Available online at www.sciencedirect.com
ScienceDirect
Journal of Sound and Vibration 320 (2009) 576–583
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

A numerical solution of the vibration equation using modified decomposition method

S. Das

Department of Applied Mathematics, Institute of Technology, Banaras Hindu University, Varanasi 221005, India

Received 31 January 2008; received in revised form 18 July 2008; accepted 25 August 2008
Handling Editor: L.G. Tham
Available online 21 October 2008

Abstract

In the present paper the well-known vibration equation for very large membrane with the help of powerful modification of Adomian decomposition method proposed by Wazwaz [A reliable modification of Adomian decomposition method, *Applied Mathematics and Computation* 102 (1999) 77–86] has been solved. By using initial value, the explicit solutions of the equation for different cases have been derived, which accelerate the rapid convergence of the series solution. The present method performs extremely well in terms of efficiency and simplicity. Numerical results for different particular cases of the problem are presented graphically.

1. Introduction

The decomposition method of Adomian has been applied to solve a wide class of non-linear differential and partial differential equations [1–4]. The advantage of the decomposition method over the other approximation methods, apart from computational simplicity, is that the method is non-perturbative and does not involve any linearization or smallness assumptions. Hence the solution obtained by this method is expected to be a better approximation.

Wazwaz [5] made further progress of this method with some modifications in the approach. The modification of the Adomian decomposition method will accelerate the rapid convergence of the series solution. This modified technique has been shown to be computationally efficient in doing several problems in applied fields [6–11].

In this paper, the modified decomposition method (MDM) is used to obtain the numerical solutions of the vibration equation for very large membrane for different particular cases. The expressions of the displacement for different time and radii of the membrane and also for various wave velocities of free vibration using the initial conditions are deduced and numerical computations are made with the help of Mathematica (Version 5.2) and presented through graphs.

E-mail address: subir_das08@hotmail.com

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2008.08.029

The elegance of the process can be attributed to its simplistic approach in seeking the solution to the problem as opposed to the complexities involved in using classical techniques like Hankel transform and its inverse.

2. Solution of the problem

The vibration equation of very large membrane is governed by the equation

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \quad r\geqslant 0, \quad t\geqslant 0 \tag{1}$$

with the initial conditions

$$u(r,0)=f(r) \tag{2}$$

$$\frac{\partial}{\partial t}u(r,0)=cg(r) \tag{3}$$

where $u(r, t)$ represents the displacement of finding a particle at the point r in the time instant t, c is the wave velocity of free vibration. We consider Eq. (1) as

$$L_{tt}u(r,t)=c^2\left[L_{rr}u(r,t)+\frac{1}{r}L_r u(r,t)\right] \tag{4}$$

where $L_{tt}\equiv \partial^2/\partial t^2$, $L_{rr}\equiv \partial^2/\partial r^2$ and $L_r\equiv \partial/\partial r$ symbolize the linear differential operations.

Applying the two-fold integration inverse operator $L_{tt}^{-1}=\int_0^t\int_0^t(\bullet)\,\mathrm{d}t\,\mathrm{d}t$ to Eq. (4), we get

$$u(r,t)=\phi_t+c^2\left[L_{tt}^{-1}L_{rr}u(r,t)+L_{tt}^{-1}\frac{1}{r}L_r u(r,t)\right]$$

where

$$\begin{aligned}\phi_t&=u(r,0)+tu_t(r,0)\\&=f(r)+ctg(r)\end{aligned} \tag{5}$$

The Adomian decomposition method [1,2] assumes an infinite series solutions for unknown function $u(r, t)$ given by

$$u(r,t)=\sum_{n=0}^{\infty}u_n(r,t) \tag{6}$$

where the components u_0, u_1, u_2, ... are usually determined recursively by

$$u_0=\phi_t$$

$$u_1=c^2\left[L_{tt}^{-1}(L_{rr}u_0)+L_{tt}^{-1}\left(\frac{1}{r}L_r u_0\right)\right]$$

$$u_2=c^2\left[L_{tt}^{-1}(L_{rr}u_1)+L_{tt}^{-1}\left(\frac{1}{r}L_r u_1\right)\right]$$

$$u_{k+1}=c^2\left[L_{tt}^{-1}(L_{rr}u_k)+L_{tt}^{-1}\left(\frac{1}{r}L_r u_k\right)\right], \quad k\geqslant 0 \tag{7}$$

Recently, Wazwaz [5] proposed that the construction of the zeroth component of the decomposition series can be defined in a slightly different way. He has proposed that if the zeroth component $u_0=\phi_t$ and the function ϕ_t is possible to divide into two parts ϕ_1 and ϕ_2 such that

$$u_0=\phi_1$$

$$u_1 = \phi_2 + c^2\left[L_{tt}^{-1}(L_{rr}u_0) + L_{tt}^{-1}\left(\frac{1}{r}L_r u_0\right)\right]$$

$$u_{k+1} = c^2\left[L_{tt}^{-1}(L_{rr}u_k) + L_{tt}^{-1}\left(\frac{1}{r}L_r u_k\right)\right], \quad k \geqslant 1 \tag{8}$$

This type of modification is giving more flexibility to the Adomian decomposition method in order to solve complicated non-linear differential equations. In many cases the modified decomposition method avoids the unnecessary complications in calculating the Adomian polynomials.

The decomposition series (6) converges very rapidly in real physical problems. The rapid convergence means that few terms are required. The practical solutions will be the n-th approximation $\alpha_n = \sum_{k=0}^{n-1} u_k(r,t)$, $n \geqslant 1$, with $\lim_{n\to\infty}\alpha_n = u(r, t)$.

3. Particular cases

Case I: Taking $f(r) = r$ and $g(r) = 1$. Using the recurrence relation (8), we find

$$u_0 = r$$

$$u_1 = ct + c^2\left[L_{tt}^{-1}L_{rr}(u_0) + L_{tt}^{-1}\left(\frac{1}{r}L_r u_0\right)\right] = ct + \frac{c^2t^2}{2r}$$

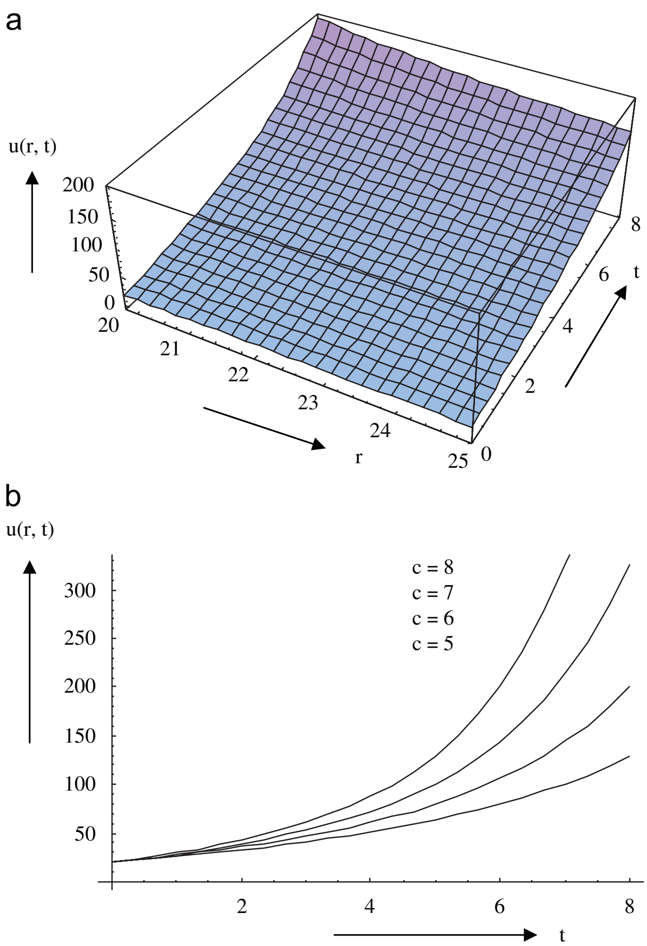

Fig. 1. (a) Plot of $u(r, t)$ with respect to r and t at $c = 6$ for Case I and (b) plot of $u(r, t)$ vs. t for different values of c at $r = 20$ for Case I.

$$u_2 = c^2 L_{tt}^{-1}\left[L_{rr}(u_1) + \frac{1}{r}L_r(u_1)\right] = \frac{c^4 t^4}{24 r^3}$$

$$u_3 = c^2 L_{tt}^{-1}\left[L_{rr}(u_2) + \frac{1}{r}L_r(u_2)\right] = \frac{c^6 t^6}{80 r^5}$$

and so on. Therefore,

$$u(r,t) = r\left[1 + c\left(\frac{t}{r}\right) + \frac{c^2}{2}\left(\frac{t}{r}\right)^2 + \frac{c^4}{24}\left(\frac{t}{r}\right)^4 + \frac{c^6}{80}\left(\frac{t}{r}\right)^6 + \cdots\right] \quad (9)$$

As evident, the above series will be convergent for the values of $|t/r| \ll 1$ i.e., for large membrane and small range of time.

Case II: Taking $f(r) = r^2$ and $g(r) = r$,

$$u_0 = r^2$$

$$u_1 = ctr + 2c^2t^2$$

$$u_2 = \frac{c^3 t^3}{6r}$$

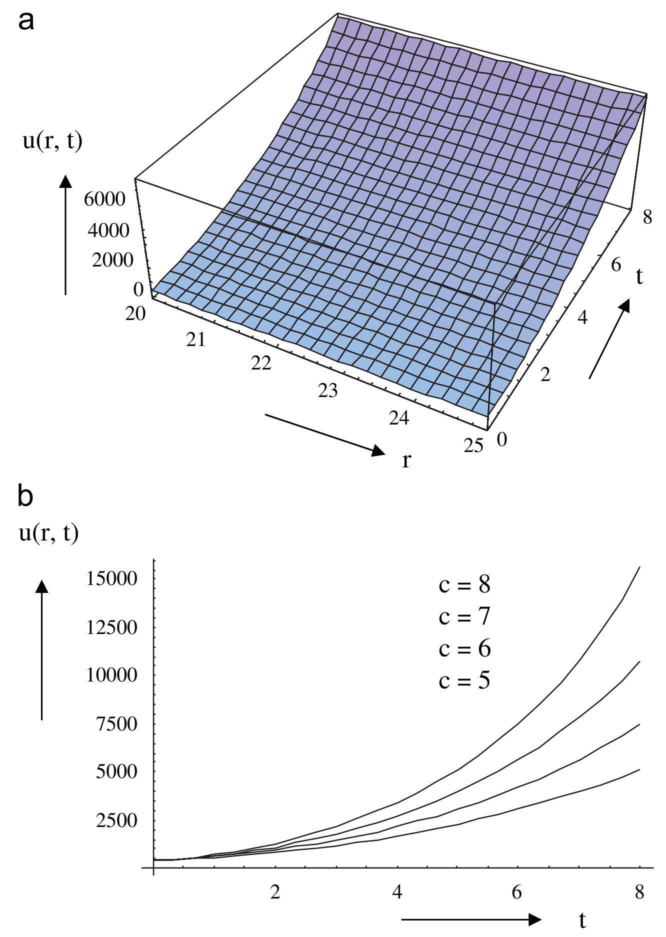

Fig. 2. (a) Plot of $u(r, t)$ with respect to r and t at $c = 6$ for Case II and (b) plot of $u(r, t)$ vs. t for different values of c at $r = 20$ for Case II.

$$u_3 = \frac{c^5 t^5}{120 r^3}$$

and so on.

Thus

$$u(r,t) = r^2\left[1 + c\left(\frac{t}{r}\right) + 2c^2\left(\frac{t}{r}\right)^2 + \frac{c^3}{6}\left(\frac{t}{r}\right)^3 + \frac{c^5}{120}\left(\frac{t}{r}\right)^5 + \cdots\right] \tag{10}$$

As of Case I, the above series is also convergent for $|t/r| \ll 1$.

Case III: Taking $f(r) = \sqrt{r}$ and $g(r) = 1/\sqrt{r}$.

Here

$$u_0 = \sqrt{r}$$

$$u_1 = \frac{ct}{\sqrt{r}} + \frac{c^2 t^2}{8 r^{3/2}}$$

$$u_2 = \frac{c^3 t^3}{24 r^{5/2}} + \frac{3 c^4 t^4}{128 r^{7/2}}$$

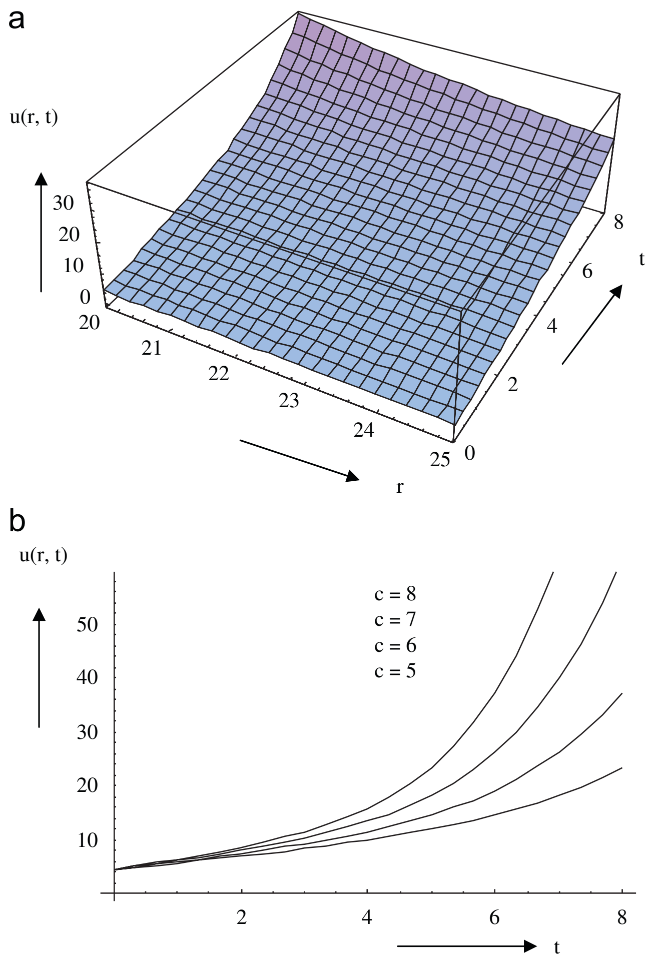

Fig. 3. (a) Plot of $u(r, t)$ with respect to r and t at $c = 6$ for Case III and (b) plot of $u(r, t)$ vs. t for different values of c at $r = 20$ for Case III.

$$u_3 = \frac{5c^5t^5}{384r^{9/2}} + \frac{49c^6t^6}{5120r^{11/2}}$$

and so on.

Finally,

$$u(r,t) = \sqrt{r}\left[1 + c\left(\frac{t}{r}\right) + \frac{c^2}{8}\left(\frac{t}{r}\right)^2 + \frac{c^3}{24}\left(\frac{t}{r}\right)^3 + \frac{3c^4}{128}\left(\frac{t}{r}\right)^4 + \frac{5c^5}{384}\left(\frac{t}{r}\right)^5 + \frac{49c^6}{5120}\left(\frac{t}{r}\right)^6 + \cdots\right] \quad (11)$$

As of Case I, the above series is convergent for $|t/r| \ll 1$.

Case IV: Taking $f(r) = r^2$ and $g(r) = 1$.

Here

$$u_0 = r^2$$

$$u_1 = ct + 2c^2t^2$$

$$u_n = 0, \quad n \geqslant 2$$

Therefore,

$$u(r,t) = r^2 + ct + 2c^2t^2 \quad (12)$$

Case V: Taking $f(r) = r^2$ and $g(r) = r^2$.

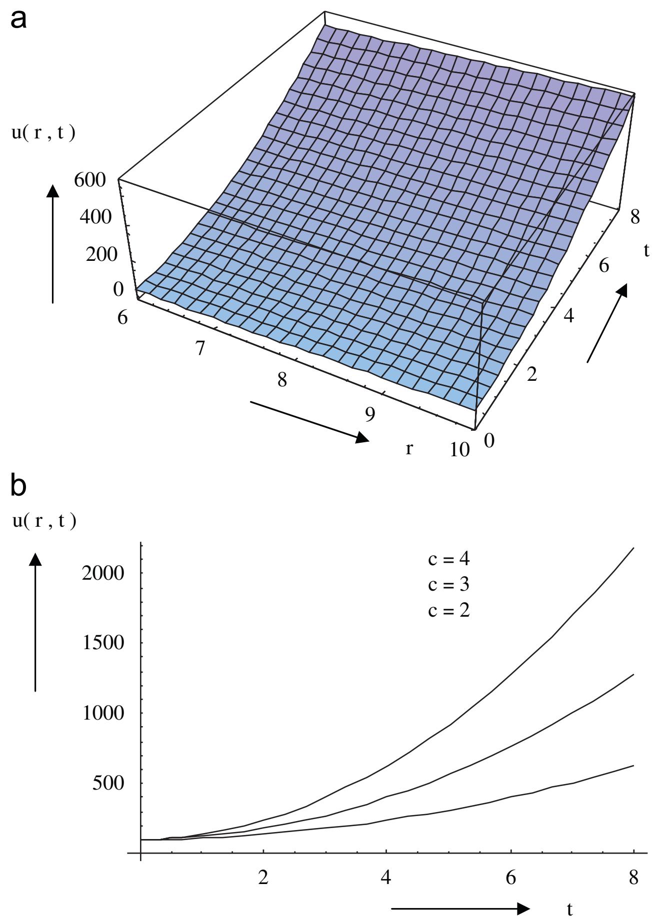

Fig. 4. (a) Plot of $u(r, t)$ with respect to r and t at $c = 2$ for Case IV and (b) plot of $u(r, t)$ vs. t for different values of c at $r = 10$ for Case IV.

Here

$$u_0 = r^2$$

$$u_1 = ctr^2 + 2c^2t^2$$

$$u_2 = \tfrac{2}{3}c^3t^3$$

$$u_n = 0, \quad n \geqslant 3$$

Therefore,

$$u(r,t) = r^2 + ctr^2 + 2c^2t^2 + \tfrac{2}{3}c^3t^3 \tag{13}$$

4. Numerical results and discussion

In this section, numerical results of the displacement for various values of radii of the membrane and time are presented through Figs. 1–5. For Cases I–III, it is kept in mind that for the convergence of the problems the ratio t/r is to be small. It is observed that for Cases I and III, the displacement decreases with the increase in r and increases with the increase in t (Figs. 1(a) and 3(a)) but for the Case II, it increases with the increase of both r and t (Fig. 2(a)) for a fixed value of wave velocity ($c = 6$).

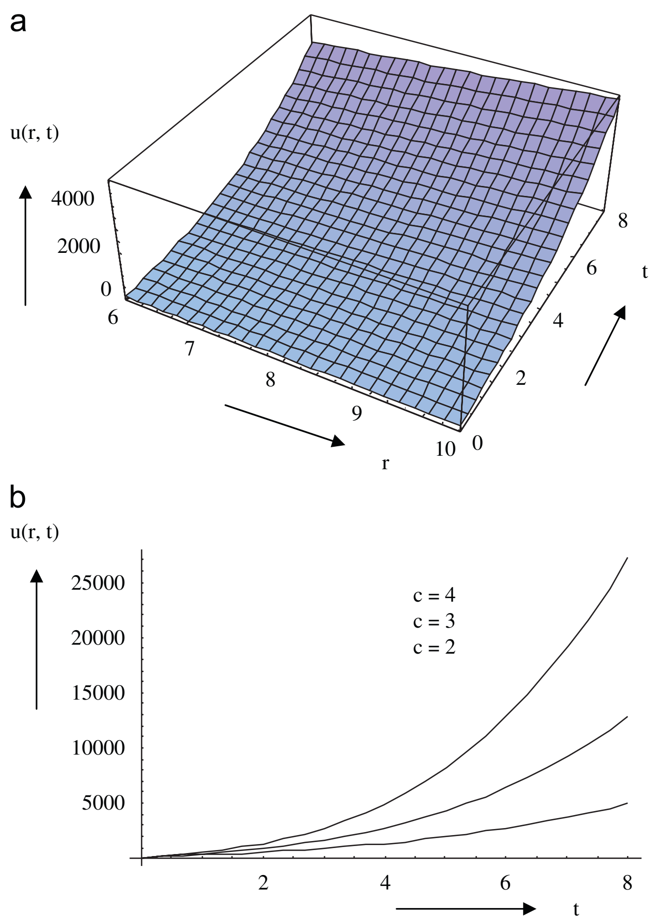

Fig. 5. (a) Plot of $u(r, t)$ with respect to r and t at $c = 2$ for Case V and (b) plot of $u(r, t)$ vs. t for different values of c at $r = 10$ for Case V.

It is also seen from Figs. 1(b), 2(b) and 3(b) that the displacement increases with the increase in t and c both at a fixed value of the radius of the membrane (for $r = 20$). The increase in displacement is faster in Case II than for Cases I and III.

For the Cases IV and V, since the expressions of displacement contain only finite number of terms, so $u(r, t)$ does not depend on the ratio t/r. It is seen that in both the cases the displacement increases with the increase in both r and t for a fixed value of $c = 2$ (Figs. 4(a) and 5(a)). Figs. 4(b) and 5(b) depict that the displacement increases with the increase in t and c for fixed value of $r = 10$. The increase in displacement in Case V is faster than that in Case IV. All the computations and figures are made using Mathematica software [12].

5. Conclusion

The modified decomposition technique is very powerful in finding solutions for various physical problems. Showing its application for vibration of very large membrane, we may conclude that the decomposition method will be very much useful for solving many engineering problems both analytically and numerically.

It is also shown that the advantage of the modified decomposition method is its fast convergence of the solution. The numerical results obtained here conform to its high degree of accuracy. Moreover, no linearization or perturbation or discretization is needed and it also avoids the accuracy of finding the inverses of Laplace and Hankel transformations.

Acknowledgments

Author is grateful to the referees for their invaluable suggestions for the improvement of the paper.

References

[1] G. Adomian, A review of the decomposition method in applied mathematics, *Journal of Mathematical Analysis and Applications* 135 (1988) 501–544.

[2] G. Adomian, *Solving Frontier Problems of Physics: The Decomposition Method*, Kluwer Academic Publishers, Boston, 1994.

[3] G. Adomian, Solutions of nonlinear PDE, *Applied Mathematics Letters* 11 (1998) 121–123.

[4] G. Adomian, R. Rach, Linear and nonlinear Schrodinger equations, *Foundation Physics* 21 (1991) 983–991.

[5] A.M. Wazwaz, A reliable modification of Adomian decomposition method, *Applied Mathematics and Computation* 102 (1999) 77–86.

[6] A.M. Wazwaz, The decomposition method applied to systems of PDE and to the reaction–diffusion Brusselator model, *Applied Mathematics and Computation* 110 (2000) 251–264.

[7] A.M. Wazwaz, Approximate solutions to boundary value problems of higher order by the modified decomposition method, *Applied Mathematics and Computation* 40 (2000) 679–691.

[8] A.M. Wazwaz, The modified decomposition method applied to unsteady flow of gas through a porous medium, *Applied Mathematics and Computation* 118 (2001) 123–132.

[9] A.M. Wazwaz, Construction of soliton solutions and periodic solution of the Boussionesq equation by the modified decomposition method, *Chaos, Solitions & Fractals* 12 (2001) 1549–1556.

[10] D. Kaya, A. Yokus, A numerical comparison of partial solutions in the decomposition method for linear and nonlinear partial differential equation, *Mathematics and Computers in Simulation* 60 (2002) 507–512.

[11] S. Saha Ray, A numerical solution of the coupled sine-Gordon equation using the modified decomposition method, *Applied Mathematics and Computation* 175 (2006) 1046–1054.

[12] S. Wolfram, Mathematica for Windows, Version 5.2.0.0, License ID: L8368-9113, Copyright 1988–2005, Wolfram research.