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Free vibrations and stability of stepped columns with cracks

S. Kukla

Częstochowa University of Technology, Dąbrowskiego 73, 42-200 Częstochowa, Poland

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Abstract

In this paper, a solution to the free vibration problem of a stepped column with cracks is presented. The open cracks occur at step changes in the cross-section of the column or at the intermediate points of the uniform segments. The cracks in the column are represented by massless rotational springs. The frequency equation is obtained by using properties of the Green's functions corresponding to the uniform segments of the column. The approach pertains to the vibration of columns consisting of an arbitrary number of uniform segments. The stability of the column, which depends on the position and size of the cracks, is numerically investigated.

1. Introduction

The presence of damage in the components of a structure can have an influence on the dynamic behaviour of the whole structure and can lead to its failure. Catastrophic failure can be caused by the fatigue cracking of a structural member. To predict the failure, vibration monitoring can be used to detect change in the dynamic characteristics of the structure. If we are to correctly interpret any observed changes, we need to be acquainted with the influence of damage on the dynamic characteristics of structures. For this reason, knowledge of the effects of damage on the vibration of the structures is of such great interest to engineers and designers. In particular, understanding the effect of cracks on the vibration and stability of columns is very important because of the practical applications.

The presence of cracks in a structure changes the structure flexibility and influences the dynamic response of this structure. Sensitivity analysis of structures to cracks was presented by Chondros and Dimarogonas [1]. The influence of crack on the dynamic characteristics of structures can be simulated by continuous crack flexibility. This approach was discussed by Chondros in paper [2]. The continuous cracked beam vibration theory for lateral vibration of beams with open cracks was developed by Chondros et al. [3], and a review of the problems to vibration of cracked structures was published by Dimarogonas in paper [4].

Determining the dynamic characteristics of beams and columns with cracks on the basis of mathematical models has been the subject of several studies [5–13]. In these studies the effect of a crack is simulated by the flexibility of a massless rotational spring. In this model, the local flexibility of the beam at the location of the

E-mail address: kukla@imi.pcz.pl

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crack is represented by a function of the crack depth. This function for one- or two-sided open cracks is derived on the basis of fracture mechanics concepts [5–8].

The effect of the cracks on the free vibration frequencies of uniform beams has been investigated by many researchers, e.g. Refs. [5–9]. In these papers, the vibrations of cracked Euler–Bernoulli beams were considered. Ostachowicz and Krawczuk in paper [5] presented a solution to a problem for a cantilever beam with two open cracks by using an exact method. The frequency equation for beams with an arbitrary number of cracks using a transfer matrix method was derived by Lin et al. [6]. In both papers the axial loads of the beams were not taken into consideration. The effect of axial loads on the vibrations of uniform, cracked beams was investigated in Refs. [7–9]. Masoud et al. [7] present experimental and theoretical results concerning an axially loaded fixed–fixed cracked beam, and Binici [8] compared the results obtained by using the proposed approach with the results obtained using the finite element method for axially loaded beams with multiple cracks. The stability behaviour of a column with a single crack, subjected to follower or vertical loads, was studied by Anifantis and Dimarogonas [9].

The vibration problems of axially loaded stepped Euler–Bernoulli beams were considered in Refs. [10–13]. In paper [10], DeRosa used an exact approach to derive the frequency equation of a stepped beam with follower forces at the step. The exact method was also applied by Naguleswaran [11] to solve the problem of the free vibration and stability of a stepped beam under different axial force in the beam segments. The method consists of dividing the beam into uniform segments and using the conditions of continuity of deflection and slope, compatibility of bending moment and shearing force at the steps. This approach can be applied to the vibration problems of stepped beams as well as of cracked ones.

The dynamic flexibility method was applied to vibration problems of stepped Euler–Bernoulli beams by Lee and Bergman [12]. The solution to the problem is obtained in terms of the dynamic Green's function. Examples of this method, being applied to the frequency analysis of axially loaded stepped beams, are presented by Kukla and Zamojska [13]. The solution to the problem is expressed by those Green's functions corresponding to the uniform segments of the stepped beam. The Green's functions are obtained as solutions of auxiliary boundary problems [12,13]. Various methods for deriving the Green's functions are presented in a book by Duffy [14].

In this paper, an analysis of stability and vibration of a column with transverse cracks is presented. The formulation and solution of the problem concern an axially loaded stepped column with an arbitrary number of open cracks. The cracks are represented by rotational springs. The solution to the free vibration problem is obtained by using the Green's function method.

2. Formulation of the problem

Consider a stepped column consisting of n segments of constant cross-sections (Fig. 1). Based on the assumption that the shear deformation and rotary inertia are negligible, the differential equations for the transverse displacements w_i of the column segments are

$$\mathrm{EI}_i \frac{\partial^4 w_i}{\partial x^4} + P_i \frac{\partial^2 w_i}{\partial x^2} + \rho A_i \frac{\partial^2 w_i}{\partial t^2} = 0, \quad i = 1, 2, \ldots, n \tag{1}$$

where P_i is the axial load acting on the i-th segment of the column, ρA_i and EI_i are the mass per unit length and the flexural rigidity of the i-th segment, respectively.

The step-wise changes of geometric and/or physical parameters of the considered column are at points x_1, $x_2, \ldots, x_n$. We assume that a crack can occur at each of these points. These points determine the uniform segments of the column. In the model it is assumed that the uniform segments are connected by a rotational spring, and as such the following conditions are satisfied at these points:

$$w_{i+1}(x_i, t) - w_i(x_i, t) = 0 \tag{2}$$

$$\frac{\partial w_{i+1}(x_i, t)}{\partial x} - \frac{\partial w_i(x_i, t)}{\partial x} = \theta_i L \frac{\partial^2 w_i(x_i, t)}{\partial x^2} \tag{3}$$

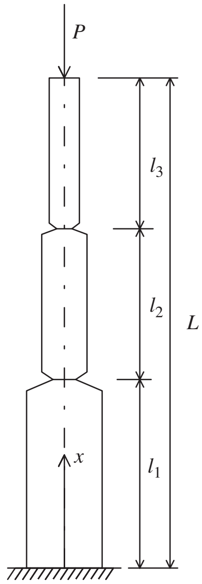

Fig. 1. A sketch of the considered system.

$$\mathrm{EI}_i \frac{\partial^2 w_i(x_i,t)}{\partial x^2} = \mathrm{EI}_{i+1} \frac{\partial^2 w_{i+1}(x_i,t)}{\partial x^2} = m_i(t) \tag{4}$$

$$\mathrm{EI}_i \frac{\partial^3 w_i(x_i,t)}{\partial x^3} + P_i \frac{\partial w_i(x_i,t)}{\partial x} = \mathrm{EI}_{i+1} \frac{\partial^3 w_{i+1}(x_i,t)}{\partial x^3} + P_{i+1} \frac{\partial w_{i+1}(x_i,t)}{\partial x} = s_i(t), \quad i = 1, 2, \ldots, n-1 \tag{5}$$

where θ_i is the parameter characterising the rotational spring which models the i-th crack, which occurs between the i-th and $(i+1)$-th segments of the column [5]. Functions $m_i(t)$ and $s_i(t)$, respectively, represent the bending moment and shear force acting on the right end of the i-th segment. A schematic diagram of the i-th segment of the considered column is shown in Fig. 2.

Functions w_1 and w_n, which describe the transverse displacements of the edge segments, satisfy boundary conditions and how the column ends are constrained. In this study it is assumed that the column is clamped at $x = 0$ and free at $x = L$, and as such the conditions for functions w_1 and w_n are

$$w_1|_{x=0} = \left.\frac{\partial w_1}{\partial x}\right|_{x=0} = 0 \tag{6}$$

$$\mathrm{EI}_n \left.\frac{\partial^2 w_n}{\partial x^2}\right|_{x=L} = 0, \quad \left.\left(\mathrm{EI}_n \frac{\partial^3 w_n(L)}{\partial x^3} + P_n \frac{\partial w_n(L)}{\partial x}\right)\right|_{x=L} = 0 \tag{7}$$

For the free vibration of the column, we assume that

$$w_i(x,t) = Y_i(x) \cos \omega t, \quad m_i(t) = \overline{M}_i \cos \omega t, \quad s_i(t) = \overline{S}_i \cos \omega t \tag{8}$$

where ω is the natural frequency of the column. Substituting Eq. (8) into Eqs. (1)–(7), we obtain

$$\frac{\mathrm{d}^4 Y_i}{\mathrm{d}x^4} + 2\mu_i p \frac{\mathrm{d}^2 Y_i}{\mathrm{d}x^2} - \kappa_i^4 \Omega^4 Y_i = 0, \quad i = 1,2,\ldots,n \tag{9}$$

$$Y_1(0) = \frac{\mathrm{d}Y_1(0)}{\mathrm{d}x} = 0 \tag{10}$$

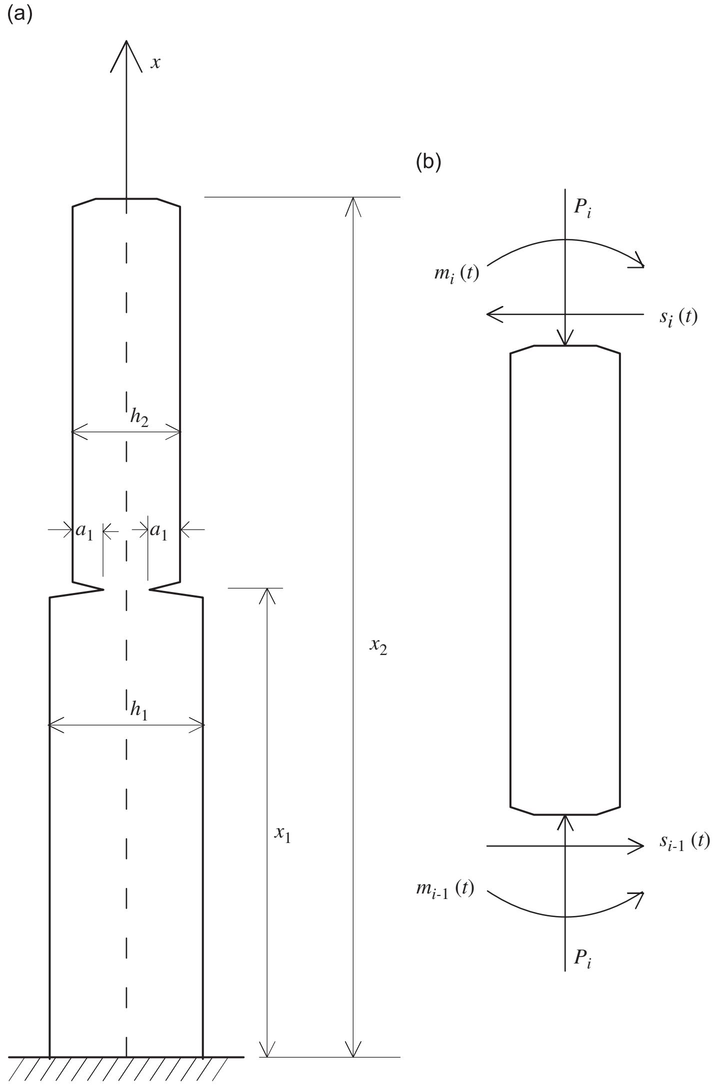

Fig. 2. Segments of the column, (a) two first segments with a crack at the step, and (b) end conditions for forces of the column segment.

$$\frac{\mathrm{d}^2 Y_n(L)}{\mathrm{d}x^2} = 0, \quad \frac{\mathrm{d}^3 Y_n(L)}{\mathrm{d}x^3} + 2\mu_n p \frac{\mathrm{d}\, Y_n(L)}{\mathrm{d}x} = 0 \tag{11}$$

$$Y_{i+1}(x_i) - Y_i(x_i) = 0 \tag{12}$$

$$\frac{\mathrm{d}\, Y_{i+1}(x_i)}{\mathrm{d}x} - \frac{\mathrm{d}\, Y_i(x_i)}{\mathrm{d}x} = \theta_i L \frac{\mathrm{d}^2 Y_i(x_i)}{\mathrm{d}x^2} \tag{13}$$

$$\frac{\mathrm{d}^2 Y_i(x_i)}{\mathrm{d}x^2} = \frac{1}{b_i} M_i, \quad \frac{\mathrm{d}^2 Y_{i+1}(x_i)}{\mathrm{d}x^2} = \frac{1}{b_{i+1}} M_i \tag{14}$$

$$\frac{\mathrm{d}^3 Y_i(x_i)}{\mathrm{d}x^3} + 2\mu_i p \frac{\mathrm{d}\, Y_i(x_i)}{\mathrm{d}x} = \frac{1}{b_i} S_i,$$

$$\frac{d^3 Y_{i+1}(x_i)}{dx^3} + 2\mu_{i+1} p \frac{dY_{i+1}(x_i)}{dx} = \frac{1}{b_{i+1}} S_i, \quad i = 1, 2, \ldots, n-1 \tag{15}$$

c_i where $\rho A_i = a_i \rho A$, $\mathrm{EI}_i = b_i \mathrm{EI}$, $P_i = c_i P$, $\sum_{i=1}^{n} = 1$, $\kappa_i^4 = a_i/b_i$, $\mu_i = c_i/b_i$, $p = P/2\mathrm{EI}$, $\Omega^4 = (\rho A/\mathrm{EI})\omega^2$, $M_i = \overline{M}_i/\mathrm{EI}$, $S_i = \overline{S}_i/\mathrm{EI}$, EI, and ρA are the reference stiffness and mass per unit length of the column, respectively.

The condition in Eq. (13) after taking into account Eq. (14a) can be rewritten as

$$\frac{dY_{i+1}(x_i)}{dx} - \frac{dY_i(x_i)}{dx} = \theta_i \frac{L}{b_i} M_i \tag{16}$$

For the open cracks considered here, θ_i is a function of the ratio between the depth of the crack a_i and half of the height h_i of the i-th segment of the column. For double-sided open cracks, the function is given by [5,7]

$$\theta_i(\gamma_i) = 6\pi\gamma_i^2 \frac{\overline{h}_i}{L}(0.5033 - 0.9022\gamma_i + 3.412\gamma_i^2 - 3.181\gamma_i^3 + 5.793\gamma_j^4) \tag{17}$$

where $\gamma_i = a_i/\overline{h}_i$, $\overline{h}_i = \min(h_i, h_{i+1})$ for $i = 1, \ldots, n-1$.

3. Solution to the problem

The solution to the free vibration problem is obtained by using the properties of the Green's functions. The Green's functions G_i $(i = 1, 2, \ldots, n)$, which are necessary in the application of this approach, satisfy differential equations [14]

$$\frac{d^4 G_i}{dx^4} + 2\mu_i p \frac{d^2 G_i}{dx^2} - \kappa_i^4 \Omega^4 G_i = \delta(x - \xi) \tag{18}$$

where $\delta(\cdot)$ is Dirac's delta function. Moreover, function G_1 satisfies the homogeneous boundary conditions corresponding to the clamped–free beam-column

$$G_1|_{x=0} = 0, \quad \left.\frac{\partial G_1}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial^2 G_1}{\partial x^2}\right|_{x=x_1} = 0, \quad \left.\left(\frac{\partial^3 G_1}{\partial x^3} + 2\mu_n p \frac{\partial G_1}{\partial x}\right)\right|_{x=x_1} = 0 \tag{19}$$

and the functions G_i $(i = 2, \ldots, n)$, satisfy the homogeneous boundary conditions corresponding to the free–free beam-column

$$\left.\frac{\partial^2 G_i}{\partial x^2}\right|_{x=x_{i-1}} = 0, \quad \left.\left(\frac{\partial^3 G_i}{\partial x^3} + 2\mu_i p \frac{\partial G_i}{\partial x}\right)\right|_{x=x_{i-1}} = 0, \quad \left.\frac{\partial^2 G_i}{\partial x^2}\right|_{x=x_i} = 0, \quad \left.\left(\frac{\partial^3 G_i}{\partial x^3} + 2\mu_i p \frac{\partial G_i}{\partial x}\right)\right|_{x=x_i} = 0 \tag{20}$$

The derivation of the Green's functions for vibration problems of axially loaded stepped beams was presented in Ref. [13]. The functions $G_1(x, \xi)$ and $G_i(x, \xi)$ for $i = 2, \ldots, n$, with the notation applied in this paper are given in Appendix A.

On the basis of Eq. (9), which corresponds to the uniform segments of the column, the following equations can be written:

$$\int_{x_{i-1}}^{x_i} \left(\frac{d^4 Y_i}{d\xi^4} + 2\mu_i p \frac{d^2 Y_i}{d\xi^2} - \kappa_i^4 \Omega^4 Y_i\right) G_i(\xi, x)\, d\xi = 0, \quad i = 1, \ldots, n \tag{21}$$

where $x_0 = 0$. Integrating the above integrals by parts and using the conditions in Eqs. (10) and (11), (14) and (15) and (19) and (20), we obtain

$$\begin{aligned}\int_{x_{i-1}}^{x_i} \left(\frac{\partial^4 G_i}{\partial \xi^4} + 2\mu_i p \frac{\partial^2 G_i}{\partial \xi^2} - \kappa_i^4 \Omega^4 G_i\right) Y_i(\xi)\, d\xi = & -\frac{1}{b_i}[S_i G_i(x, x_i) - S_{i-1} G_i(x, x_{i-1})] \\ & + \frac{1}{b_i}\left[M_i \frac{\partial G_i(x, x_i)}{\partial \xi} - M_{i-1} \frac{\partial G_i(x, x_{i-1})}{\partial \xi}\right]\end{aligned} \tag{22}$$

The left-hand side of Eq. (22) after using Eq. (18) assumes the form

$$\int_{x_{i-1}}^{x_i}\left(\frac{\partial^4 G_i}{\partial \xi^4}+2\mu_i p\frac{\partial^2 G_i}{\partial \xi^2}-\kappa_i^4\Omega^4 G_i\right)Y_i(\xi)\,\mathrm{d}\xi=\int_{x_{i-1}}^{x_i}\delta(\xi-x)Y_i(\xi)\,\mathrm{d}\xi \tag{23}$$

The property of the Dirac delta function gives [14]

$$\int_{x_{i-1}}^{x_i}\delta(\xi-x)Y_i(\xi)\,\mathrm{d}\xi=Y_i(x),\quad x_{i-1}<x<x_i \tag{24}$$

Taking into account Eqs. (23) and (24) in Eq. (22), we have

$$Y_i(x)=\frac{1}{b_i}\left[-\frac{\partial G_i(x,x_{i-1})}{\partial \xi}M_{i-1}+G_i(x,x_{i-1})S_{i-1}+\frac{\partial G_i(x,x_i)}{\partial \xi}M_i-G_i(x,x_i)S_i\right] \tag{25}$$

where $M_0=M_n=0$, $S_0=S_n=0$.

The functions $Y_i(x)$ given by Eq. (25) are expressed by the Green's functions and their derivatives. An unknown frequency parameter Ω, which occurs in the equation, can be determined by using the conditions in Eqs. (12) and (16). After substituting functions $Y_i(x)$ into these conditions, a set of equations is obtained. This set of equations may be written in the following matrix form:

$$\mathbf{A}(\omega)\,\mathbf{x}=\mathbf{0} \tag{26}$$

where $\mathbf{x}=[M_1\ldots S_1\ldots M_{n-1}\ldots S_{n-1}]^{\mathrm{T}}$ and $\mathbf{A}(\omega)=[a_{ij}]_{1\leqslant i,\ j\leqslant 2(n-1)}$. The non-zero elements a_{ij} of matrix $\mathbf{A}$ are given in Appendix B.

A nontrivial solution to Eq. (26) exists when the determinant of matrix $\mathbf{A}$ is set equal to zero, yielding the frequency equation of the stepped column with n cracks:

$$\det \mathbf{A}(\omega)=0 \tag{27}$$

Frequency equation (27) with respect to the circular frequencies ω_k is then solved numerically using an approximate method. The mode shapes corresponding to the frequencies are given by Eq. (25) when the $(2n-1)$ coefficients M_1, S_1, …, M_{n-2}, S_{n-2}, and M_{n-1}, in dependence with S_{n-1}, are determined from Eq. (26).

It should be noted that the presented formulation and solution to the vibration problem concern the case of a column with cracks located at intermediate points on the uniform segments. (If two adjacent segments have

Table 1
First two critical forces of a one-stepped column with crack at the step

A_2/A_1	$\gamma_1=0$		$\gamma_1=0.25$		$\gamma_1=0.5$		$\gamma_1=0.75$	
	p_1	p_2	p_1	p_2	p_1	p_2	p_1	p_2
$l_1=0.375L$								
1.0	2.4674	22.2066	2.4187	22.1820	2.1451	22.0228	1.3069	21.1497
0.8	***1.8546***	***14.5763***	1.8323	14.5699	1.7007	14.5295	1.2128	14.3291
0.6	1.0594	8.6943	1.0540	8.6735	1.0210	8.5451	0.8696	7.9652
0.4	0.3744	3.3308	0.3740	3.3272	0.3713	3.3046	0.3572	3.1886
$l_1=0.5L$								
1.0	2.4674	22.2066	2.4320	21.8878	2.2236	20.0457	1.4793	14.8092
0.8	***2.0839***	***14.6739***	2.0626	14.6371	1.9348	14.4058	1.4303	13.3155
0.6	1.3918	9.1850	1.3844	9.1845	1.3388	9.1814	1.1318	9.1645
0.4	0.5549	4.6677	0.5541	4.6631	0.5493	4.6342	0.5244	4.4861
$l_1=0.625L$								
1.0	2.4674	22.2066	2.4454	21.6013	2.3095	18.3958	1.7181	11.2621
0.8	2.2809	15.7419	2.2648	15.5579	2.1648	14.4837	1.7149	10.9501
0.6	1.8227	9.4312	1.8140	9.4147	1.7598	9.3116	1.5057	8.8163
0.4	0.8952	5.5777	0.8937	5.5772	0.8843	5.5737	0.8359	5.5544

The values written in bold-italic are in accordance with the results presented in Ref. [11].

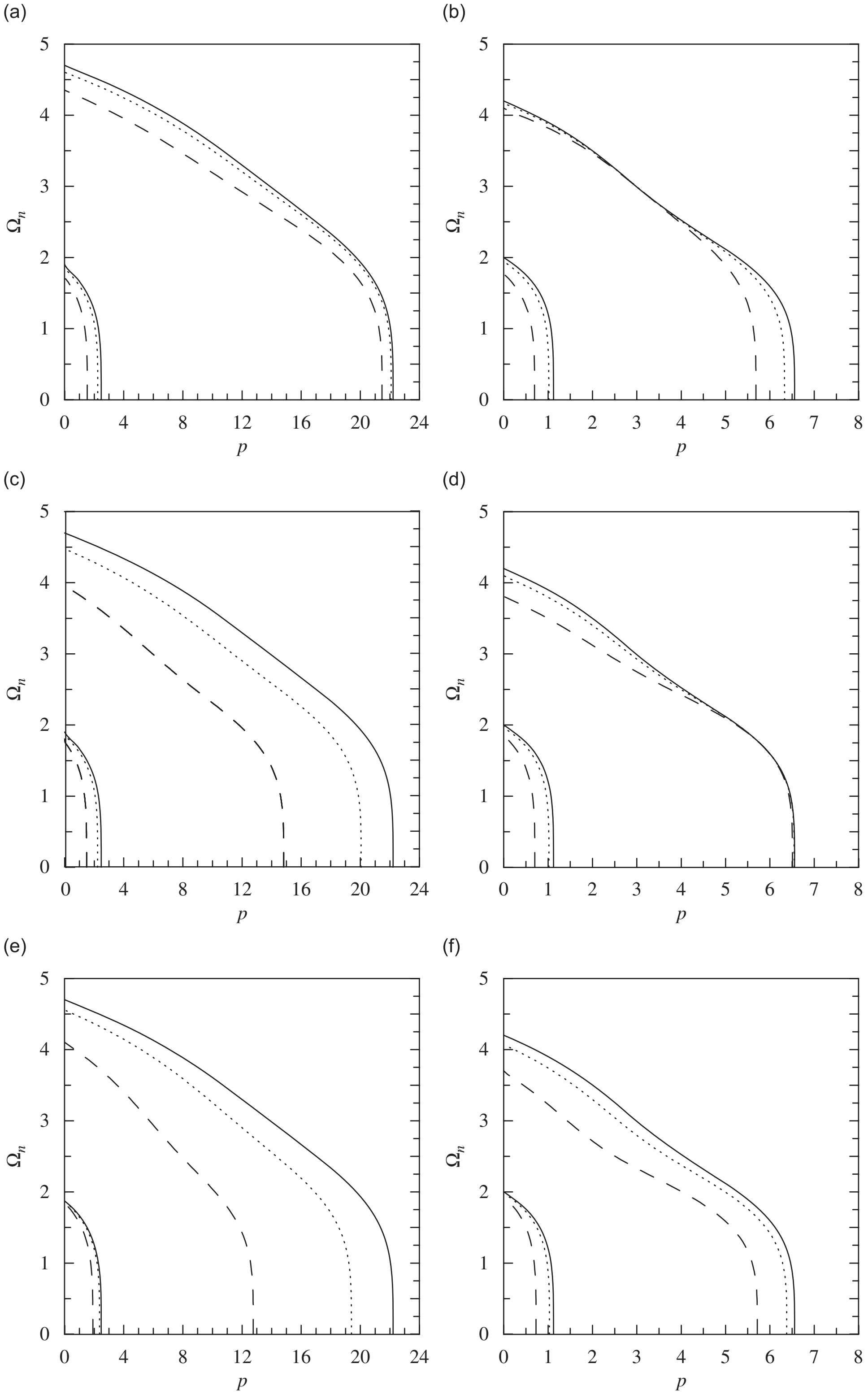

Fig. 3. Frequency curves for uniform (a, c, e) and tapered (b, d, f) columns with one open crack at location: $x_1 = 0.375L$ (a, b), $x_1 = 0.5L$ (c, d), $x_1 = 0.625L$ (e, f); $\gamma_1 = 0$ ———, $\gamma_1 = 0.5$ · · · · · · · · · ·, $\gamma_1 = 0.75$ – – – – – – –.

the same geometrical dimensions and physical properties, then the two parts of the column can be treated as one segment with a crack).

4. Numerical examples

The presented solution was used in the numerical analysis of free vibration and stability of stepped columns with cracks. The first example concerns a one-stepped column with a crack at the step. The calculations were performed for columns with a step in three different positions: $l_1 = 0.375$; 0.5; and $0.675L$. For each considered column, four cases of the crack depths ($\gamma_1 = 0$; 0.25; 0.5; and 0.75) and four values of the ratios of the width of the column segments ($A_2/A_1 = 0.4$; 0.6; 0.8; and 1.0) are assumed. The two critical forces calculated for these columns are presented in Table 1. The values of the critical forces obtained here for $\gamma_1 = 0$, $A_2/A_1 = 0.8$; $l_1 = 0.375L$ and $l_1 = 0.5L$, are in accordance with the results presented by Naguleswaran [11]. (The remaining results are calculated for other data).

The free vibration frequencies of a non-uniform column can be approximated by frequencies of the multi-stepped columns. The approach presented in this paper enables an arbitrary number of uniform segments,

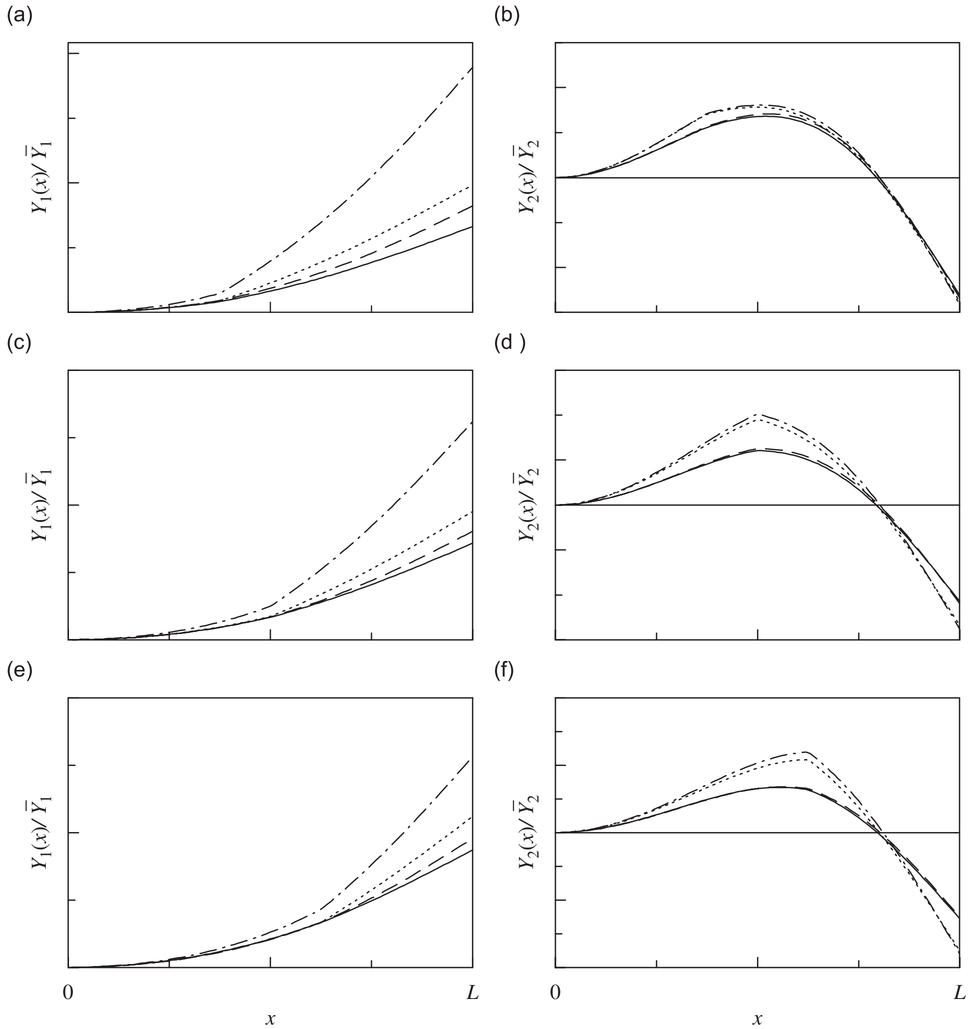

Fig. 4. First and second mode shapes of the column with a double-sided crack located at $x_1/L = 0.375$ (a, b), $x_1/L = 0.5$ (c, d), $x_1/L = 0.625$ (e, f) for different axial loads and crack ratio, $\gamma_1 = 0.5$, $p = 0.15$ ——— , $\gamma_1 = 0.7$, $p = 0.15$ · · · · · · · · · · , $\gamma_1 = 0.5$, $p = 0.3$ −−−−−−−, $\gamma_1 = 0.7$, $p = 0.3$ − · − · − · − · − · −.

which create a stepped column, to be taken into account. The approximation of the shape of the non-uniform column by the stepped column with a suitably large number of uniform segments leads to sufficiently accurate numerical results. The results are valid when the assumptions concerning the crack model are satisfied.

The frequency curves presented in Fig. 3 concern columns with one open crack which are axially loaded by a force that is represented by parameter p. Numerical calculations were performed for two cases of the cross-section of the column: (1) quadratic cross-section with a constant side: $h_0 = 0.035L$ (Fig. 3(a, c, e)), (2) rectangular cross-section, one side of which is constant and the second varies as a linear function of the axial coordinate: $h_0 = 0.05L$ at the clamped end and $h_I = 0.02L$ at the free end of the column (Fig. 3(b, d, f)). Three locations of the crack on the column are assumed: $x_1 = 0.375L$ (Fig. 3(a, b)), $x_1 = 0.5L$ (Fig. 2 (c, d)), $x_1 = 0.625L$ (Fig. 3 (e, f)), and three values of the parameter characterising the flexibility of the column at the crack location are considered: $\gamma_1 = 0$ (solid lines in Fig. 3), $\gamma_1 = 0.5$ (dotted lines), and $\gamma_1 = 0.75$ (dashed lines). The results of calculations show that the non-uniformity of the column as well as the location and size of the crack can significantly affect the eigenfrequencies of the column. In particular, a crack in the column causes a decrease in the critical forces.

The mode shapes corresponding to the first two frequencies of the tapered column with one double-sided open crack are presented in Fig. 4. The solid and dashed lines are obtained for crack ratio $\gamma_1 = 0.5$ and the axial loads are characterised by dimensionless parameters $p = 0.15$ and $S = 0.3$, respectively. Similarly, the dotted and dashed–dotted lines are plotted for the same p parameter, but the crack ratio is assumed as $\gamma_1 = 0.7$. In the calculations, the location of the crack on the column are assumed as: $x_1 = 0.375L$ (Fig. 4(a, b)), $x_1 = 0.5L$ (Fig. 4(c, d)), and $x_1 = 0.625L$ (Fig. 4(e, f)). The curves show changes in the ratios $Y_i(x)/\overline{Y}_i$, where $Y_i(x)$ is the mode shape of the i-th segment of the cracked column and $\overline{Y}_i$ is the amplitude of displacement of the non-cracked column at $x = x_1$. The effect of the crack on the mode shape is rather small. Therefore, it is necessary to use the information on changes in the eigenfrequencies and mode shapes to predict the crack position and the crack ratio.

5. Conclusions

In this paper a solution to the free vibration problem of axially loaded stepped columns with open cracks has been presented. The frequency equation was obtained by using the properties of the Green's functions. The Green's functions corresponding to the clamped–free and free–free beam columns were used. The formulation and solution of the problem concern stepped columns with an arbitrary number of cracks. The local flexibilities introduced into the structure by the cracks are represented by rotational springs. Stepped columns with cracks, which consist of any number of uniform segments, may be used as an approximation of non-uniform columns, with cracks whose cross-sectional area continuously varies.

The changes in the natural frequencies observed in vibration monitoring of structures can indicate damage to their elements. Identifying damage at a primary stage when it first arises is vitally important. However, the small changes in the eigenfrequencies are difficult to investigate in practice. In this paper, the exact solution to the vibration problem can be used to investigate the influence of the crack on the column vibration in the case of small changes in the eigenfrequencies.

The local flexibilities of the column introduced by cracks cause a decrease in the eigenfrequencies and the critical forces of this column. Numerical example shows that the non-uniformity of the column as well as the location and size of the crack can significantly affect the eigenfrequencies and critical forces of the column. Although the examples presented in this paper concern stepped columns with one open crack, the approach may be used in the vibration analysis of the columns with an arbitrary number of cracks.

Appendix A

In this paper Green's functions G_i, which are solutions to boundary problems in Eqs. (18 and 19) or Eqs. (18 and 20) are used. To present the solutions to these problems, four linearly independent solutions to a homogeneous differential equation associated with the four-order Eq. (18) are introduced:

$$\phi_i^{(0)}(x) = \cosh\beta_i x - \cos\alpha_i x, \quad \phi_i^{(1)}(x) = \beta_i \sinh\beta_i x + \alpha_i \sin\alpha_i x,$$
$$\phi_i^{(2)}(x) = \beta_i^2 \cosh\beta_i x + \alpha_i^2 \cos\alpha_i x, \quad \phi_i^{(3)}(x) = \beta_i^3 \sinh\beta_i x - \alpha_i^3 \sin\alpha_i x$$

where $\alpha_i = \sqrt{\mu_i p + \sqrt{\mu_i^2 p^2 + \kappa_i^4 \Omega^4}}$, $\beta_i = \sqrt{-\mu_i p + \sqrt{\mu_i^2 p^2 + \kappa_i^4 \Omega^4}}$. To abbreviate the notation, the following functions are used:

$$\overline{\phi}_i^{(1)}(x) = \frac{\sinh\beta_i x}{\beta_i} - \frac{\sin\alpha_i x_i}{\alpha_i}, \quad \overline{\phi}_i^{(2)}(x) = \frac{\cosh\beta_i x}{\beta_i^2} + \frac{\cos\alpha_i x}{\alpha_i^2}, \quad \overline{\phi}_i^{(3)}(x) = \frac{\sinh\beta_i x}{\beta_i^3} + \frac{\sin\alpha_i x_i}{\alpha_i^3}$$

Using the functions $\phi_i^{(j)}(x)$ and $\overline{\phi}_i^{(j)}(x)$, the Green's function G_1, corresponding to a clamped–free beam, may be written in the form [13]

$$G_1(x,\xi;\Omega) = \frac{1}{\alpha_1^2 + \beta_1^2}[C_{11}(\xi)\phi_1^{(0)}(x) + C_{12}(\xi)\bar{\phi}_1^{(1)}(x) + \bar{\phi}_1^{(1)}(x-\xi)H(x-\xi)] \tag{A1}$$

where H denotes a Heaviside function and $C_{11}(\xi)$ and $C_{12}(\xi)$ are

$$C_{11}(\xi) = \frac{1}{D_1}\left\{\phi_1^{(1)}(x_1)\overline{\phi}_i^{(2)}(x_1-\xi) - \overline{\phi}_1^{(2)}(x_1)\phi_1^{(1)}(x_1-\xi)\right\}$$

$$C_{12}(\xi) = \frac{1}{D_1}\left\{\bar{\phi}_1^{(1)}(x_1)\phi_1^{(1)}(x_1-\xi) - \phi_1^{(2)}(x_1)\bar{\phi}_1^{(2)}(x_1-\xi)\right\}$$

where $D_1 = \phi_1^{(2)}(x_1)\overline{\phi}_1^{(2)}(x_1) - \phi_1^{(1)}(x_1)\overline{\phi}_1^{(1)}(x_1)$.

The Green's function G_i, corresponding to free–free beams, can be written in the form

$$G_i(x,\xi;\Omega) = C_{i1}(\xi)\phi_i^{(1)}(x-x_{i-1}) + C_{i2}(\xi)\bar{\phi}_i^{(2)}(x-x_{i-1})$$
$$+ \bar{\phi}_i^{(1)}(x-\xi)H(x-\xi), \quad i = 2,3,\ldots,n \tag{A2}$$

where functions $C_{i1}(\xi)$, $C_{i2}(\xi)$ are

$$C_{i1}(\xi) = \frac{1}{D_i}\{\phi_i^{(0)}(l_i)\overline{\phi}_i^{(2)}(x_i-\xi) - \overline{\phi}_i^{(3)}(l_i)\phi_i^{(1)}(x_i-\xi)\}$$

$$C_{i2}(\xi) = \frac{1}{D_i}\{\phi_i^{(0)}(l_i)\phi_i^{(1)}(x_i-\xi) - \phi_i^{(3)}(l_i)\overline{\phi}_i^{(2)}(x_i-\xi)\}$$

where $D_i = \phi_1^{(3)}(l_i)\overline{\phi}_1^{(3)}(l_i) - [\phi_i^{(0)}(l_i)]^2$.

Appendix B

The non-zero elements a_{ij} of the matrix **A** which occur in Eq. (25) are as follows (n is the number of uniform segments of the stepped beam):

- for $i = 1, 2, \ldots, n-1$

$$a_{2i-1,\,2i-1} = -\frac{\partial G_i(x_i,x_i)}{\partial\xi} - \mu_i\frac{\partial G_{i+1}(x_i,x_i)}{\partial\xi}, \quad a_{2i-1,2i} = G_i(x_i,x_i) + \mu_i G_{i+1}(x_i,x_i),$$
$$a_{2i,\,2i-1} = -\frac{\partial^2 G_i(x_i,x_i)}{\partial\xi\partial x} - \mu_i\frac{\partial^2 G_{i+1}(x_i,x_i)}{\partial\xi\partial x} - \theta_i, \quad a_{2i,2i} = \frac{\partial G_i(x_i,x_i)}{\partial x} + \mu_i\frac{\partial G_{i+1}(x_i,x_i)}{\partial x}$$

- for $i = 1, 2, \ldots, n-2$

$$a_{2i-1,\,2i+1} = \mu_i \frac{\partial G_{i+1}(x_i, x_{i+1})}{\partial \xi}, \quad a_{2i-1,2i+2} = -\mu_i G_{i+1}(x_i, x_{i+1})$$

$$a_{2i,\,2i+1} = \mu_i \frac{\partial^2 G_{i+1}(x_i, x_{i+1})}{\partial \xi \partial x}, \quad a_{2i,2i+2} = -\mu_i \frac{\partial G_{i+1}(x_i, x_{i+1})}{\partial x}$$

- for $i = 2, 3, \ldots, n-1$

$$a_{2i-1,\,2i-3} = \frac{\partial G_i(x_i, x_{i-1})}{\partial \xi}, \quad a_{2i-1,2i-2} = -G_i(x_i, x_{i-1})$$

$$a_{2i,\,2i-3} = \frac{\partial^2 G_i(x_i, x_{i-1})}{\partial \xi \partial x}, \quad a_{2i,2i-2} = -\frac{\partial G_i(x_i, x_{i-1})}{\partial x}$$

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