t)$ can also be obtained from Eqs. (48) and (49), respectively, as follows:

$$S(x, y, \omega) = \Im[s(x, y, t)] = \sum_{s=1}^{\infty}\sum_{r=1}^{\infty} \phi^S_{rs}(x, y) Q_{rs}(\omega) \tag{157}$$

$$S_{ss}(x, y, \omega) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{s=1}^{\infty}\sum_{r=1}^{\infty} \phi^S_{rs}(x, y)\phi^S_{mn}(x, y) S_{q_{rs}q_{mn}}(\omega) \tag{158}$$

where $S_{q_{rs}q_{mn}}(\omega)$ is the PSD function of $q_{rs}(t)$ and can be determined from Eq. (57) as follows:

$$S_{q_{rs}q_{mn}}(\omega) = H^*_{rs}(\omega) H_{mn}(\omega) S_{N_{rs}N_{mn}}(\omega) \tag{159}$$

where

$$H_{rs}(\omega) = \frac{Q_{rs}(\omega)}{N_{rs}(\omega)} = \frac{1}{(\omega^2_{rs} - \omega^2) + \mathrm{i}(2\zeta_{rs}\omega_{rs}\omega)} \tag{160}$$

and $S_{N_{rs}N_{mn}}(\omega)$ is the PSD function of $N_{rs}(t)$ and can be determined from Eq. (56)

$$S_{N_{rs}N_{mn}}(\omega) = \phi^F_{rs}(x_j, y_j)\phi^F_{mn}(x_j, y_j) S_{f_j f_j}(\omega) \tag{161}$$

where $\phi^F_{rs}(x_j, y_j)$ is the generic force mode shape function. Table 4 shows some typical force expressions.

Finally, from Eq. (59) and with the imposition of the above equations, the PSD of system response variable $s(x, y, t)$ can be finally expressed:

$$S_{ss}(x,y,\omega) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{s=1}^{\infty}\sum_{r=1}^{\infty} \phi_{rs}^{S}(x,y)\phi_{mn}^{S}(x,y)\phi_{rs}^{F}(x_j,y_j)\phi_{mn}^{F}(x_j,y_j)H_{rs}^{*}(\omega)H_{mn}(\omega)S_{f_jf_j}(\omega) \tag{162}$$

The PSD function of $s_i = s(x_i, y_i, t)$ at (x_i, y_i) location can be obtained:

$$\begin{aligned} S_{s_is_i}(\omega) &= S_{ss}(x_i,y_i,\omega) \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{s=1}^{\infty}\sum_{r=1}^{\infty} \phi_{rs}^{S}(x_i,y_i)\phi_{mn}^{S}(x_i,y_i)\phi_{rs}^{F}(x_j,y_j)\phi_{mn}^{F}(x_j,y_j)H_{rs}^{*}(\omega)H_{mn}(\omega)S_{f_jf_j}(\omega) \end{aligned} \tag{163}$$

From Eq. (60), the rms value of $s_i = s(x_i, y_i, t)$ can then be obtained:

$$\begin{aligned} s_{i,\mathrm{rms}}^2 = \overline{s_i^2} &= \int_{-\infty}^{\infty} S_{s_is_i}(\omega)\,\mathrm{d}\omega \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{s=1}^{\infty}\sum_{r=1}^{\infty} \phi_{rs}^{S}(x_i,y_i)\phi_{mn}^{S}(x_i,y_i)\phi_{rs}^{F}(x_j,y_j)\phi_{mn}^{F}(x_j,y_j) \\ &\quad\times \int_{-\infty}^{\infty} H_{rs}^{*}(\omega)H_{mn}(\omega)S_{f_jf_j}(\omega)\,\mathrm{d}\omega \end{aligned} \tag{164}$$

and

$$\begin{aligned} \sigma_{s_i}^2 &= \overline{s_i^2} - (\bar{s}_i)^2 \\ &= (s_{i,\mathrm{rms}})^2 - (\bar{s}_i)^2 \end{aligned} \tag{165}$$

For the assumption of white noise excitation $S_{f_jf_j}(\omega) = S_0$ with zero mean $\bar{f}_j = E[f_j(t)] = 0$ and the plate with light damping and well-separated modes, Eq. (164) can be simplified as follows according to Eq. (64):

$$\sigma_{s_i}^2 = s_{i,\mathrm{rms}}^2 \approx \sum_{s=1}^{\infty}\sum_{r=1}^{\infty}[\phi_{rs}^{S}(x_i,y_i)\phi_{rs}^{F}(x_j,y_j)]^2 \frac{\pi S_0}{2\zeta_{rs}\omega_{rs}^3} \tag{166}$$

If $\bar{f}_j \neq 0$, the mean of $s_i(t) = s(x_i, y_i, t)$ can be derived from Eq. (69) as follows:

$$\bar{s}_i = \sum_{s=1}^{\infty}\sum_{r=1}^{\infty} \frac{\phi_{rs}^{S}(x_i,y_i)\phi_{rs}^{F}(x_j,y_j)}{\omega_{rs}^2}\bar{f}_j \tag{167}$$

In summary, the statistical quantities of $s_i(t) = s(x_i, y_i, t)$, in terms of mean $\bar{s}_i$ and standard deviation σ_{si}, can be determined and thus the Gaussian distribution function as shown in Eq. (70) can be obtained and used to evaluate the structural response.

In summary, this section presents the general formulation for analytical solutions of plates for four types of vibration analyses. When the force function is identified in terms of spatial function $\Gamma(x, y)$ as shown by some examples in Table 4 and the typical responses of interest are specified as shown in Table 5, both $\phi_r^F(x_j)$ and $\phi_r^S(x_i)$ can be determined, respectively, as shown in Tables 4 and 5. The solutions in terms of generic force/actuator and response/sensor mode shape functions can be fully obtained. The above formulation is suitable for various boundary conditions of plates whose mode shape functions can be defined to maintain the orthogonal properties as shown in Eqs. (130)–(132). Although only the simply supported uniform plate is illustrated, the systematic solutions for transient, harmonic and spectrum analyses can be easily adapted for complex plates as well, such as with non-uniform thickness and different boundary conditions.

5. Conclusions

This work generalizes the theoretical solution of four types of vibration analyses for continuous structure systems with damping consideration subject to various forms of actuation forces and sensing responses. Both one-dimensional beam and two-dimensional rectangular plate problems are presented to illustrate the mathematical derivation through the general formulation. The generic formulation can provide a systematic approach to deal with the mathematical derivation of system responses of interest for complex combinations of different types of structures, boundary conditions and forcing functions. The formulation is straightforward and can be easily adapted to different structures with various boundaries and force conditions. The solutions for transient dynamic analysis, harmonic response analysis and spectrum analysis, considering various forms of actuation forces and sensing response, can be expressed in a concise format by generic force/actuator and response/sensor mode shape functions, respectively. The physical quantities can be well interpreted and are useful for engineering design analysis as well as for other applications such as active control, damage detection and force prediction. Although only beam and plate structures are shown, the theoretical formulation in this work is applicable to other continuous structure systems, such as strings, bars, shafts and membranes, as well.

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