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Identification of multiple cracks in a beam using natural frequencies

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Abstract

A simple method to identify multiple cracks in a beam is presented. The cracks are modeled as rotational springs and the forward problem is solved using the finite element method. The inverse problem is solved iteratively for the locations and sizes of the cracks using the Newton–Raphson method. Numerical examples are provided for the identification of triple cracks in a cantilever beam as well as double cracks. The detected crack locations and sizes are in excellent agreement with the actual ones.

1. Introduction

Due to its theoretical and practical importance, the crack identification problem in structures has been extensively investigated and many methods were proposed as can be found in the survey of Dimarogonas [1].

The majority of studies concerning crack identification in a beam dealt with a single crack case. The frequency contour plot method [2–8] had been one of the most favored tools to identify a single crack using the lowest three natural frequencies. Liang et al. [2] proposed that the location and the size of a crack could be identified through finding the intersection point of three frequency contour lines. The beam was based on the Euler–Bernoulli beam theory and the crack was modeled as a massless rotational spring. The scheme was adopted in the crack detection in stepped beams [3] and truncated wedged beams [4,5]. The crack was assumed to be open and normal to the beam surface in most studies. Nandwana and Maiti [6] studied the crack identification problem when the crack was inclined edge type or when the crack was beneath the beam surface. Lele and Maiti [7] and Nikolakopoulos et al. [8] extended the frequency contour plot method to the crack identification in beams based on the Timoshenko beam theory and in plane frame, respectively. In many cases, however, the three curves of frequency contour plot did not intersect because of inaccuracies in the modeling as compared to measured results, and the zero-setting procedure was recommended [3–7]. Owolabi et al. [9] used the changes in frequencies and amplitudes to detect a crack in a beam where the experimentally obtained

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frequency response functions were curve-fitted and the frequency contours were plotted using the lowest three natural frequencies.

It was also suggested that two measurements were sufficient to detect a crack in a beam. Rizos et al. [10] developed a crack identification method which needed amplitude measurements at two positions of the beam. Narkis [11] showed that if the crack was very small the only information required for the crack identification was the variation of the first two natural frequencies due to a crack. Dado [12] presented a direct mathematical model to detect a crack in a beam, where the lowest two natural frequencies were required as input data.

Hu and Liang [13] introduced a technique to detect multiple cracks. The continuum damage model was used first to identify the discretizing elements of a structure that contained the cracks, and then the spring damage model was used to quantify the location and size of the discrete crack in each damaged element. Patil and Maiti [14] presented a method that combined the vibration modeling through transfer matrix method and the approach given by Hu and Liang [13]. The identification of multiple cracks in beams was regarded as an optimization problem by Ruotolo and Surace [15] who selected the combination of fundamental functions as the objective function and utilized a solution procedure employing generic algorithms. Shifrin and Ruotolo [16] proposed that $n+2$ equations were sufficient to form the system determinant for a beam with n cracks.

The objective of this study is to present a simple method based on the massless rotational spring model for the crack, the finite element method and the Newton–Raphson method to identify multiple cracks in a beam, which requires $2n$ natural frequencies to detect n cracks in a beam.

2. Forward problem

The geometry of a beam with a crack and its finite element model are given in Fig. 1. Parameters $\alpha = a/h$ and $\beta = s/L$ denote the normalized crack size and the normalized crack location. The finite element equation of a beam segment based on the Euler–Bernoulli theory of length ΔL is given as

$$[\mathbf{M}]^e\{\ddot{W}\}^e + [\mathbf{K}]^e\{W\}^e = \{f\}^e \tag{1}$$

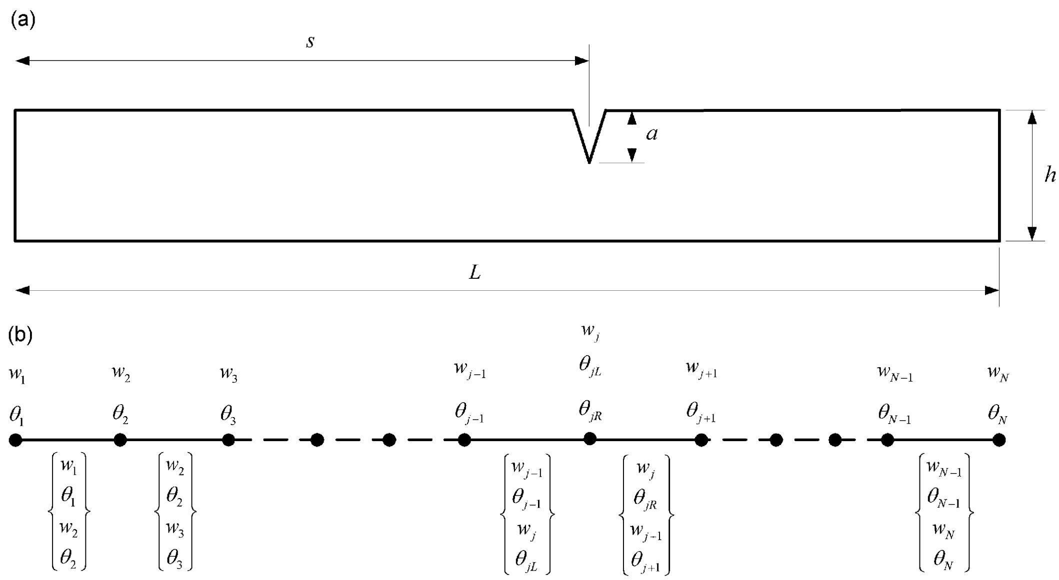

Fig. 1. Beam with a crack and its finite element model.

where matrices $[\mathbf{M}]^e$ and $[\mathbf{K}]^e$ are the element mass and stiffness matrices defined as

$$[\mathbf{M}]^e = \frac{\rho A\,\Delta L}{420}\begin{bmatrix} 156 & 22\,\Delta L & 54 & -13\,\Delta L \\ & 4(\Delta L)^2 & 13\,\Delta L & -3(\Delta L)^2 \\ & & 156 & -22\,\Delta L \\ \mathrm{SYM} & & & 4(\Delta L)^2 \end{bmatrix}, \quad [\mathbf{K}]^e = \frac{EI}{(\Delta L)^3}\begin{bmatrix} 12 & 6\,\Delta L & -12 & 6\,\Delta L \\ & 4(\Delta L)^2 & -6\,\Delta L & 2(\Delta L)^2 \\ & & 12 & -6\,\Delta L \\ \mathrm{SYM} & & & 4(\Delta L)^2 \end{bmatrix} \tag{2}$$

and $\{f\}^e$ is the generalized element load vector. E, I, A and ρ are Young's modulus, the second moment of area, the cross sectional area and the density.

As shown in Fig. 1(b) the j-th node that represents the crack has three degrees of freedom $(w_j, \theta_{jL}, \theta_{jR})$ while each of other nodes has two degrees of freedom (w_i, θ_i). The element variable vector of the i-th element that does not include the crack node is $\{W\}^e = \{w_i \;\; \theta_i \;\; w_{i+1} \;\; \theta_{i+1}\}^{\mathrm{T}}$. The element variable vectors that belong to the elements adjoining the crack node to the left and to the right are $\{W\}^e = \{w_{j-1} \;\; \theta_{j-1} \;\; w_j \;\; \theta_{jL}\}^{\mathrm{T}}$ and $\{W\}^e = \{w_j \;\; \theta_{jR} \;\; w_{j+1} \;\; \theta_{j+1}\}^{\mathrm{T}}$, respectively. The rotations θ_{jL} and θ_{jR} are connected through the cracked stiffness matrix:

$$[\mathbf{K}]_c = \begin{bmatrix} K_t & -K_t \\ -K_t & K_t \end{bmatrix} \tag{3}$$

where K_t, the torsional stiffness per unit width at the crack, is given by Nandwann and Maiti [3] by

$$K_t = \frac{h^2 E}{72\pi\alpha^2 f(\alpha)} \tag{4a}$$

$$f(\alpha) = 0.6384 - 1.035\alpha + 3.7201\alpha^2 - 5.1773\alpha^3 + 7.553\alpha^4 - 7.332\alpha^5 + 2.4909\alpha^6 \tag{4b}$$

Matrices $[\mathbf{M}]^e$, $[\mathbf{K}]^e$ and $[\mathbf{K}]_c$ are assembled to form the global mass and stiffness matrices $[\mathbf{M}]$ and $[\mathbf{K}]$. After the application of the boundary conditions the global equations of motion of the free vibration of the beam is

$$[\mathbf{K}]\{W\} = \omega^2[\mathbf{M}]\{W\} \tag{5}$$

from which the natural frequencies ω's are computed.

The finite element model of present study can easily be extended to a beam with multiple cracks. Fig. 2 shows a cantilever beam with double cracks. Parameters $\alpha_i = a_i/h$ and $\beta_i = s_i/L$ $(i = 1{,}2)$ represent the normalized size and the normalized location of the i-th crack. A finite element mesh is generated, where the crack nodes with three degrees of freedom $(w_j, \theta_{jL}, \theta_{jR})$ are placed at the crack locations and two degrees of freedom (w_i, θ_i) are allocated to the other nodes.

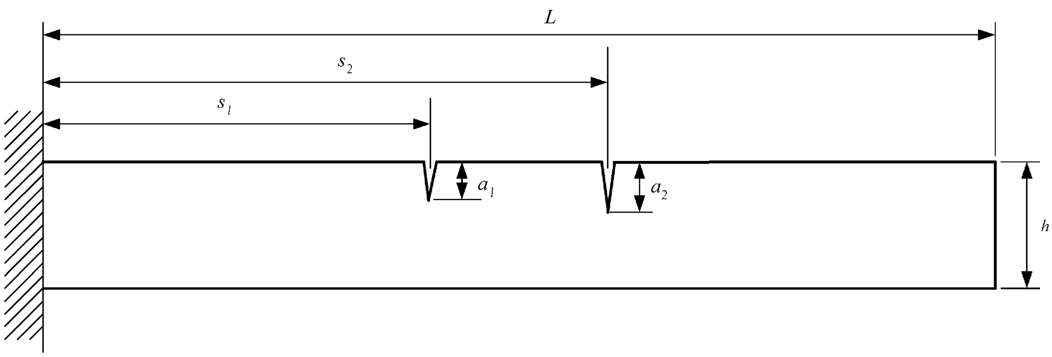

Fig. 2. Cantilever beam with double cracks.

Table 1
Natural frequencies of a beam with double cracks.

β_1	β_2	ω_1 (rad/s)	ω_2 (rad/s)	ω_3 (rad/s)	ω_4 (rad/s)
$\alpha_1 = 0.1$, $\alpha_2 = 0.1$					
0.2	0.4	417.0794	2622.389	7341.322	14368.22
	0.6	417.6436	2619.704	7337.863	14370.04
	0.8	417.8164	2628.122	7332.257	14313.00
0.4	0.6	418.7587	2612.191	7333.411	14407.20
	0.8	418.9328	2620.495	7327.831	14350.39
0.6	0.8	419.5051	2617.800	7325.045	14349.95
$\alpha_1 = 0.1$, $\alpha_2 = 0.2$					
0.2	0.4	414.9444	2600.723	7303.517	14330.75
	0.6	417.1079	2590.404	7290.658	14337.80
	0.8	417.7751	2622.266	7267.328	14124.72
0.4	0.6	418.2175	2583.284	7285.600	14374.36
	0.8	418.8908	2614.722	7262.398	14162.31
0.6	0.8	419.4628	2612.009	7260.865	14158.15
$\alpha_1 = 0.2$, $\alpha_2 = 0.1$					
0.2	0.4	411.8595	2622.059	7314.768	14229.40
	0.6	412.4035	2619.383	7310.739	14233.96
	0.8	412.5698	2627.816	7305.520	14177.14
0.4	0.6	416.5980	2590.929	7294.442	14371.91
	0.8	416.7702	2598.935	7289.354	14316.15
0.6	0.8	418.9616	2588.579	7278.944	14314.41
$\alpha_1 = 0.2$, $\alpha_2 = 0.2$					
0.2	0.4	409.8045	2600.313	7278.547	14187.64
	0.6	411.8884	2590.033	7263.595	14204.94
	0.8	412.5293	2621.953	7241.723	13992.57
0.4	0.6	416.0653	2563.053	7245.001	14340.79
	0.8	416.7289	2593.382	7223.669	14132.44
0.6	0.8	418.9196	2582.962	7217.882	14116.76

$L = 0.5\,\text{m}$; $E = 210\,\text{GPa}$; $h = 0.02\,\text{m}$; $\rho = 7860\,\text{kg/m}^3$; cantilever beam.

Table 2
Natural frequencies of a beam with triple cracks.

$\alpha_1 = \alpha_2 = \alpha_3 = 0.1$								
β_1	β_2	β_3	ω_1 (rad/s)	ω_2 (rad/s)	ω_3 (rad/s)	ω_4 (rad/s)	ω_5 (rad/s)	ω_6 (rad/s)
0.2	0.4	0.6	416.8933	2612.065	7323.879	14356.68	23589.91	35603.94
0.2	0.4	0.8	417.0652	2620.375	7318.436	14299.97	23600.29	35573.62
0.2	0.6	0.8	417.6291	2617.683	7315.436	14300.48	23601.47	35573.74
0.4	0.6	0.8	418.7431	2610.199	7310.806	14337.70	23575.09	35597.99

$L = 0.5\,\text{m}$; $E = 210\,\text{GPa}$; $h = 0.02\,\text{m}$; $\rho = 7860\,\text{kg/m}^3$; cantilever beam.

A program is written in Matlab, and the lowest four natural frequencies of a cantilever beam with double cracks for various crack sizes and crack locations are computed and given in Table 1. The length, the height, Young's modulus and the density of the beam are $L = 0.5\,\text{m}$, $h = 0.02\,\text{m}$, $E = 210\,\text{GPa}$ and $\rho = 7860\,\text{kg/m}^3$, respectively, throughout this study. Also the lowest six natural frequencies of a cantilever beam with triple cracks for various crack locations are given in Table 2.

3. Inverse problem

For the identification of n cracks there are $2n$ unknown crack parameters: α_1,β_1,α_2,β_2,…,α_n and β_n. To match the number of equations and that of the unknowns it is assumed that $2n$ natural frequency measurements $(\omega_1^0, \omega_2^0, \ldots, \omega_{2n}^0)$ are available. The Newton–Raphson procedure is applied as follows:

(a) assume initial values of $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_n, \beta_n$;
(b) locate the nodes that represent the cracks according to the new crack position parameters $\beta_1, \beta_2, \ldots, \beta_n$ and generate the finite element mesh of the beam;
(c) solve the forward problem for $\omega_1, \omega_2, \ldots, \omega_{2n}$, with the crack parameters α_1,β_1,α_2,β_2,…,α_n,β_n, and evaluate the Jacobian matrix $[J]$:

$$[J] = \begin{bmatrix} \dfrac{\partial\omega_1}{\partial\alpha_1} & \dfrac{\partial\omega_1}{\partial\beta_1} & \dfrac{\partial\omega_1}{\partial\alpha_2} & \dfrac{\partial\omega_1}{\partial\beta_2} & \cdots & \dfrac{\partial\omega_1}{\partial\alpha_n} & \dfrac{\partial\omega_1}{\partial\beta_n} \\ \dfrac{\partial\omega_2}{\partial\alpha_1} & \dfrac{\partial\omega_2}{\partial\beta_1} & \dfrac{\partial\omega_2}{\partial\alpha_2} & \dfrac{\partial\omega_2}{\partial\beta_2} & \cdots & \dfrac{\partial\omega_2}{\partial\alpha_n} & \dfrac{\partial\omega_2}{\partial\beta_n} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \dfrac{\partial\omega_{2n}}{\partial\alpha_1} & \dfrac{\partial\omega_{2n}}{\partial\beta_1} & \dfrac{\partial\omega_{2n}}{\partial\alpha_2} & \dfrac{\partial\omega_{2n}}{\partial\beta_2} & \cdots & \dfrac{\partial\omega_{2n}}{\partial\alpha_n} & \dfrac{\partial\omega_{2n}}{\partial\beta_n} \end{bmatrix} \tag{6}$$

and the residuals

$$\begin{cases} \Re_1 = \omega_1 - \omega_1^0 \\ \Re_2 = \omega_2 - \omega_2^0 \\ \vdots \\ \Re_{2n} = \omega_{2n} - \omega_{2n}^0 \end{cases} \tag{7}$$

(d) solve the equation

$$[J]\begin{Bmatrix} d\alpha_1 \\ d\beta_1 \\ d\alpha_2 \\ d\beta_2 \\ \vdots \\ d\alpha_n \\ d\beta_n \end{Bmatrix} = -\begin{Bmatrix} \Re_1 \\ \Re_2 \\ \vdots \\ \Re_{2n} \end{Bmatrix} \tag{8}$$

for $\{d\alpha_1\ \ d\beta_1\ \ d\alpha_2\ \ d\beta_2\ \ \cdots\ \ d\alpha_n\ \ d\beta_n\}^T$,
(e) update the crack parameters

$$(\alpha_i)_{new} = (\alpha_i)_{old} + d\alpha_i, \quad (\beta_i)_{new} = (\beta_i)_{old} + d\beta_i \ \ (i = 1, 2, \ldots, n) \tag{9}$$

(f) iterate the procedures (b)–(e) until the residuals become sufficiently small.

In fact, almost any damage identification method based on the optimization theory is reduced to a linearlized system of equations similar to that of Eq. (8). The elements of the Jacobian matrix are the sensitivities of the natural frequencies with respect to the crack parameters. Morassi [17] developed an explicit expression of the frequency sensitivity to damage assuming that the size of the crack was small enough. In this study, however, the cracks are not assumed to be of small size and the elements of the Jacobian matrix are computed numerically. For example, $\partial\omega_1/\partial\alpha_1$ is computed by

$$\frac{\partial\omega_1}{\partial\alpha_1}=\frac{\omega_1(\alpha_1+\delta,\beta_1,\alpha_2,\ldots,\beta_n)-\omega_1(\alpha_1,\beta_1,\alpha_2,\ldots,\beta_n)}{\delta}\quad(|\delta|\ll 1)\tag{10}$$

The forward problem is solved $2n+1$ times per iteration to build the Jacobian matrix and the residuals. To suppress overshoots in the early stage an underrelaxation is performed during the first three or four iterations

$$(\alpha_i)_{\text{new}}=(\alpha_i)_{\text{old}}+0.25\,\mathrm{d}\alpha_i,\quad(\beta_i)_{\text{new}}=(\beta_i)_{\text{old}}+0.25\,\mathrm{d}\beta_i\ \ (i=1,2,\ldots,n)\tag{11}$$

The inverse problem of identifying double cracks in a cantilever beam is solved for six simulation cases A, B, C, D, E and F, presented in Table 3. The computational results from the forward problem with actual crack positions and sizes are input as measurements. Proper selection of the initial guesses is important for the convergence of the solution. The detected crack locations and crack sizes are found practically identical to the actual crack parameters. The numbers of iterations required for the convergence of the crack parameters to the four significant digits are 9 (case A), 11 (case B), 14 (case C), 18 (case D), 10 (case E) and 18 (case F), respectively. The normalized residuals $(\omega_k-\omega_k^0)/\omega_k^0(k=1,2,3,4)$ of each iteration are plotted in Fig. 3, which shows that the crack parameters converge very rapidly as the iteration proceeds.

Present method is also applied to the identification of triple cracks in a cantilever beam. The actual and the detected crack parameters are given in Table 4. The error in the detection of crack location is less than 5 percent and it is less than 3 percent in the case of detection of crack sizes. The numbers of iterations required for the convergence of the crack parameters to the four significant digits for cases G and H are 22 and 23. The normalized residuals $(\omega_k-\omega_k^0)/\omega_k^0(k=1,2,3,4,5,6)$ are plotted in Fig. 4.

Like all other inverse problems present method is strongly influenced by input data noise and only input data essentially free of noise are considered in cases A–H. Let us assume that the input data of case A are slightly biased and given as $\omega_1^0=412\,\text{rad/s}$, $\omega_2^0=2620\,\text{rad/s}$, $\omega_3^0=7310\,\text{rad/s}$ and $\omega_4^0=14200\,\text{rad/s}$ instead of

Table 3
Comparison of actual and detected crack parameters after 10 iterations.

Case	α_1	α_2	β_1	β_2
A				
Actual	0.2	0.1	0.2	0.4
Initial guess	0.3	0.3	0.3	0.5
Detected	0.2000	0.1000	0.2000	0.4000
B				
Actual	0.1	0.1	0.2	0.6
Initial guess	0.3	0.3	0.3	0.7
Detected	0.1000	0.1000	0.1999	0.6000
C				
Actual	0.1	0.1	0.2	0.8
Initial guess	0.3	0.3	0.1	0.9
Detected	0.0997	0.0995	0.2015	0.8025
D				
Actual	0.2	0.2	0.4	0.6
Initial guess	0.3	0.3	0.5	0.8
Detected	0.2029	0.1971	0.4028	0.6028
E				
Actual	0.1	0.2	0.4	0.8
Initial guess	0.3	0.3	0.5	0.6
Detected	0.1000	0.2000	0.4000	0.8000
F				
Actual	0.2	0.2	0.6	0.8
Initial guess	0.3	0.3	0.5	0.7
Detected	0.1997	0.1988	0.5996	0.7982

Double cracks in a cantilever beam; $L=0.5\,\text{m}$; $E=210\,\text{GPa}$; $h=0.02\,\text{m}$; $\rho=7860\,\text{kg/m}^3$.

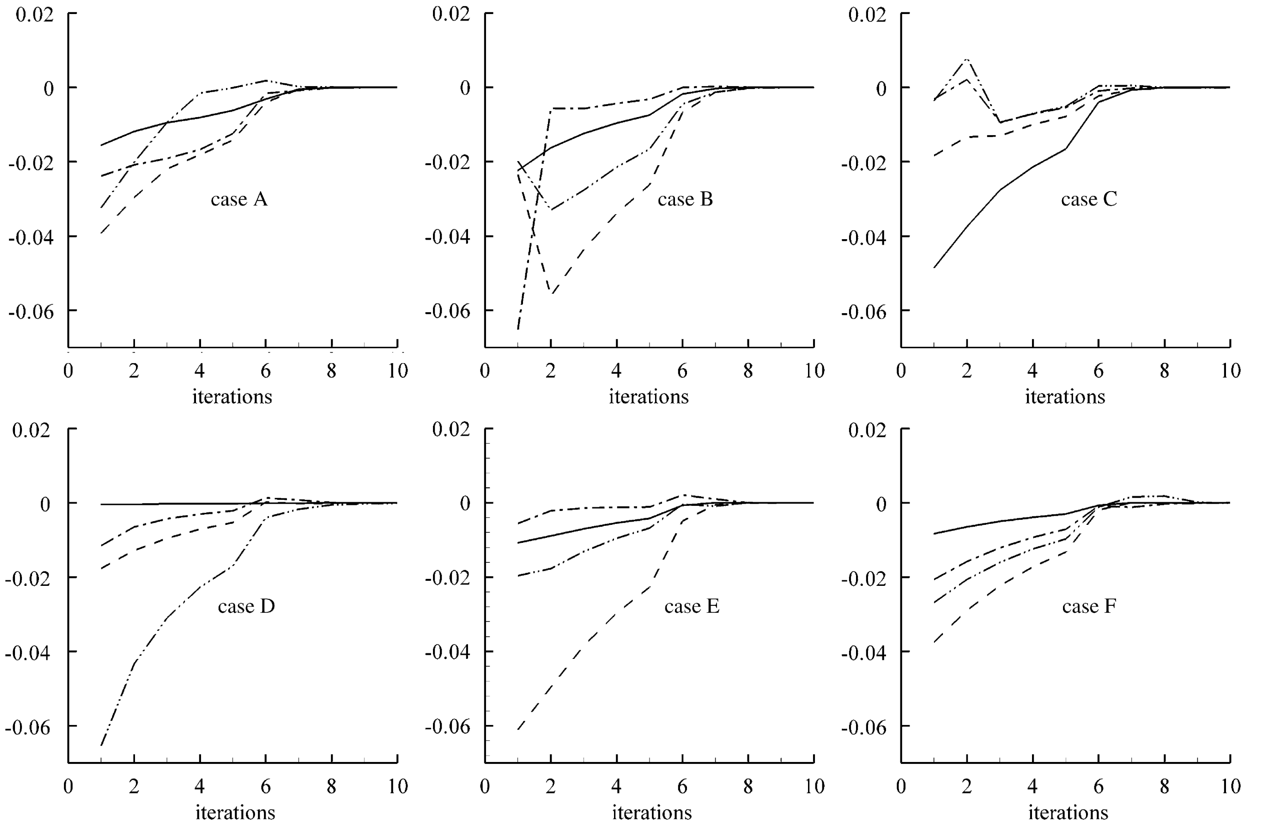

Fig. 3. Normalized residuals of the natural frequencies in a beam with double cracks (——: $(\omega_1-\omega_1^0)/\omega_1^0$, – – – –: $(\omega_2-\omega_2^0)/\omega_2^0$, —·—·: $(\omega_3-\omega_3^0)/\omega_3^0$, —··—: $(\omega_4-\omega_4^0)/\omega_4^0$).

Table 4
Comparison of actual and detected crack parameters after 20 iterations.

Case	α_1	α_2	α_3	β_1	β_2	β_3
G						
Actual	0.1	0.1	0.1	0.2	0.4	0.8
Initial guess	0.3	0.3	0.3	0.1	0.3	0.7
Detected	0.1011	0.1024	0.0975	0.2094	0.4010	0.8076
H						
Actual	0.1	0.1	0.1	0.2	0.6	0.8
Initial guess	0.3	0.3	0.3	0.1	0.5	0.7
Detected	0.1014	0.1016	0.0977	0.2077	0.5992	0.8065

Triple cracks in a cantilever beam; $L = 0.5\,\mathrm{m}$; $E = 210\,\mathrm{GPa}$; $h = 0.02\,\mathrm{m}$; $\rho = 7860\,\mathrm{kg/m^3}$.

the natural frequencies listed in Table 1. The inverse problem is solved and the crack parameters are estimated to be $\alpha_1 = 0.2016$, $\alpha_2 = 0.1066$, $\beta_1 = 0.2093$ and $\beta_2 = 0.4247$, which deviate considerably from those of actual values.

Present method has several merits over other crack identification methods. First of all, it can be extended to detect any number of cracks when the natural frequency measurements $(\omega_1^0, \omega_2^0, \ldots, \omega_{2n}^0)$ are available. Also the application of the Newton–Raphson iteration method is much simpler than the generic algorithms or the frequency contour plot method. Moreover, present method can be used to identify cracks in a short beam

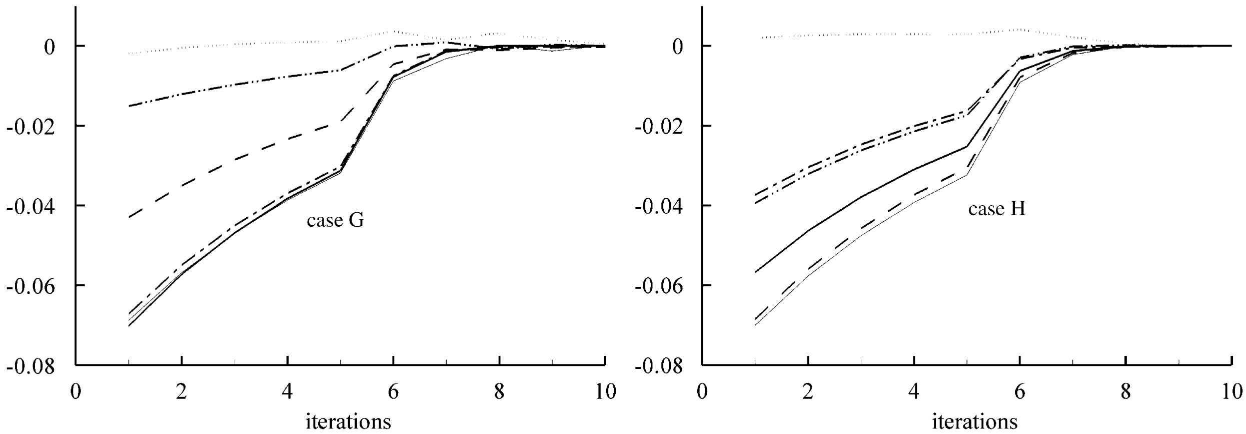

Fig. 4. Normalized residuals of the natural frequencies in a beam with triple cracks (——: $(\omega_1-\omega_1^0)/\omega_1^0$, – – – –: $(\omega_2-\omega_2^0)/\omega_2^0$, —·—·: $(\omega_3-\omega_3^0)/\omega_3^0$, —··—: $(\omega_4-\omega_4^0)/\omega_4^0$, ·········: $(\omega_5-\omega_5^0)/\omega_5^0$, ——: $(\omega_6-\omega_6^0)/\omega_6^0$).

when the element mass matrix $[\mathbf{M}]^e$ and the element stiffness matrix $[\mathbf{K}]^e$ of Eq. (2) are modified to take account of the effects of the transverse shear and the rotary inertia according to the Timoshenko beam theory.

The number of cracks present in a beam is usually unknown, and present method which assumes that the number of cracks is known *a priori* has a serious limitation in its application. In their continuum damage model Hu and Liang [13] introduced a damage index. They divided the beam into several segments with each segment assumed to have a respective damage index, and they developed and solved a set of equations to determine how many segments contained cracks. Such an approach may serve as an excellent preprocessor for present method to provide the initial guesses of crack parameters not to mention the number of cracks.

4. Conclusions

A simple and efficient method to detect multiple cracks in a beam is presented. The crack is modeled as a massless rotational spring and the forward problem is solved by using the finite element method based on the Euler–Bernoulli beam theory. In the finite element model the node that represents the crack has three degrees of freedom while each of other nodes has two degrees of freedom. The rotations of the node that represents the crack are connected through the cracked stiffness matrix. The inverse problem is solved iteratively for the crack locations and sizes by the Newton–Raphson method. The $2n$ natural frequency measurements are required to identify n cracks in a beam. Numerical examples are provided for the identification of triple cracks in a cantilever beam as well as double cracks. The identified crack locations and sizes are in excellent agreement with the actual ones.

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