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Prediction condensed models adapted to the nonlinear structures in time domain

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Abstract

This paper deals with a method to study closely the stationary solution of nonlinear dynamic systems in time domain. This method is based on the exploitation of Karhunen–Loève decomposition with or without parametric modifications as well as on the characteristics of localized nonlinearities. With the application of this method at first on linear models initially condensed by Karhunen–Loève, the predictions of nonlinear responses can be obtained rapidly. This method is adapted to a condensed linear model used in the first optimization procedure of the nonlinear dynamic behaviour. This robust basis will be used as condensation basis of the modified model local per zone, which leads to a prediction of vibratory responses of complex structures modified and affected by localized nonlinearities.

1. Introduction

The purpose of this study is, on the one hand, to predict at low numerical cost the nonlinear vibratory condensed responses of structures made up of one or more sub-structures, and on the other hand to implement condensed models adapted to the optimization procedures. Into the assemblies, one can introduce some localized nonlinearities in the behaviour of the complete structure. These nonlinearities originate essentially from slipping at the interfaces between two structures assembled by bolts rivets, etc.

The control of the dynamic behaviour of these complete, often complex, structures requires the construction of Finite Elements Models (FEM) of large size. In order to reduce the numerical simulations cost, many reduction methods of linear models have been proposed [1–4]. However, one can note that these methods remain currently insufficient and maladjusted to optimization problems, which require reduced models that can provide predictions of good quality. This is the case in particular for structural modifications and nonlinear dynamics problems.

The only choices that are generally offered to the designer are either to re-use the reduction transformation of the nominal model [5–7] or to recalculate the eigenvectors to reactualize the condensation basis of each

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modified component [8], or to use the equivalent linearization method or the harmonic balance [9–11]. The first option generally leads to inaccurate results whereas the second is typically not practical for reasons of prohibitive calculation cost. The third one is effective in the frequency domain only; it rests on the resolution of a costly iterative algorithm for the structures of large size. The difficulty of any reduction procedure is the optimal condensation. In the proposed method, one can seek to supplement the representation of the Ritz basis with the aim of compensating the truncation effect.

Nowadays, among the standard condensation basis is that of Karhunen–Loève (KL), also known as the decomposition POD. This POD basis is powerful. It consists of providing a basis for the modal decomposition of a whole lot of functions. These functions are commonly called empirical eigenfunctions, empirical orthogonal functions or orthogonal eigenmodes. In general, there are two interpretations for the POD. The first interpretation relates to the POD like KL decomposition [12–16] and the second one considers that the POD consists of three methods: the KLD method, the principal component analysis (PCA) and the singular values decomposition (SVD) [17–19]. In the recent years, multiple applications of POD methods have been developed in various engineering fields. The POD has been integrated in several disciplines concerning image processing, analysis of signals, in vibration of structures, dynamic chaotic systems [20–22], etc.

In this paper, a robust condensation method is proposed for use in a context of optimization, which requires the reanalysis of modified structures and its extension to nonlinear dynamics. It is based on the exploitation of the KL condensation method combined with a Newmark integration method (unconditionally stable). This proposed method improves the standard reduction method by taking into account a priori knowledge of the potential modifications of the design variables.

The proposed approach consists of supplementing the standard transformation basis of a structure by good selected optimized residual static vectors. These vectors depend on the various design variables retained for structure of study. Then, this approach is extended to nonlinear dynamics. In fact, the presence of one or more nonlinearities can be comparable to a continuous family of the structures locally modified. This work shows the effectiveness of this method in the case of one or more localized nonlinearities:

This study in the time domain comprises two parts:

- The first part relates to the basis formulation in linear dynamics KL, in the time domain $\mathbf{T}^0_{\mathrm{KL}}$. Then, this basis is extended to structures modified locally by zone, leading to a new enriched basis by static residues representing these structural modifications: $\mathbf{T}^l_{\mathrm{KLE}} = \left[\mathbf{T}^0_{\mathrm{KL}} \middle| \Delta\mathbf{T}^l_{\mathrm{KL}}\right]$.
- In the second part, an extension of the enriched basis is proposed for the case of nonlinear structures. In fact, the idea consists of enriching the time KL transformation by static residues due to the nonlinear modifications: $\mathbf{T}^{\mathrm{nl}}_{\mathrm{KLE}} = \left[\mathbf{T}^0_{\mathrm{KL}} \middle| \Delta\mathbf{T}^l_{\mathrm{KL}} \middle| \Delta\mathbf{T}^{\mathrm{nl}}_{\mathrm{KL}}\right]$.

Finally, this method is compared with the standard modal condensation method (CM), which is enriched by residues associated with the parametric modifications and to localized nonlinearities, based on the modal properties of the associated linear system.

2. Formulation of the reanalysis problem of nonlinear structures

The second-order differential equation corresponding to the movement of a nonlinear modified system is expressed in the following matrix form:

$$(\mathbf{M} + \Delta\mathbf{M})\ddot{\mathbf{y}}(t) + \mathbf{B}\dot{\mathbf{y}}(t) + (\mathbf{K} + \Delta\mathbf{K})\mathbf{y}(t) + \mathbf{f}^{\mathrm{nl}}((\dot{\mathbf{y}}(t), \mathbf{y}(t)) = \mathbf{f}^{e}(t) \tag{1}$$

Eq. (1) can be written under a new form as follows:

$$\begin{aligned} &\mathbf{M}\ddot{\mathbf{y}}(t) + \mathbf{B}\dot{\mathbf{y}}(t) + \mathbf{K}\mathbf{y}(t) + \mathbf{f}^{\mathrm{nl}}((\dot{\mathbf{y}}(t), \mathbf{y}(t)) = \mathbf{f}^{e}(t) + \mathbf{f}^{\Delta}(t) \\ &\text{with } \ \mathbf{f}^{\Delta}(t) = -(\Delta\mathbf{M}\ddot{\mathbf{y}}(t) + \Delta\mathbf{K}\mathbf{y}(t)) \end{aligned} \tag{2}$$

where **M**, **K** and **B** are, respectively, the mass, stiffness and damping matrices, which are real, symmetric, positive definite; $\mathbf{f}^{\Delta}(t)$ is the parametric modifications forces; $\mathbf{f}^{\mathrm{nl}}((\dot{\mathbf{y}}(t), \mathbf{y}(t))$ represents the nonlinear forces of stiffness and damping in the time domain; and $\mathbf{f}^{e}(t)$ is the external applied force.

One is particularly interested in Eq. (2) comprising of several nonlinearities of the cubic stiffness type. Each nonlinearity results in the presence of a force proportional to the cubic displacement in this model:

$$\mathbf{f}_{ij}^{\mathrm{nl}}(\mathbf{y}(t)) = \gamma(\mathbf{y}(t))^3 \tag{3}$$

where γ is a coefficient of the nonlinearity in N/m^3.

The real nonlinear force of the system (2) defined between two points (*i,j*) of the structure is written as follows:

$$\mathbf{f}^{\mathrm{nl}}(\mathbf{y}(t)) = \left[0\ldots0\ldots\gamma(y_i - y_j)^3\ldots0\ldots\gamma(y_j - y_i)^3\ldots0\right]^{\mathrm{T}} \tag{4}$$

where y_i and y_j are, respectively, displacement at point i and displacement at point j.

The solutions of this nonlinear problem can be obtained only using one iterative numerical algorithm and the calculation of these solutions can be considered only in the discrete case, thus leading to a prohibitory calculation cost.

As such, an approach in the time domain is proposed based on KL condensation, allowing one to treat condensed models having modified design parameters in an optimization context, without carrying out a new reanalysis of the complete model on each modification level. Then, this approach is extended to a nonlinear context. The presence of one or more nonlinearities can be comparable to a continuous family of locally modified structures; the nonlinear design variables are then used for the generation of the static residues basis.

3. Reanalysis of nonlinear structures by the enriched KL condensation method

3.1. Formulation of the standard transformation KL

The decomposition of KL is largely used in order to determine the eigenfunctions of a linear system calculated starting from a set of data generated through this system [12]. In fact, any set of data can be broken up into an optimal series of orthogonal empirical eigenfunctions using the KL procedure. These optimal eigenfunctions are extracted starting from the temporal or frequency samples from the responses of the system. The extraction of a minimal number of functions, constituting it possible to collect the maximum energy of the system, constitutes a criterion of selection of KL transformation. Consequently this transformation is used for the linear systems as a tool for condensation [13,14].

The KL procedure [15] consists of selecting a series of optimal modes [$\phi_1(x)$, $\phi_2(x)$,…,$\phi_M(x)$] starting from a series of temporal responses, temporal condensation $y(x, t)$. These functions $\phi(x)$ are obtained by maximizing the mean of the internal product between $y(x, t)$ and $\phi(x)$:

$$\text{Maximiser} \quad \frac{\left\langle(\phi, y)\right\rangle}{(\phi, \phi)} \tag{5}$$

where $(\phi, y) = \int_\Omega \phi(x)y(x,t)\,\mathrm{d}\Omega$ is the internal product defined in space Ω. $\langle\bullet\rangle$ is the mean operator.

The mean is defined as a correlation function of two points as follows:

$$\begin{aligned} E(x,x') &= \left\langle y(x,t)y(x',t)\right\rangle \\ &= \frac{1}{N}\sum_{i=1}^{N} y(x,t)y^{\mathrm{T}}(x',t) \end{aligned} \tag{6}$$

The normalization condition, $(\phi, \phi) = 1$, is imposed in order to obtain a unique solution.

Consequently, the condition on Eq. (5) is $\phi(x)$, which represents the eigenfunction of the eigenvalues problem according to

$$\int_\Omega K(x,x')\phi(x',t)\mathrm{d}x' = \lambda\phi(x) \tag{7}$$

where $K(x, x')$ is a non-negative Hermitain operator.

The solution of the optimization problem (5) is given by the eigenfunctions $\phi_i(x)$ of the integral equation (7), also named proper orthogonal modes (POMs). The corresponding eigenvalues $\lambda_i(\lambda_i \geqslant 0)$ are named proper orthogonal values (POVs).

Among the resolution methods of the problem (8), one distinguishes that which was developed by Taehyoun [13]. This method consists in exploiting the sampling procedure for which POMs can be represented as follows:

$$\phi(x) = \sum_{k=1}^{N} \alpha_k y_k(t) \tag{8}$$

The substitution of the relation (8) in Eq. (7) led to the eigenvalues problem according to

$$\mathbf{C}\boldsymbol{\alpha} = \lambda\boldsymbol{\alpha} \tag{9}$$

$$\mathbf{c}_{nk} = \frac{1}{N}(y_n, y_k) = \frac{1}{N}\int_{\Omega} y_n(x')y_k^{\mathrm{T}}(x')\mathrm{d}x' \tag{10}$$

where $\mathbf{c}_{nk}$ is the symmetric, positive definite covariance matrix and $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_N)$ are the eigenvectors of the problem (9).

The eigenvectors of Eq. (9) are substituted in Eq. (8), generating the eigenfunctions $\phi(x)$.

The order of the eigenfunctions $\phi_1(x)$, $\phi_2(x),\ldots,\phi_N(x)$ corresponds to the order of the amplitude of the eigenvalues classified in the decreasing order ($\lambda_1 > \lambda_2 > \cdots > \lambda_N$).

The majority of the structural or energetic characteristics are collected by the first eigenfunctions [14]. These eigenfunctions satisfy the following orthonormality relations:

$$(\phi_n, \phi_k) = \begin{cases} 1 & (n = k) \\ 0 & (n \neq k) \end{cases} \tag{11}$$

Each member of the unit is reproduced by a modal decomposition of the functions $\phi(x)$:

$$y_n(x) = \sum_k a_k \phi_k(x) \tag{12}$$

Eq. (11) is the decomposition of KL. The series $\phi_k(x)$ constitutes the empirical base:

$$\mathbf{T}_{\mathrm{KL}}^{0} = [\phi_1, \ldots, \phi_k] \tag{13}$$

The importance of the particular mode KL is determined by the amplitude of λ_i, which represents $\langle ||(\phi_i, y)||^2 \rangle$ in the time domain. The retained (r) eigenfunctions ($r \ll M$) for the continuation of this study correspond to the first (r) eigenvalues λ_i ($\lambda_1 > \lambda_2 > \cdots > \lambda_r$).

3.2. Proposition of an enriched KL transformation

It is possible to use an alternative solution that consists of supplementing the transformation matrix for a new basis of optimized vectors built starting from the design variables that are modified. The innovation of this approach lies in its use of a priori knowledge of the design problem and in the localization of the potential modifications by retaining the amplitude of modifications like one of the variables. The objective is to build an optimal set, system of Ritz vectors while being based on the nominal model to enrich the matrix of standard transformation (Fig. 1).

3.2.1. Calculation of the static residues vectors for the structural modifications

In order to avoid starting the design cycle of the modified structures in an optimization procedure from the beginning, it is necessary to enrich the KL standard basis of the linear model by a best selection of static vectors. In fact, one distinguishes in this work two sets of parameters: parameters (k_i) and (m_i) acting on the stiffness and the mass. The initial FEM is represented by its matrices **K** and **M**. The FEM of the modified

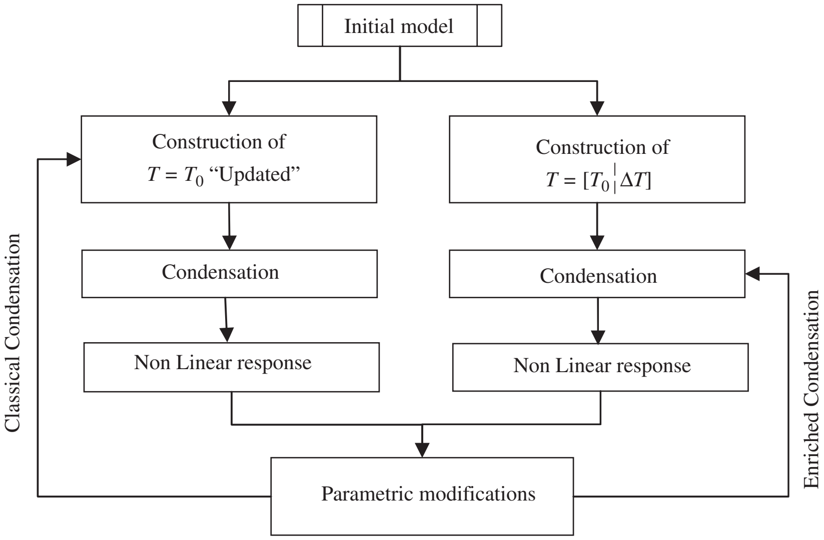

Fig. 1. Global algorithm of the prevision nonlinear response with the enriched condensation method.

system is represented by its matrices $\mathbf{K}_m$ and $\mathbf{M}_m$ such as

$$\begin{aligned} \mathbf{K}_m(k_i) &= \mathbf{K} + \Delta\mathbf{K}(k_i) \\ \mathbf{M}_m(m_i) &= \mathbf{M} + \Delta\mathbf{M}(m_i) \end{aligned} \tag{14}$$

The initial model is not entire modified, but at least in large influential zones of the structure or substructure. To account for the effect of the modifications, the first step consists of fixing, estimating, or localizing the modifiable zones of the model and their associate design parameters:

$$\begin{aligned} \mathbf{K}^{\text{zone}} &= \sum_{e=1}^{n\text{elem}} \mathbf{K}_e^{\text{elem}} \\ \mathbf{M}^{\text{zone}} &= \sum_{e=1}^{n\text{elem}} \mathbf{M}_e^{\text{elem}} \end{aligned} \tag{15}$$

One can write the relation (15) in the form:

$$\begin{aligned} \Delta\mathbf{K} &= \sum_{i=1}^{np} \mathbf{K}_i^{\text{zone}}(\Delta k_i) \\ \Delta\mathbf{M} &= \sum_{i=1}^{np} \mathbf{M}_i^{\text{zone}}(\Delta m_i) \end{aligned} \tag{16}$$

where Δk_i and Δm_i are the parameter variations of k_i and m_i. In general, the parameters k_i, m_i intervene in a nonlinear way in the matrices of corrections.

For the plate model, the parameter setting concerning the stiffness is carried out by uncoupling the effects of membrane and flexion. One can decompose the total elementary stiffness matrix into membrane stiffness $\mathbf{K}_m^{\text{zone}}$ and flexion stiffness $\mathbf{K}_f^{\text{zone}}$. The thickness, noted "$e$", with a linear contribution in membrane and cubic contribution in flexion, results in

$$\mathbf{K}^{\text{zone}} = e\mathbf{K}_m^{\text{zone}} + e^3\mathbf{K}_f^{\text{zone}} \tag{17}$$

A weak variation thickness "Δe" involves a variation of stiffness $\mathbf{K}^{\text{zone}}(\Delta e)$ defined by

$$\mathbf{K}^{\text{zone}}(\Delta e) = \frac{\Delta e}{e}(e\mathbf{K}_m^{\text{zone}} + 3e^3\mathbf{K}_f^{\text{zone}}) \tag{18}$$

In the absence of the nonlinearities and external forces, the conservative system (2) becomes

$$\mathbf{M}\ddot{\mathbf{y}}(t)+\mathbf{K}\mathbf{y}(t)=\mathbf{f}^{\Delta}(t)$$
$$\text{with}\quad \mathbf{f}^{\Delta}(t)=-(\Delta\mathbf{M}\ddot{\mathbf{y}}(t)+\Delta\mathbf{K}\mathbf{y}(t)) \tag{19}$$

The enrichment of the standard transformation KL (Eq. (13)) by $\Delta\mathbf{T}^{l}_{\mathrm{KL}}$, the static residues associated with the potential structural modifications, is based on the approximation

$$\mathbf{f}^{\Delta}(t)\simeq-\left(\Delta\mathbf{M}\ddot{\mathbf{y}}_0(t)+\Delta\mathbf{K}\mathbf{y}_0(t)\right) \tag{20}$$

where $\mathbf{y}_0(t)$ is the response of the initial system.

For each zone ($i=1,\ 2,\ldots,\ np$), a force sub-basis $\mathbf{F}^{\Delta}$ can be defined from the initial modal properties (deterministic eigensolutions) and the stiffness and mass matrices of the zones:

$$\mathbf{F}_{\Delta}=[\mathbf{F}^1_{\Delta},\mathbf{F}^2_{\Delta},\ldots\mathbf{F}^{np}_{\Delta}] \tag{21}$$

The representative force basis of the modifications group by the concatenation of the sub-basis $\mathbf{F}_{\Delta}$, followed by singular values decomposition (SVD), leads to linear independence of the columns of this basis. From this, the static vectors can be constructed:

$$\mathbf{R}_{\Delta}=\mathbf{K}^{-1}\mathbf{F}_{\Delta} \tag{22}$$

In practice, the resolution of the problem (2) without condensation leads to high numerical costs and sometimes it can be very difficult. The condensation of this model by a standard reduction method is proving insufficient in terms of robustness towards parametric modifications. Therefore, it is proposed that a dynamic condensation method can be exploited by adapting it to nonlinear models.

The reduction basis $\mathbf{T}^{l}_{\mathrm{KLE}}$, which is common to both the initial and the modified systems, is constructed by the nominal basis $\mathbf{T}^{0}_{\mathrm{KL}}$ and the static displacements, which are associated with a set of static loads $\mathbf{F}_{\Delta}$ that are representative of the potential perturbations (Eq. (20)):

$$\mathbf{T}^{l}_{\mathrm{KLE}}=\left[\mathbf{T}^{0}_{\mathrm{KL}}\middle|\Delta\mathbf{T}^{l}_{\mathrm{KL}}\right] \tag{23}$$

where $\mathbf{T}^{0}_{\mathrm{KL}}$ is the nominal reduction basis, $\Delta\mathbf{T}^{l}_{\mathrm{KL}}$ is the correction basis due to structural modifications and $\mathbf{T}^{l}_{\mathrm{KLE}}$ is the robust extended basis.

In practice, the nominal reduction basis $\mathbf{T}^{0}_{\mathrm{KL}}$ can be a simply truncated modal basis in the direct dynamic condensation (CM).

3.2.2. *Calculation of the residues static vectors for localized nonlinearities*

In this case, one is located in the nonlinear field in absence of structural modifications. The nonlinear stiffness matrix $\Delta\mathbf{K}_{\mathrm{nl}}$ is added to the matrix of initial stiffness in the form:

$$\mathbf{K}_{\mathrm{nl}}=\mathbf{K}+\Delta\mathbf{K}_{\mathrm{nl}} \tag{24}$$

In the same manner, one can apply the robust condensation method developed previously. In fact, it is possible to build static residues associated with the nonlinear forces. For that, it is enough to know the elements that are submitted to stiffness modifications. Either $\Delta\mathbf{K}_{\mathrm{nl}}$ the stiffness matrix or only the coefficients associated with the dof of the nonlinear elements with the same types are not null:

$$\Delta\mathbf{K}_{\mathrm{nl}}=\begin{bmatrix}0&0&\cdots&0&\cdots&0\\0&0&\cdots&0&\cdots&0\\0&0&\cdots&k^{\mathrm{nl}}&\cdots&0\\0&0&\cdots&0&\cdots&0\end{bmatrix} \tag{25}$$

Then, one can calculate an associated forces basis $\mathbf{F}^{\mathrm{nl}}_{\Delta}=\Delta\mathbf{K}_{\mathrm{nl}}\mathbf{B}$, then the static residues associated with the forces basis $\mathbf{R}^{\mathrm{nl}}_{\Delta}=\mathbf{K}^{-1}\mathbf{F}^{\mathrm{nl}}_{\Delta}$. The response vectors basis $\mathbf{B}$ can be either a modal basis resulting from the associated conservative linear system (SCA), or a KL basis. In fact, if one uses a basis modal truncated with first modes $\mathbf{B}=\mathbf{Y}_0$ and if a basis KL is used, $\mathbf{B}=\mathbf{T}^{0}_{\mathrm{KL}}$.

3.2.3. Enrichment of the KL basis by static residue vectors in the presence of structural modifications and localized nonlinearities

Following the structural modifications and the nonlinear modifications, the enrichment of the KL basis is written as

$$\mathbf{T}^{\mathrm{nl}}_{\mathrm{KLE}} = \left[\mathbf{T}^{0}_{\mathrm{KL}} \middle| \Delta\mathbf{T}^{l}_{\mathrm{KL}} \middle| \Delta\mathbf{T}^{\mathrm{nl}}_{\mathrm{KL}}\right] \tag{26}$$

4. Nonlinear response of the modified model condensed by enriched KL

The equations governing the modified nonlinear responses condensed by the enriched transformation KL, $\mathbf{T}^{\mathrm{nl}}_{\mathrm{KLE}}$, at the time t_{n+1} can be written in the following form:

$$\mathbf{M}^{c}_{m}\ddot{\mathbf{y}}^{c}_{n+1}(t) + \mathbf{B}^{c}_{m}\dot{\mathbf{y}}^{c}_{n+1}(t) + \mathbf{K}^{c}_{m}\mathbf{y}^{c}_{n+1}(t) + \mathbf{f}^{c}_{\mathrm{nl}}((\dot{\mathbf{y}}^{c}_{n+1}(t), \mathbf{y}^{c}_{n+1}(t)) = \mathbf{f}^{c}_{e}(t) \tag{27}$$

where $\mathbf{M}^{c}_{m}$, $\mathbf{K}^{c}_{m}$ and $\mathbf{B}^{c}_{m}$ are, respectively, the condensed modified mass, stiffness and damping matrices. $\mathbf{f}^{c}_{\mathrm{nl}}((\dot{\mathbf{y}}^{c}_{n+1}(t), \mathbf{y}^{c}_{n+1}(t))$ and $\mathbf{f}^{c}_{e}(t)$ are, respectively, the vector of the condensed nonlinear forces and the applied condensed external forces.

In this paper, the nonlinearity introduced is of cubic stiffness type (Eq. (28))

$$\mathbf{f}^{c}_{\mathrm{nl}}((\dot{\mathbf{y}}^{c}_{n+1}(t), \mathbf{y}^{c}_{n+1}(t)) = \gamma\left(\mathbf{y}^{c}_{n+1}\right)^{3} \tag{28}$$

where γ is the nonlinear stiffness coefficient.

Eq. (29)then makes it possible to define the residues vector $\Re(\mathbf{y}^{c}_{n+1})$ as follows:

$$\Re(\mathbf{y}^{c}_{n+1}) = \mathbf{M}^{c}_{m}\ddot{\mathbf{y}}^{c}_{n+1}(t) + \mathbf{f}((\dot{\mathbf{y}}^{c}_{n+1}(t), \mathbf{y}^{c}_{n+1}(t)) - \mathbf{f}^{c}_{\mathrm{ext}}(t) \tag{29}$$

Also $\left(\mathbf{y}^{c}_{n+1}\right)^{k}$ an approximation of the solution $\mathbf{y}^{c}_{n+1}$ obtained with the iteration k, the residual equation can be written with a satisfactory precision as follows:

$$\Re(\mathbf{y}^{c}_{n+1})^{k+1} - \Re(\mathbf{y}^{c}_{n+1})^{k} = \mathbf{S}(\mathbf{y}^{c}_{n+1})^{k}.\left((\mathbf{y}^{c}_{n+1})^{k+1} - (\mathbf{y}^{c}_{n+1})^{k}\right) \tag{30}$$

$\mathbf{S}(\mathbf{y}^{c}_{n+1})^{k}$ is the matrix of the gradients matrix at the iteration k.

The nonlinear condensed modified system (Eq. (30)) is solved iteratively by exploiting the Newton–Raphson method.

5. Comparison criteria of the time responses

In order to quantify the comparison of the obtained predictions, the temporal moments are used as a comparison criterion. These temporal moments $M_i(t_s)$ have been proposed for the transitory analysis [23]. They are similar to the static moments and are calculated as balanced summations of the quadratic temporal signal:

$$M_i(t_s) = \int_{-\infty}^{+\infty} (t - t_s)^i \left(y(t)^2\right) \mathrm{d}t \tag{31}$$

where t_s corresponds to a temporal shift and the index i represents the order of the moment. For more simplicity, the temporal moments M_i are defined for $T_s = 0$.

These temporal moments, $M_i(t_s)$, can be normalized to generate what is referred to in the literature as the central moments [23,24]. The central moments are defined and calculated as follows:

$$\begin{aligned}
&E = M_0: \quad \text{energy} \ (\mathrm{m^2 s}),\\
&T = \tfrac{M_1}{M_0}: \quad \text{central time ``Centroid'' (s)},\\
&A_e^2 = \tfrac{E}{D}: \quad \text{root energie amplitude} \ (\mathrm{m^2}),\\
&D^2 = \tfrac{M_2}{M_0} - \left(\tfrac{M_1}{M_0}\right)^2\\
&D: \text{rms duration (s)}
\end{aligned} \tag{32}$$

Eqs. (31) and (32) show that the calculation of the temporal moments starting from the temporal signals is traditional. The temporal moments with a limited band have been proposed to analyse the signals in a specific frequency band [24].

The first central moment E represents the total energy of the signal. It is also equal to the value of the autocorrelation function of Eq. (5) for null times. The amplitude of the root of energy A_e represents the amplitude through the entire signal. It differs from E by the normalization factor equal to the inverse characteristic duration D. The central time T or centroide is defined as the time indexed t_s that produces a null first moment, which is t_s such as $M_1(t_s) = 0$.

While looking to the distribution of the temporal energy, T represents a point of balance, which is the moment where half of the energy is passed and the other half has just arrived to the sensor. The duration within the meaning of least square D describes the dispersion of the wave form. A basic rule is a significant part of the transitory energy must be with 2 or 3 the rms duration in the vicinity of the centroide T it is identical to the standard deviation in static.

6. Numerical simulations

Example 1. The considered structure (Fig. 2) is a system with three beams linked between them by two nonlinear springs K_1 and K_2, respectively, at the points F, G, and H. The structure is embedded at points A, B, C and with two springs nonlinear K_3 and K_4 at the points D and E.

The FEM contains 288 dof. The nominal mechanical characteristics are: $E_0 = 2.1 \times 10^{11}\,\mathrm{N/m^2}$; $\rho_0 = 7800\,\mathrm{kg/m^3}$. The nonlinear coefficients $\gamma_{(i=1,\ldots,4)}$ are defined in Table 1.

Six modifications per zone are defined (Fig. 2). The Young modulus of the six modification zones $E_{(i=1,\ldots,6)}$ is equal to $2.1 \times 10^{11}\,\mathrm{N/m^2}$. The density for the same modified zones $\rho_{(i=1,\ldots,6)}$ is equal to $7800\,\mathrm{kg/m^3}$.

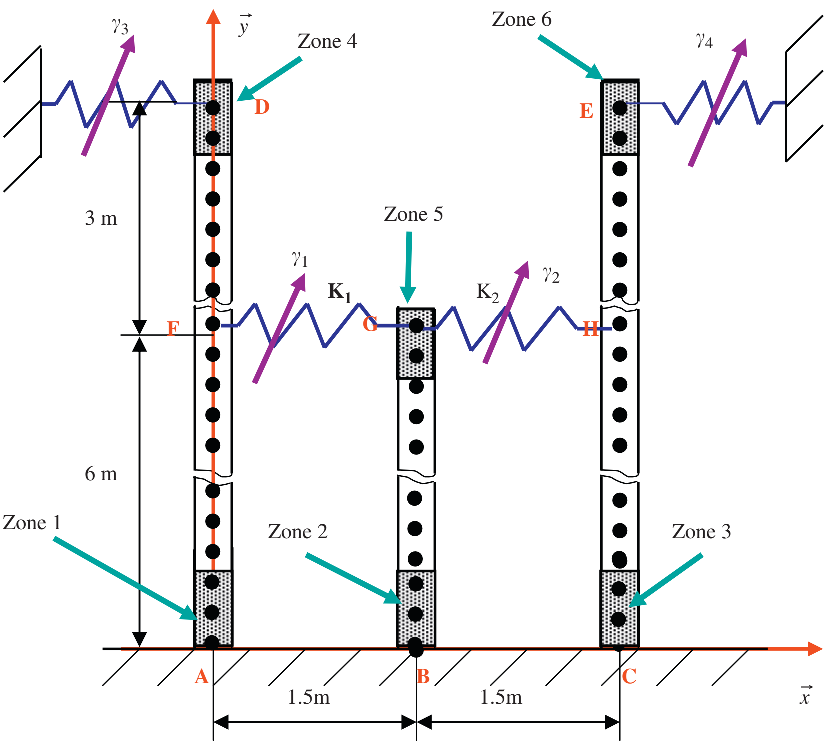

Fig. 2. Finite Elements Model of the structure with three beams.

Table 1
Nonlinear parameters

	Nominal values	Description
$\gamma_1 = \gamma_4$	10^8	Nonlinear stiffness of the spring 1 and 4 (N/m^3)
$\gamma_2 = \gamma_3$	10^6	Nonlinear stiffness of the spring 2 and 3 (N/m^3)

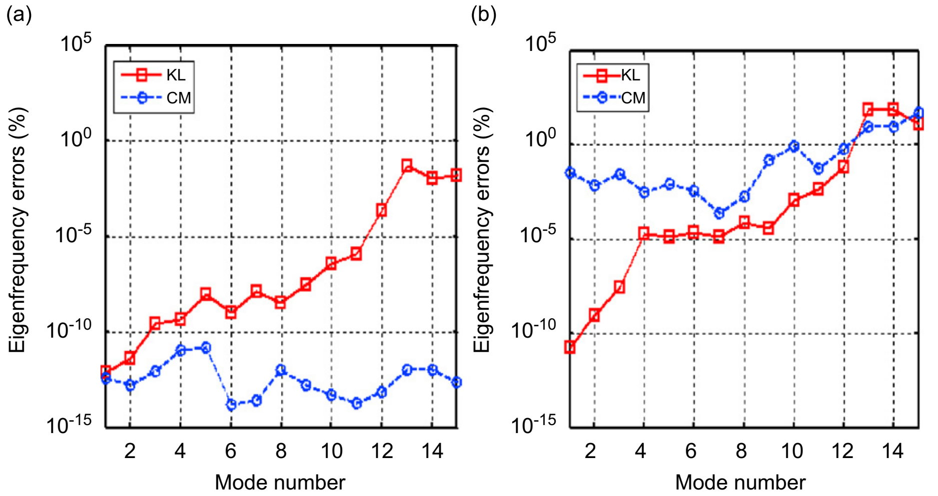

Fig. 3. Errors on the eigensolutions: (a) relative eigenfrequency errors and (b) relative eigenvectors errors.

The frequency band of analysis [0,500 Hz] contains the first 15 modes. The structure is excited with dof no. 71 (point F) by a force of 10^4 N. The observation point is dof no. 179 (point G).

For this first example, one can select 100 samples "snapshots". The basis is then truncated to the first 30 eigenfunctions. In the same way, the condensed matrices by the standard CM method have identical size as those condensed by KL.

The curves in Figs. 3(a) and (b) illustrate, respectively, the eigenfrequency errors and the eigenvectors errors on the eigenfrequencies and eigenvectors between the linear condensed model by the KL method or by the CM method and the linear reference model.

It is noted that the errors on the eigensolutions resulting from the two methods are acceptable on the first 12 modes.

On the last three modes, the CM method always presents a good coherence with the reference, contrary to as well as the KL method.

Figs. 4(a) and (b) illustrate, respectively, the nonlinear responses without structural modifications for the same observation position (point F: dof no. 71) and the zoom for the same responses between [0, 0.05] s. One can note that the nonlinear condensed response issued from the direct condensation method is consistent with reference to the beginning and somewhat consistent on the levels of the peaks between [0, 0.1] s. On the other hand, the nonlinear response condensed by the KL method is always consistent with the reference.

Table 2 shows that the energetic criteria for the reference and the CM method are not identical. On the contrary, the values resulting from the KL method are confused with those of the reference. Table 3 shows the parametric modifications of the structure per zone.

In the case of parametric modifications, the initial basis $\mathbf{T}^0_{KL}$ or $\mathbf{T}^0_{CM}$ is not capable of representing correctly the dynamic behaviour of modified structures. Therefore, it is necessary to enrich this initial basis by static residues representing the error forces. These residues are well selected, thus avoiding the systematic use of all the linearly independent columns after SVD.

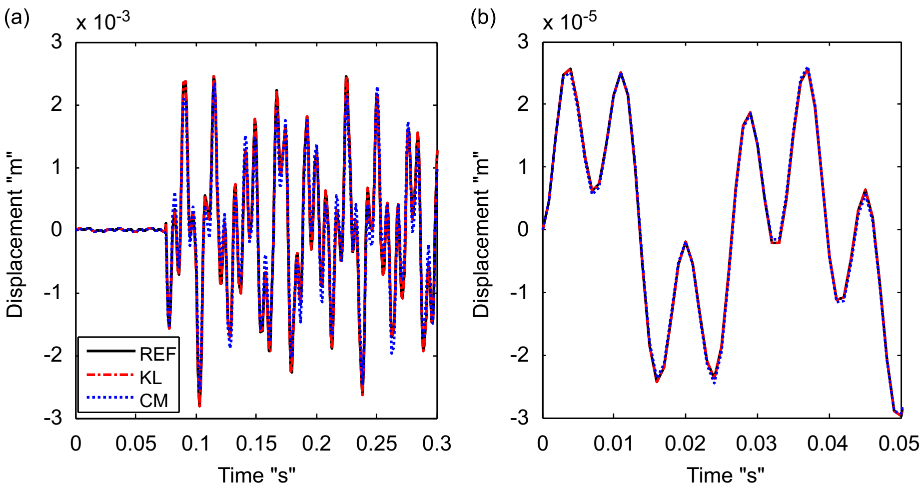

Fig. 4. Nonlinear responses resulting from the reference and the condensed models (dof no. 71): (a) nonlinear responses at dof no. 71 and (b) zoom between [0 0.05 s] for the same responses.

Table 2
Energy criteria (case: without structural modification)

	E (m^2 s)	T (s)	D (s)
Exact response	2.61×10^{-2}	0.090	0.0716
CM response	2.54×10^{-2}	0.093	0.0713
KL response	2.61×10^{-2}	0.09	0.0716

Table 3
Parametric modifications

	Structure 1: three beams structure					
	Zone 1	Zone 2	Zone 3	Zone 4	Zone 5	Zone 6
E	× 50	× 50	× 50	× 15	× 15	× 15
ρ	× 20	× 20	× 20	× 50	× 50	× 50

The KL method is first applied and the initial model with 288 dof is reduced to a condensed one with 30 dof (KL_30). The KL method is then enriched by 10 static residual vectors, resulting in a new robust condensed model with 40 dof (KLE). These residual vectors are portioned as 6 vectors due to structural modifications and 4 vectors due to nonlinear modifications.

Fig. 5 represents the MAC matrix between modes of the reference modified system and those of the reference initial system. This MAC illustrates the level of the applied modifications. Also, Table 4 shows the gaps in frequencies.

The non-enriched standard condensation cannot allow making the predictive calculation of the modified structure with a sufficient precision. This is illustrated by the MAC matrices (Figs. 6(a)–(c)) given for this example, which show the distance in form (eigenvectors) between the reference modified model (without condensation) and the condensed modified model using the enriched condensation method (KLE): Fig. 6(a) standard condensation KL_30, Fig. 6(b) standard condensation KL_40 (same size as enriched basis) and Fig. 6(c) enriched condensation KLE. The last enriched basis allows a good prediction of the first 13 global modes.

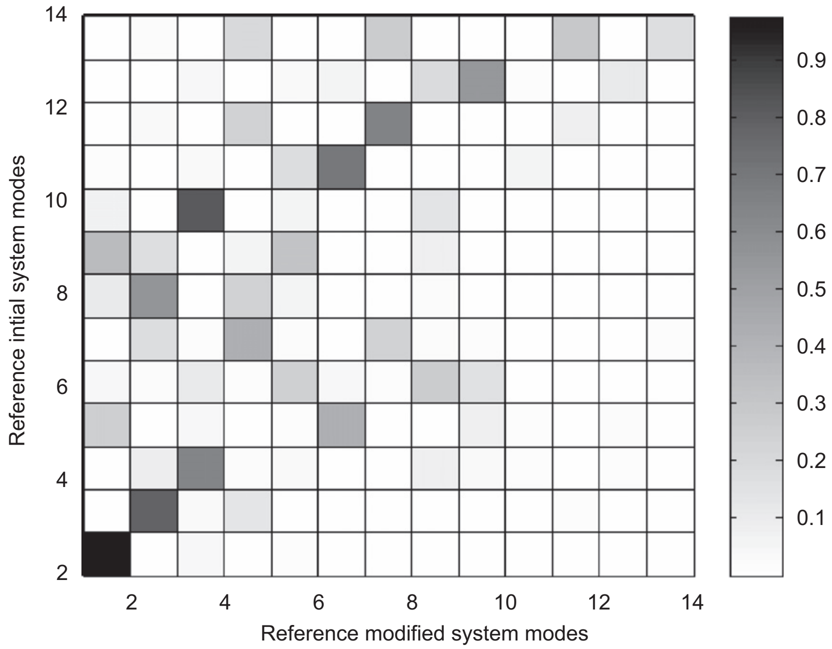

Fig. 5. MAC matrix between modes of the reference modified system and those of the initial reference system.

Table 4
Eigenfrequency of the initial/modified models

	Eigenfrequency (Hz)	
	Initial model	Modified model
1	67.976	46.231
2	95.644	64.702
3	133.735	65.851
4	142.261	72.930
5	142.295	82.232
6	185.138	83.623
7	201.836	111.046
8	208.937	112.206
9	262.264	150.647
10	314.906	193.494
11	318.564	206.731
12	404.587	273.620
13	423.465	287.004
14	423.692	300.708
15	438.267	376.921

Table 5 shows the composition of the transformation matrices according to the used method.

Figs. 7(a) and (b) illustrate the condensed modified nonlinear responses resulting from the reference, the CME and KLE methods at the excitation position (dof no. 71) and the observation position (dof no. 179). The condensed modified nonlinear responses concern, respectively, the excitation point and the observation point. It is noted that these responses resulting from the proposed KLE method are well consistent with the reference. The responses resulting from the CME method are inconsistent with the reference.

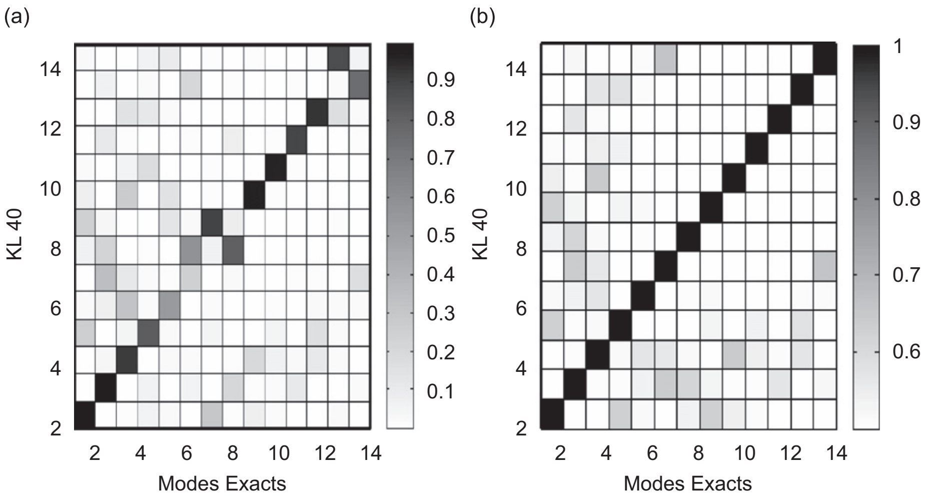

Fig. 6. Distance in form (MAC): (a) exacts/KL modes and (b) modes exacts/KLE.

Table 5
Composition of the condensation basis

	KL40 method (initial)	CME method	KLE method
Modes POM	40	30	30
Residues	0	10	10
Total	40	40	40

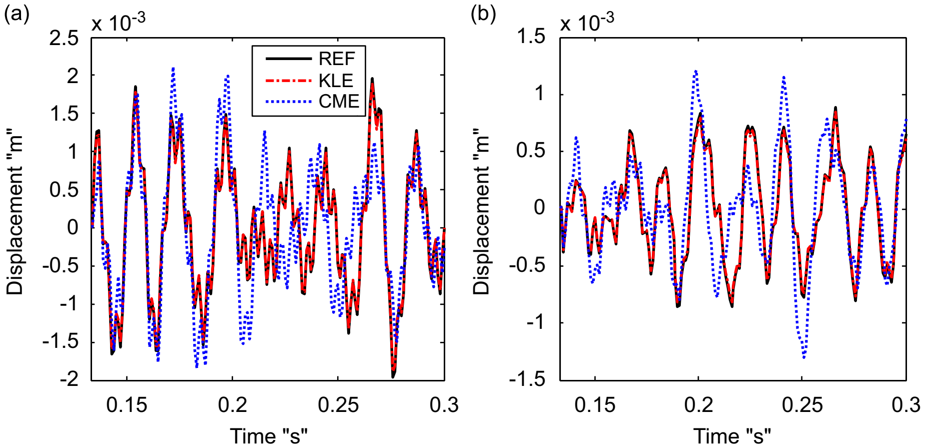

Fig. 7. Nonlinear responses resulting from the reference and the condensed modified models (dof no. 71: excitation position and no. 179: observation position): (a) responses at the excitation position (dof no. 71) and (b) responses at the observation position (dof no. 179).

Table 6 illustrates the eigensolutions calculated by the KL_40, CME and KLE methods. It is shown that the KLE method gives good accuracy when predicting the 15 eigenmodes, compared with the standard KL_40 method, which gives more limited prediction level.

In order to highlight the performances of the reduced models in terms of calculation costs, the CPU time between the condensed and reference models can be compared. Examination of Table 7 shows the good

Table 6
Error on the eigensolutions resulting from the three methods

Mode number	KL initial		CME		KLE	
	40		40		40	
	εf	εv	εf	εv	εf	εv
1	0.309	1.104	0.000	0.001	0.000	0.000
2	0.765	2.958	0.006	0.008	0.000	0.000
3	1.737	8.723	0.003	0.023	0.000	0.000
4	0.353	4.765	0.005	0.015	0.000	0.000
5	0.805	7.114	0.008	0.114	0.000	0.000
6	0.447	3.416	0.044	0.416	0.000	0.000
7	0.052	0.685	0.052	0.385	0.000	0.000
8	0.270	2.768	0.027	0.410	0.000	0.000
9	0.206	3.313	0.026	0.373	0.000	0.000
10	1.629	10.624	0.029	1.124	0.000	0.000
11	0.486	4.615	0.486	0.615	0.000	0.000
12	1.403	18.047	0.403	1.047	0.000	0.001
13	0.736	21.476	0.736	2.293	0.000	0.002
14	7.477	36.074	2.390	1.302	0.000	0.009
15	6.016	47.394	0.070	2.140	0.001	0.220

Table 7
CPU time–reduction ratio

	CPU times (min)		
	REF	CME	KLE
	100	3	2
Reduction ratio (%)		97	98

Table 8
Energetic criteria (case: with structural modification)

	E (m^2 s)	T (s)	D (s)
Exact response	10.1×10^{-3}	0.064	0.0555
CME response	13.5×10^{-3}	0.067	0.0135
KLE response	10.1×10^{-3}	0.064	0.0555

performances of the proposed method, which exploits the KLE method or the CME method. In fact, the reduction ratio in CPU time is 98% compared to the reference.

After the structural modifications carried out on the structure and in the presence of localized nonlinearities, the energetic criteria for the reference and CME method are not identical (Table 8). But those for the reference and KLE method have a good concordance.

In the second example, one is interested only in the KLE method because the CME method is limited in this first numerical example.

Example 2. The structure considered (Fig. 8) is a system of plates in the shape of a square. Its FEM contains 4140 dof. The geometrical and mechanical characteristics are: $e = 1\times10^{-3}\,\mathrm{m}$; $E_0 = 2.1\times10^{11}\,\mathrm{N/m^2}$; $\rho_0 = 7800\,\mathrm{kg/m^3}$. Six modification zones are defined (Fig. 8 and Table 9). The frequency band of analysis

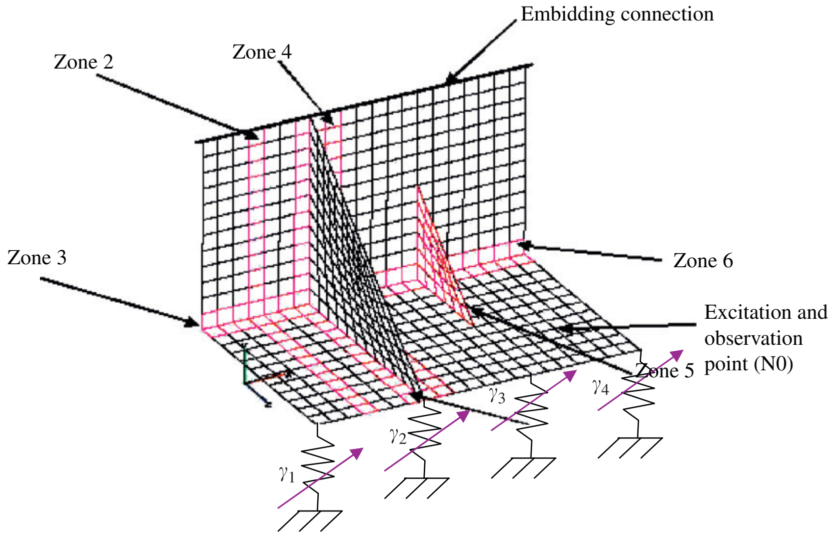

Fig. 8. Square FEM—definitions of the modifications per zone.

Table 9
Parametric modifications

	Structure 2: assembly of plates					
	E: zone 1	*e*: zone 2	*E*: zone 3	*E*: zone 4	*e*: zone 5	*E*: zone 6
Modifications per zones	× 1.5	× 0.25	× 1.5	× 1.5	× 0.25	× 1.5

contains the first 10 modes. The structure is excited at three points (dof no. 842, dof no. 2127 and dof no. 3770) at the same time with a force of 100 N. The nonlinear coefficients $\gamma_{(i=1,\dots,4)}$ are, respectively, defined as follows: $\gamma_1 = \gamma_4 = 10^{10}\,\text{N/m}^3$ and the $\gamma_2 = \gamma_3 = 10^8\,\text{N/m}^3$.

By applying the KL method, one can consider 550 samples "snapshots" with 200 modes in the useful frequency band.

One can present in Fig. 9 the distance in form between the first eigenvectors of the initial model and the modified model. The gaps in frequencies are shown in Table 10.

The enrichment of the transformation basis KLE (Table 11) is realized by using 80 static residues (76 residues vectors due to structural modifications and four residues vectors due to nonlinear modifications).

The modified eigenvectors calculated from the modified reduced models are reconstituted through the respective reduction basis and then compared with the modified eigenvectors obtained from the reference model. One can illustrate the distance in form between these vectors for the KL and KLE methods. Figs. 10 and 11 present, respectively, the relative errors on the eigensolutions and the distance in form between these reconstituted modified eigenvectors obtained using the KL and KLE methods and the reference modified eigenvectors.

In Figs. 12(a) and (b), the results of the condensed model (enriched basis: KLE) are compared to those of the reference. The examination of these results shows that the exploitation of the KL basis allows a good dynamic representation in time domain.

Examination of Table 12 shows the good performances of the proposed method, which exploits the KLE method. In fact, the reduction ratio in CPU time is 80% compared to the reference.

Note that the calculation of the nonlinear responses is carried out on a PC Intel® Pentium® 4 CPU 3.20 GHz with 1 GB of RAM.

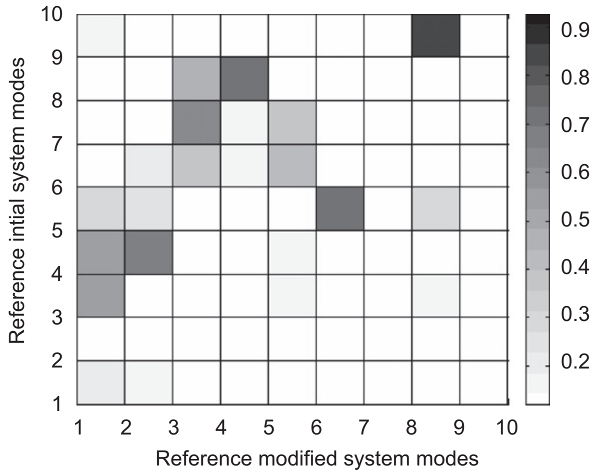

Fig. 9. MAC matrix between modes of the reference modified system and those of the initial reference system.

Table 10
Eigenfrequency for initial/modified models

	Eigenfrequency (Hz)	
	Initial model	Modified model
1	53.570	355.68
2	66.977	409.24
3	189.38	632.70
4	307.91	750.54
5	508.81	812.72
6	550.34	829.06
7	605.41	862.43
8	666.84	985.20
9	759.94	1338.10
10	803.32	1461.10

Table 11
Composition of the condensation basis

	KL280 initial model	KLE model
POM modes	280	200
Residue	0	80
Total	280	280

After the structural modifications carried out on the structure and in the presence of localized nonlinearities, the energetic criteria for the reference KLE method have a good concordance (Table 13).

Example 3. In this example, we consider a linear steel beam embedded on the left and linked with a localized nonlinear system on the right. The nominal mechanical characteristics are $E = 2.1 \times 10^{11}\,\mathrm{N/m^2}$ and $\rho = 7800\,\mathrm{kg/m^3}$. The length of the beam is $L = 0.3\,\mathrm{m}$, the width is $b = 0.0366\,\mathrm{m}$ and the thickness is $h = 0.005\,\mathrm{m}$.

A mass ($m_1 = 0.2\,\mathrm{kg}$) is attached to the free end of the beam with a linear spring stiffness ($K_1 = 2500\,\mathrm{N/m}$). It is also attached to a nonlinear 1-dof system (cubic stiffness $K_2 = 10^8\,\mathrm{N/m^3}$, linear damper $C_1 = 0.25\,\mathrm{N\,m/s}$.

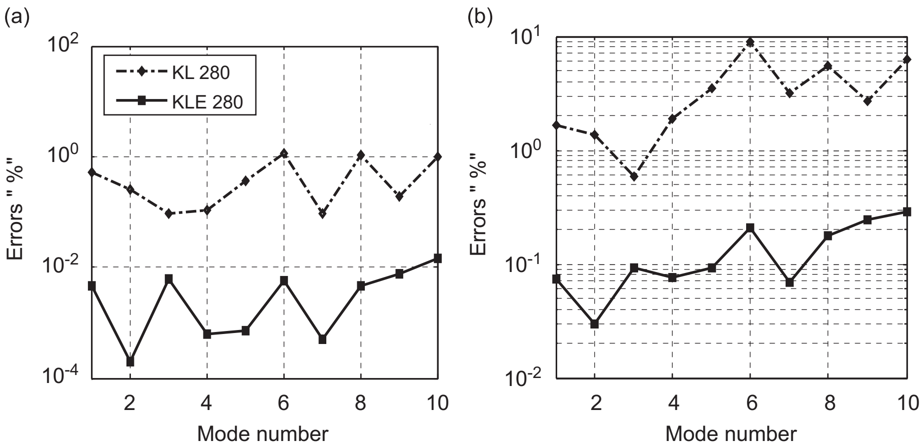

Fig. 10. Error on the eigensolutions of the modified model (KL280—KLE): (a) relative eigenfrequency errors and (b) relative eigenvectors errors.

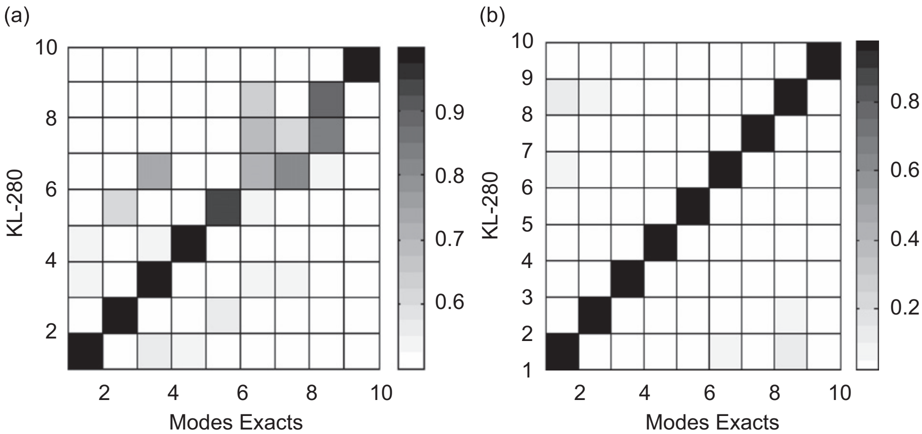

Fig. 11. Distance in forms (MAC): (a) exacts/KL280 and (b) modes exacts/KLE.

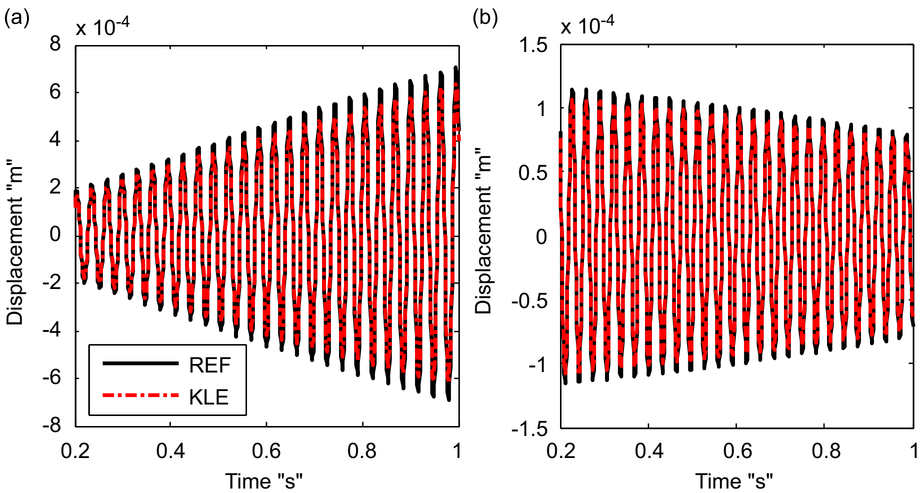

Fig. 12. Nonlinear responses resulting from the reference and the condensed modified model with the KLE method. Responses at the (a) excitation position (Ne) and (b) observation position (No).

Table 12
CPU time–reduction ratio

	CPU times (h)	
	REF 19	KLE 4
Reduction ratio (%)		80

Table 13
Energy criteria (case: with structural modification)

	E (m^2 s)	T (s)	D (s)
Exact response	45.47×10^{-2}	0.509	0.2322
KLE response	41.04×10^{-2}	0.509	0.2322

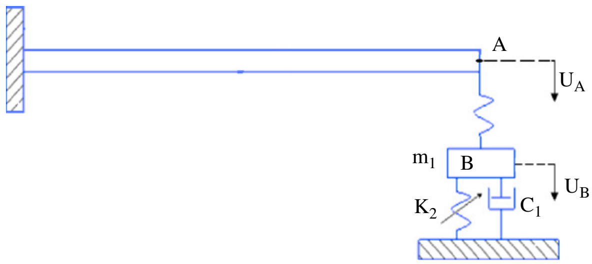

Fig. 13. Complete system: beam with nonlinear system.

The frequency band of analysis is [0–4000 Hz] and contains the first 10 elastic modes. The structure is excited at point A by a harmonic force ($F(t) = F_0$*cos(ωt)). One can also simulate the initial condition with velocity V_0 at the same point A (Fig. 13). The structure is discretized with a 2D beam finite element (2 dof/node). The full FEM contains 101 dof.

In this case, one can select 80 samples "snapshots". The initial KL basis is then truncated to the first 15 eigenfunctions defined by Eq. (13). The curves in Fig. 14 illustrate the eigenfrequencies and the eigenvectors errors between the linear condensed model by the initial KL method containing 15 vectors and the linear reference model. It is noted that the errors on the eigensolutions resulting from the KL method is acceptable on the first 8 modes. Table 14 illustrates the reference eigenfrequencies of the initial linear model ($K_2 = 0$).

To study the robustness of the proposed KLE transformation (16 vectors) versus changes of initial (velocity: V_0) or forcing conditions (amplitude F_0 of excitation) in the nonlinear system, two configurations A and B are studied.

6.1. Initial conditions

6.1.1. Configuration A

In this configuration, we propose to study the chaotic motion with initial conditions $V_0 = 6$ m/s. Fig. 15 shows the reference of the dynamic response and the diagram of phase at dof U_B.

Fig. 16 compares the nonlinear dynamic responses of the full model (reference) and the 16 dof condensed model by the KLE method (16 vectors in the basis).

It is noted that in this regime, the proposed method can correctly predict the reference response in a time interval selected a priori.

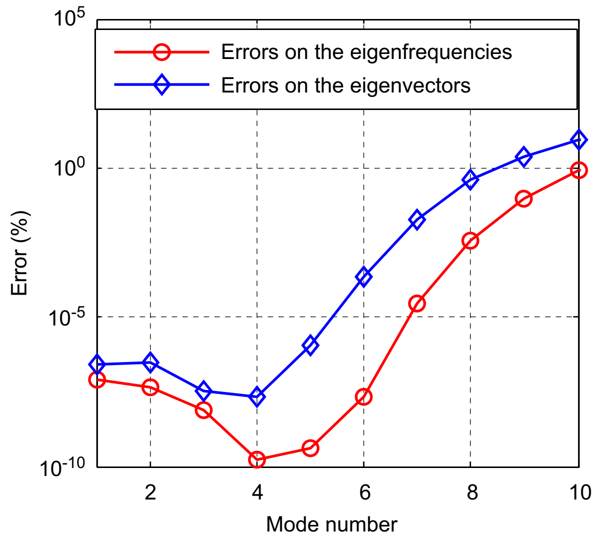

Fig. 14. Comparison of full model–reduced model: (a) eigensolutions and (b) eigenfrequencies.

Table 14
Eigenfrequencies

	Initial eigenfrequency linear model (Hz)
1	10.3
2	28.7
3	106.8
4	294.8
5	576.8
6	953.1
7	1423.6
8	1988.3
9	2647.1
10	3400.1

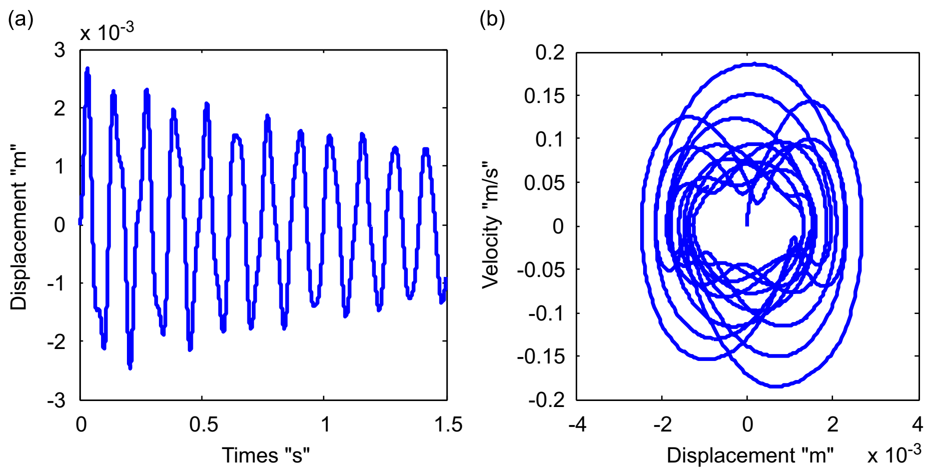

Fig. 15. (a) Nonlinear response at dof U_B and (b) diagram of phase at dof U_B.

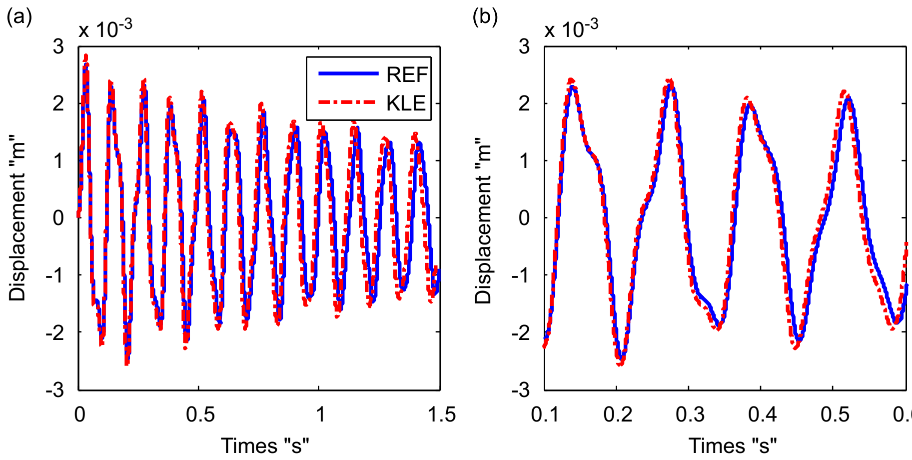

Fig. 16. Nonlinear responses calculated with a reference and an enriched KLE model at dof U_B.

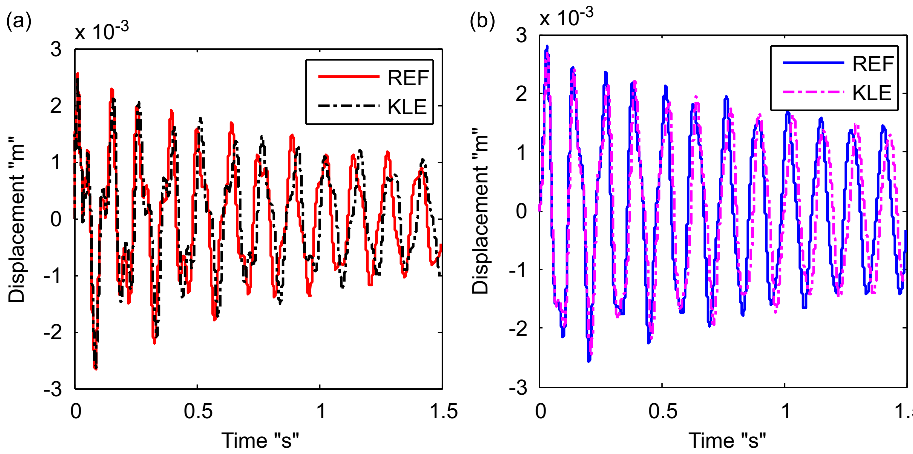

Fig. 17. Nonlinear responses calculated with a reference and an enriched condensed model: (a) dof U_A and (b) dof U_B.

6.1.2. Configuration B

In this new configuration, two levels of perturbation of initial conditions are considered:

- Small perturbation of the initial velocity ($V_0 = 6\,\text{m/s}$) equal to $+5\%$ is introduced. The previous enriched basis (KLE) is used to represent the new condensed nonlinear response (Fig. 17).
  The dynamic prediction is satisfactory for this type of nonlinear regime and the level of change introduced on the initial conditions.
- Important perturbation of initial velocity ($V_0 = 6\,\text{m/s}$) equal to $+20\%$ is introduced. Fig. 18 illustrates the nonlinear responses at points A and B. We note that the previous KLE method cannot correctly predict the reference of nonlinear regime with this level modification introduced on the initial conditions.

6.2. Forcing conditions

6.2.1. Configuration A

In this configuration, one can propose to study the periodic and quasi-periodic motions with $F_0 = 10\,\text{N}$ ($V_0 = 0\,\text{m/s}$).

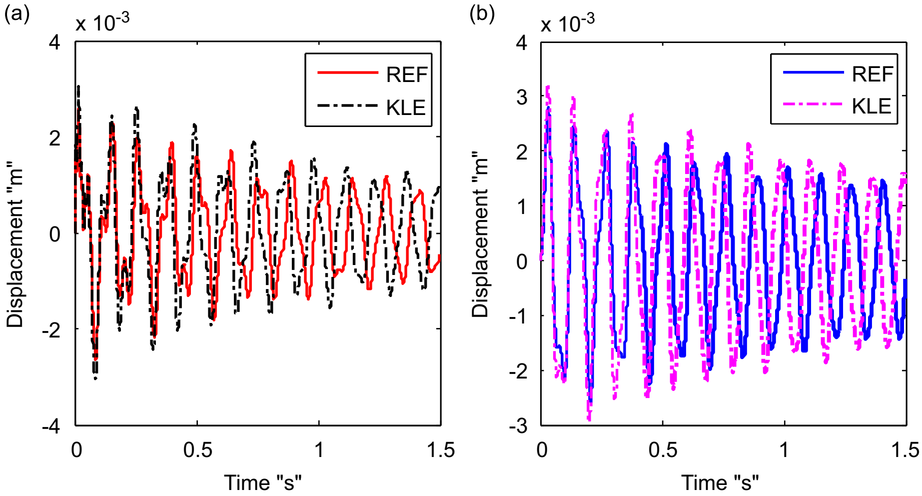

Fig. 18. Nonlinear responses calculated with a reference and an enriched condensed KL.

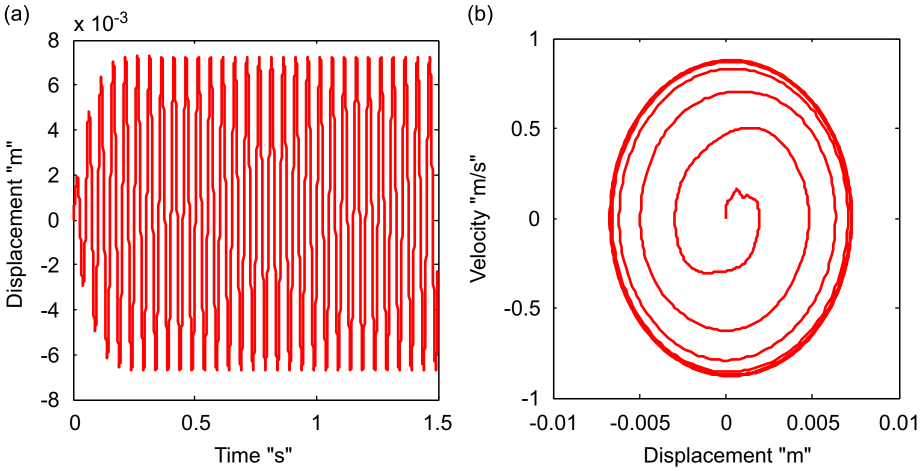

Fig. 19. Dynamic response of nonlinear reference at dof U_A: (a) nonlinear response of reference at dof U_A and (b) diagram of phase at dof U_A.

To obtain this nonlinear comportment, the value of the viscous damper $C = 2.5\,\mathrm{N\,m/s}$ and $K_2 = 6.25 \times 10^6\,\mathrm{N/m^3}$. Fig. 19 shows the reference of the dynamic response of U_A. It is notable that a limit of cycle is clearly illustrated in the diagram of phase.

Fig. 20 shows the response at point U_B. The diagram of phase illustrates a clearly cycle limit.

Fig. 21 illustrates the nonlinear responses of dof U_A and U_B calculated with the full and the reduced KLE model.

It is noted that in periodic or quasi-periodic regime, the proposed method can predict with great precision the reference response in a time interval fixed a priori.

6.2.2. *Configuration B*

In this new configuration, two levels of perturbation of forcing conditions are considered:

- Small perturbation of the excitation force ($F_0 = 10\,\mathrm{N}$) equal to -5% is introduced. The previous enriched basis (KLE) is used to represent the new condensed nonlinear response (Figs. 22a and b).

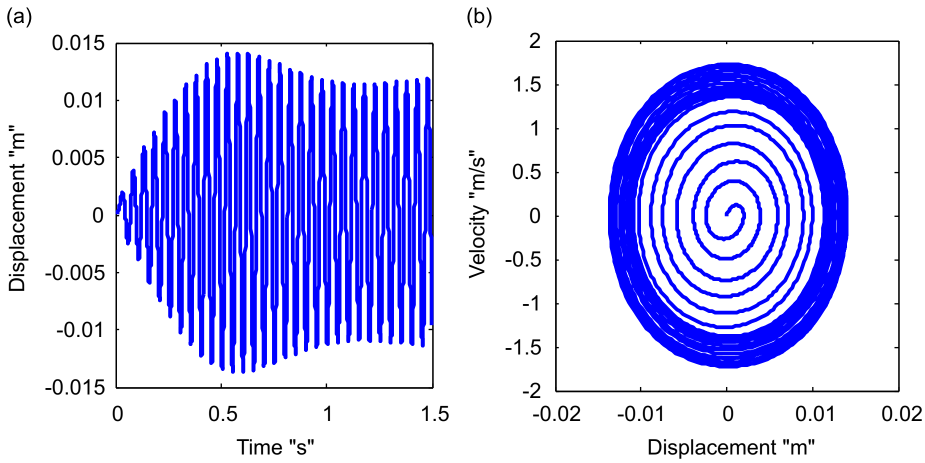

Fig. 20. Dynamic response of nonlinear reference at dof U_B: (a) nonlinear response of reference at dof U_B and (b) diagram of phase at dof U_B.

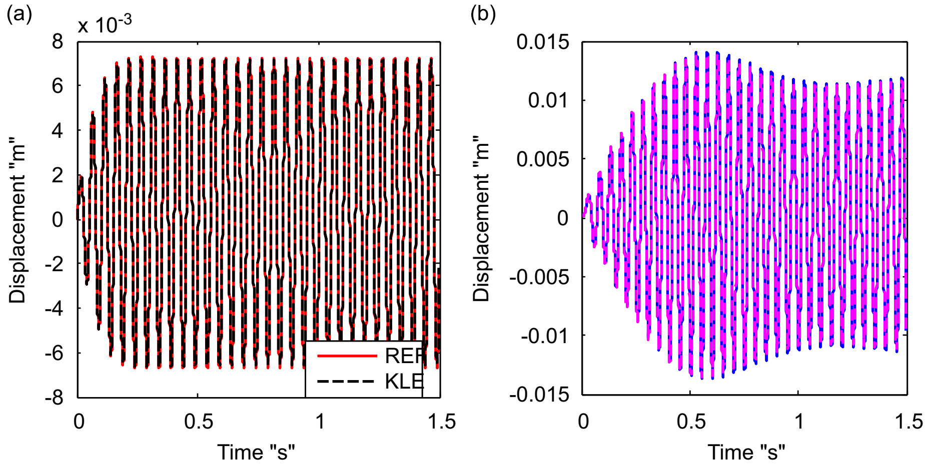

Fig. 21. Comparison of nonlinear responses: reference model/enriched KLE model: (a) nonlinear response at dof U_A and (b) nonlinear response at dof U_B.

The prediction is satisfactory for this type of nonlinear regime and the level of change introduced on the initial conditions.

- Important perturbation of the excitation force ($F_0 = 10\,\mathrm{N}$) equal to $+20\%$ is now introduced. Fig. 23 illustrates the nonlinear responses of dof U_B.

The various simulations shows that the enriched basis KLE is able to predict the behaviour of the perturbed or modified system (parametric modifications, initial or forcing conditions perturbations, etc.) provided they are within reasonable limits (small to moderate modifications) of change in the design phase. This is also verified in linear structural using an approximate reduced linear basis.

The proposed method can be applied mainly to nonlinear structures with parametric modifications.

It can also be used to predict the behaviour of nonlinear structures as a result of changes in initial conditions and loads, taking precautions as regards the level of modifications considered.

Regarding the validity domain of the proposed method, it is difficult to establish a quality criterion for this domain. The reduction approach based on an approximate Ritz or KL basis has generally empirical

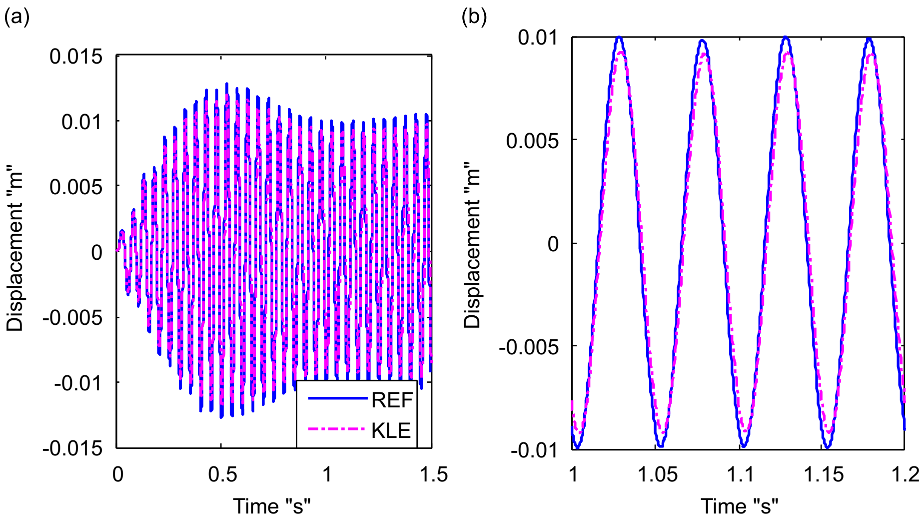

Fig. 22. Comparison of nonlinear responses: reference model/enriched KLE model: (a) nonlinear response at dof U_B and (b) nonlinear response at dof U_B.

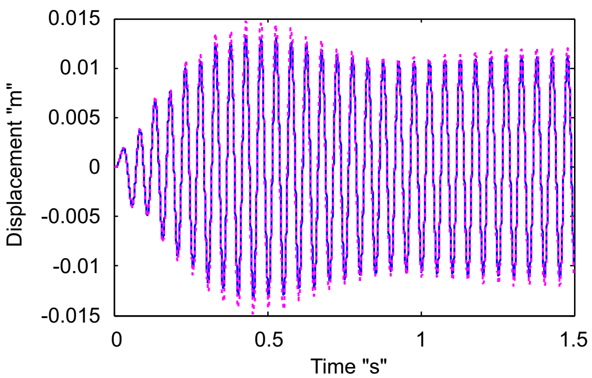

Fig. 23. Nonlinear responses at dof U_B.

qualitative criteria based on the number of normal modes taken into account in the basis and the level of modifications introduced.

In the case of nonlinear structural dynamics, it is also possible as shown by the previous simulations that the same criteria can also be used.

7. Conclusions

In the nonlinear time domain, a method has been proposed for the purpose of studying the behaviour of models with or without structural modifications in the presence of localized nonlinearities of models. This proposed method gives satisfactory results.

Before nonlinear analysis, the study of robustness by the modal condensation method or the Karhunen–Loève method versus parametric modifications shows that neither can be privileged.

In the case of the local modifications per zone, it is necessary to enrich the classical basis by additional vectors obtained from static loads representing the modification forces. These additional vectors are properly selected. This enriched basis allows the construction of robust condensed models.

Condensed modified model by the KLE method is conformed with the reanalysis needs in the optimization procedures of nonlinear dynamic behaviour. In fact, the extension of these robust models in nonlinear dynamic leads to the obtention of predictive nonlinear condensed modified models. The comparison of CPU time necessary for the calculation of nonlinear reference in time domain and condensed models show the advantages of the KLE method applied in structures in the presence of localized nonlinearities.

These robust condensed nonlinear models can be used in the multi-objectives optimization procedures.

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