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Oscillator with fraction order restoring force

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Abstract

In this paper a new analytical method for solving the differential equations which describe the motion of the oscillator with fraction order elastic force is introduced. Using the first integral of motion the exact period of vibration in the form of the Euler beta function is obtained. Based on that value the approximate solution of the strong nonlinear fraction order differential equation is formed. The suggested procedure is applied for various fraction values, but also for the pure quadratic, cubic and quintic oscillators. The advantage of the suggested method is that is valid for all fraction values $\alpha \geqslant 1$, the accuracy of the approximate solution is very high as the period of vibration is exactly analytically determined and is independent on the time.

In the paper the fraction order differential equation with small linear and nonlinear terms is also considered. The Krylov–Bogolubov method is extended due to the fact that frequency of vibration depends on the amplitude of vibration. The approximate solution is with time variable amplitude, phase and frequency. The linear damped fraction order oscillator is widely discussed. The approximate solution of the differential equation which describes the vibrations of such oscillator is compared with exact numerical one and shows a good agreement.

1. Introduction

The theory of linear vibrations is based on the assumption that the elastic force in the oscillatory system is a linear function of deflection. Recently, investigations show that the elastic force in the oscillator is a nonlinear deflection function. It requires the modification of the mathematical model of the restoring force. Due to mathematical simplicity, the nonlinear elastic force F is usually assumed as a polynomial deflection function (see Refs. [1,2], for example)

$$F(x) = -(c_1 x + c_2 x|x| + \cdots), \tag{1}$$

where c_1 and c_2 are constant coefficients. The result of Eq. (1) is that the force is an odd parity one, i.e., $F(-x) = -F(x)$, and all solutions of

$$\ddot{x} + c_1 x + c_2 x|x| + \cdots = 0, \tag{2}$$

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oscillate. Based on this assumption the theory of nonlinear oscillations is developed. The models of the nonlinear oscillators with linear and quadratic term

$$\ddot{x}+\omega^2 x+c_2^* x|x|=0, \tag{3}$$

and also linear and cubic term

$$\ddot{x}+\omega^2 x+c_3^* x|x|^2=0, \tag{4}$$

where c_2^* and c_3^* need not to be a small, are widely investigated (see for example Refs. [3,4]). The exact analytical solutions of Eqs. (3) and (4) have the form of Jacobi elliptic function [5]. This type of solution exists also when the linear term with ω^2 is zero in Eq. (4) and the oscillator is pure-cubic [6]. For the pure-quadratic oscillator, with $\omega^2=0$ in Eq. (3), the exact solution is not given, yet [7,8].

In this paper the extension to the model of the restoring elastic force is done. It is assumed that the elastic force is a non-polynomial deflection function which need not to have a linear term

$$F(x)=-x|x|^{\alpha-1}, \quad 1\leqslant\alpha\leqslant 2. \tag{5}$$

The mathematical model of the oscillator considered in this paper is

$$\ddot{x}+c_1^2 x|x|^{\alpha-1}=0, \quad x(0)=A, \quad \dot{x}(0)=0. \tag{6}$$

Eq. (6) represents a pure nonlinear fraction order differential equation.

Mickens [9] was the first to consider the oscillator with elastic force with fractional order deflection function

$$\ddot{x}+x^{1/(2n+1)}=0 \quad \text{for } n=1,2,\ldots. \tag{7}$$

He applied the harmonic balance method for solving the equation

$$\ddot{x}+x^{1/3}=0, \tag{8}$$

i.e., the rewritten equation

$$(\ddot{x})^3+x=0, \quad x(0)=A, \quad \dot{x}(0)=0. \tag{9}$$

The correction to the approximate solution of Eq. (9) is obtained using a functional and the generalized harmonic balance method, as it is suggested by Cooper [10]. The application of the second harmonic balance method [11] gives the correction to the previous solution. It is concluded that the latter method provides only small corrections to the periodic solution obtained in the first approximation. In the paper [12] an iteration technique is used to calculate a higher-order approximation to the periodic solutions of Eq. (8). Comparison between the iteration procedure and the harmonic balance methods shows that the two techniques are in excellent agreement, especially with regard to the calculated values of the angular frequency. Recently, Ozis et al. [13] and Belendez et al. [14] using the modified He's Lindstedt–Poincare method [15,16] give the approximate solution for Eq. (8). They rewrite Eq. (8) in the form

$$\ddot{x}+0x+px^{1/3}=0 \tag{10}$$

or

$$\ddot{x}+x=px-px^{1/3}, \tag{11}$$

where $p\in[0,1]$ is a constant parameter. The method is based on expanding both the solution x and the constant 0 and 1 in Eqs. (10) and (11) into series. Processing as the standard perturbation method and requiring of no secular term gives the frequency value in the first-order and higher-order approximations.

The extension of the fraction order model (7) is done by Hu [17]. Using the harmonic balance method the equation

$$\ddot{x}+x^{(2(m-n)+1)/(2n+1)}=0 \quad \text{for } m\geqslant n, \ m,n=0,1,2,\ldots \tag{12}$$

is rewritten as

$$(\ddot{x})^{2n+1}+(x)^{2(m-n)+1}=0, \quad x(0)=A, \quad \dot{x}(0)=0, \tag{13}$$

and the first-order approximate solution is determined

$$x = A\cos(\omega_n t), \tag{14}$$

where

$$\omega_n = \omega_n(A) = \left[\frac{2^{2(2n-m)}\begin{pmatrix} 2(m-n)+1 \\ m-n \end{pmatrix}}{A^{2(2n-m)}\begin{pmatrix} 2n+1 \\ n \end{pmatrix}}\right]^{1/(4n+2)}. \tag{15}$$

The oddness of both the numerator $(2m+1)$ and the denominator $(2n+1)$ of the exponent is important. If one of the parts in this ratio is even then Eq. (12) is not an oscillator equation. Based on the first integral, van Horssen [18] gives the exact analytical expression for period of the periodical solution of Eq. (7) and the approximate numerical solutions of the period for various values of n, while Belendez et al. [14] calculated the exact value of period of vibration for $n = 1$. Gottlieb [19] considered not only the oscillators where the force function is understood to be an odd function (7), but also the functions with even nominator or denominator. He introduced the sign function which enables the change of sign of the force according to deflection sign change

$$\ddot{x} + \text{sign}(x)|x|^{\alpha} = 0, \quad \alpha < 1, \tag{16}$$

where

$$\text{sign} = \begin{matrix} +1 & \text{for } x > 0, \\ -1 & \text{for } x < 0 \end{matrix} \tag{17}$$

and gives the exact frequency values for various values in the interval $\alpha = [0, 1]$. Eq. (16) enables one to consider a more general class of oscillators where the exponent is allowed to take any positive real value (such as odd, even, rational or irrational and so on). The correctness of the assumed model (16) is proved for $\alpha = 0$ when the differential equation approaches $\ddot{x} + \text{sign}(x) = 0$ which has periodic solutions as investigated by Awrjewitz and Andrianov [20]. Besides, in the paper of Andrianov [21] the special cases when $\alpha \gg 1$, $\alpha \ll 1$ and $\alpha \to 1$ are investigated.

Analyzing the aforementioned results the following can be concluded:

1. A system under the influence of a fraction order elastic restoring force described with an equation such as Eq. (7) or (16) has only periodic solutions.
2. The main investigations are done for the oscillatory systems where the fraction value of the deflection function of the elastic force is $\alpha = [0, 1]$. The exact and approximate values of the angular frequency and period of vibration are determined.
3. Only the approximate solution for a special group of oscillators with the elastic force which depends on $x^{(2(m-n)+1)/(2n+1)}$ for $m \geqslant n$, where $m, n = 0, 1, 2, \ldots$ is considered.

The results of the previous investigations are limited (see 2 and 3). To eliminate this disadvantage, in this paper a new mathematical model of the nonlinear elastic force (6) is suggested. Model (6) represents the generalization of Eqs. (12) and (16) and is discussed in this paper:

1. The exact period of vibrations for the oscillator with fraction order (6) is calculated.
2. The approximate solution of Eq. (6) is assumed in the form of the circular function. The angular frequency is calculated using the exact value of the period of vibration. The main attention is given to oscillators with $1 < \alpha < 2$ and also $\alpha = 2$ which are not considered, yet.
3. The first-order approximate solution of the equation

$$\ddot{x} + c_1^2 x|x|^{\alpha-1} = \varepsilon f(x, \dot{x}), \quad x(0) = A, \quad \dot{x}(0) = 0, \tag{18}$$

   where $\varepsilon \ll 1$ is a small parameter, is given. The Krylov–Bogolubov method [22] is adopted for solving Eq. (18). The trial solution for Eq. (18) is supposed in the form of the generating solution of Eq. (6), but with the amplitude, frequency and phase now taken to be functions of time. The special attention is given to the fraction order oscillator with linear damping. The analytical solution is compared with numerical one.

2. Analytically obtained period of vibration

Integrating Eq. (6) and using the initial conditions, the first integral of energy type is obtained

$$\frac{\dot{x}^2}{2} + \frac{c_1^2}{\alpha+1}|x|^{\alpha+1} = \frac{c_1^2}{\alpha+1}|A|^{\alpha+1}. \tag{19}$$

The both terms on the left side are positive and the motion is periodic (see Refs. [9,14,19]) with period

$$T_{\text{ex}} = 4\int_0^A \frac{\mathrm{d}x}{|\dot{x}|} = 4\sqrt{\frac{\alpha+1}{2c_1^2}}\int_0^A \frac{\mathrm{d}x}{\sqrt{|A|^{\alpha+1} - |x|^{\alpha+1}}}. \tag{20}$$

Substituting the new variable $|x| = |A||u|^{1/(\alpha+1)}$ into Eq. (20), the transformed version of period is

$$T_{\text{ex}} = \frac{4|A|^{(1-\alpha)/2}}{c_1\sqrt{2(\alpha+1)}}\int_0^1 (1-|u|)^{-1/2}u^{-\alpha/(\alpha+1)}\,\mathrm{d}u. \tag{21}$$

Introducing the Euler beta function $B(m,n)$ (see Ref. [23])

$$B(m,n) = \int_0^1 (1-|u|)^{n-1}u^{m-1}\,\mathrm{d}u, \tag{22}$$

relation (21) can be rewritten as follows:

$$T_{\text{ex}} = \frac{4|A|^{(1-\alpha)/2}}{c_1\sqrt{2(\alpha+1)}}B\left(\frac{1}{\alpha+1},\frac{1}{2}\right). \tag{23}$$

Due to

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \tag{24}$$

the exact period is

$$T_{\text{ex}} = \frac{4|A|^{(1-\alpha)/2}}{c_1\sqrt{2(\alpha+1)}}\frac{\Gamma\left(\frac{1}{\alpha+1}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3+\alpha}{2(\alpha+1)}\right)}, \tag{25}$$

where Γ is the Euler gamma function [24].

Using $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ (see Ref. [24]) the period expression is finally

$$T_{\text{ex}} = \frac{1}{c_1|A|^{(\alpha-1)/2}}\left(\frac{2\sqrt{2}\pi}{\sqrt{(\alpha+1)}}\right)\left(\frac{\Gamma\left(\frac{1}{\alpha+1}\right)}{\sqrt{\pi}\Gamma\left(\frac{3+\alpha}{2(\alpha+1)}\right)}\right). \tag{26}$$

3. The approximate solution

The approximate solution of Eq. (6) is assumed in the form of a circular function

$$x = A\cos(\omega t), \tag{27}$$

where A is the initial amplitude (6) and ω is the angular frequency of vibration. Relation (27) is an approximation to the exact solution of Eq. (6) which corresponds to a truncated Fourier expansion where only the first term is retained. Based on the exact period of vibration (25) and the assumption of periodical harmonic function (27) the explicit expression for the angular frequency follows:

$$\omega = \frac{2\pi}{T_{\rm ex}} = \frac{c_1\pi\sqrt{2(\alpha+1)}}{2|A|^{(1-\alpha)/2}} \frac{\Gamma\left(\dfrac{3+\alpha}{2(\alpha+1)}\right)}{\Gamma\left(\dfrac{1}{\alpha+1}\right)\Gamma\left(\dfrac{1}{2}\right)}, \tag{28}$$

i.e., for Eq. (26)

$$\omega = \sqrt{\frac{\alpha+1}{2}} \frac{\sqrt{\pi}\Gamma\left(\dfrac{3+\alpha}{2(\alpha+1)}\right)}{\Gamma\left(\dfrac{1}{\alpha+1}\right)} c_1|A|^{(\alpha-1)/2}. \tag{29}$$

The oscillations are periodical with the constant amplitude A. The angular frequency of vibration depends not only on the fraction value α but also on the initial amplitude A, as it is well known for nonlinear oscillators.

3.1. Examples and conclusions

Analyzing relations (26), (27) and (29) it is obvious:

1. For the linear oscillator, when $\alpha = 1$, the period of vibration (26) is for $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\Gamma(1) = 1$

$$T_{\rm ex} = \frac{2\pi}{c_1} = \frac{6.2832}{c_1|A|^0}, \tag{30}$$

as it is well known in the linear theory of harmonic vibration.

2. For $\alpha = \frac{5}{3}$ and the corresponding differential equation (6)

$$\ddot{x} + c_1^2 x|x|^{2/3} = 0, \tag{31}$$

the approximate solution is

$$x = A\cos(0.94081 c_1 t|A|^{1/3}). \tag{32}$$

The analytical solution, x, Eq. (32) is compared with the numerical one, x_N, obtained by solving Eq. (31) using the Runge–Kutta procedure. For the parameter values $A = 0.5$ and $c_1 = 1$ the $x - t$ and $x_N - t$ diagrams are plotted (Fig. 1). Comparing the curves it is seen that the difference is negligible.

For Eq. (29) the angular frequency of vibration for $\alpha = \frac{5}{3}$ is calculated

$$\omega = \frac{2}{\sqrt{3}} \frac{\sqrt{\pi}\Gamma(\frac{7}{8})}{\Gamma(\frac{3}{8})} c_1|A|^{1/3} = 0.94081 c_1|A|^{1/3}. \tag{33}$$

Hu and Xiong [17] give the approximate angular frequency determined by using the harmonic balance method, when $c_1 = 1$

$$\omega_a = (\tfrac{5}{6}|A|^2)^{1/6} x = 0.97007|A|^{1/3}. \tag{34}$$

For comparison of ω (33) with ω_a (34) the percentage error is calculated

$$\left|\frac{\omega - \omega_a}{\omega}\right| \cdot 100\% = \left|\frac{0.94081 - 0.97007}{0.94081}\right| \cdot 100\% = 3.0163\%.$$

To prove the correctness of the suggested approximate procedure two other oscillators with fraction order deflection are calculated: one, with even denominator ($\alpha = \frac{3}{2}$) and the other with even nominator ($\alpha = \frac{4}{3}$). For $\alpha = \frac{4}{3}$ and the differential equation

$$\ddot{x} + c_1^2 x|x|^{1/3} = 0, \tag{35}$$

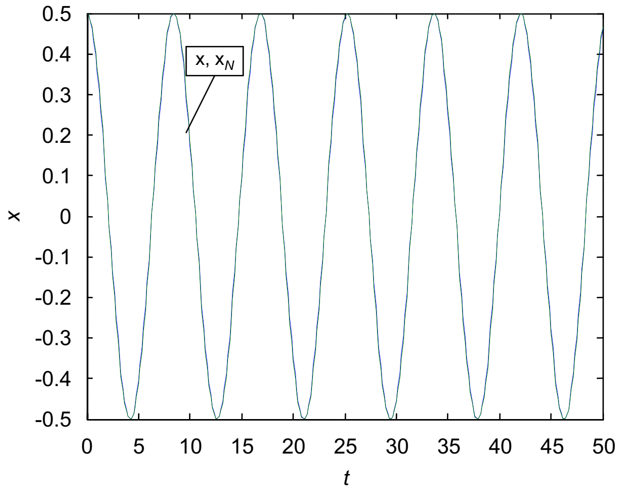

Fig. 1. Plots of Eq. (32) and the corresponding numerical solution, x_N, versus t for $\alpha = \frac{5}{3}$.

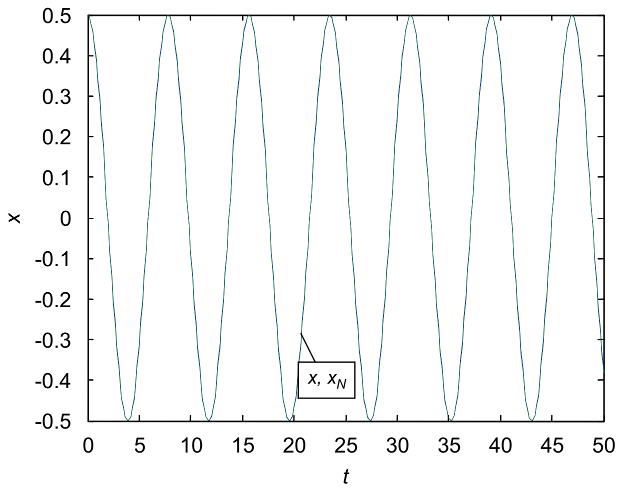

Fig. 2. Plots of Eq. (38) and the corresponding numerical solution, x_N, versus t for $\alpha = \frac{3}{2}$.

the approximate solution is

$$x = A\cos(0.96915tc_1|A|^{1/6}), \tag{36}$$

and for $\alpha = \frac{3}{2}$ and

$$\ddot{x} + c_1^2 x|x|^{1/2} = 0, \tag{37}$$

the result is

$$x = A\cos(0.95469c_1|A|^{1/4}). \tag{38}$$

In Figs. 2 and 3 the time evolution diagrams for $\alpha = \frac{3}{2}$ and $\frac{4}{3}$, with parameter values $A = 0.5$ and $c_1 = 1$, are plotted, respectively. The approximated solutions (36) and (38) are in good agreement with numerical solutions of Eqs. (35) and (37), respectively.

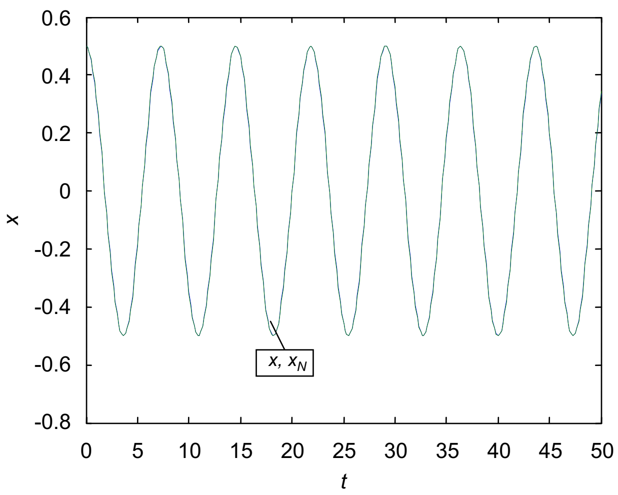

Fig. 3. Plots of Eq. (36) and the corresponding numerical solution, x_N, versus t for $\alpha = \frac{4}{3}$.

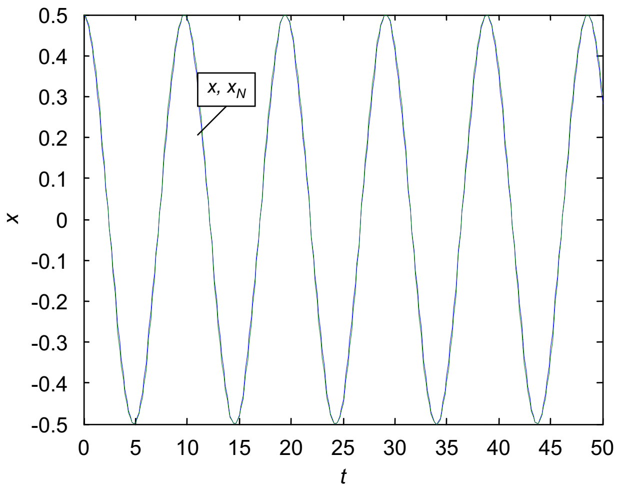

Fig. 4. Plots of Eq. (41) and the corresponding numerical solution, x_N, versus t for $\alpha = 2$.

3. For $\alpha = 2$ the exact period of oscillation Eq. (26) becomes

$$T_{\text{ex}} = \frac{2\sqrt{2\pi}}{\sqrt{3}c_1|A|^{1/2}} \frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{5}{6})} = \frac{6.8693}{c_1|A|^{1/2}}. \tag{39}$$

According to Eq. (39) the angular frequency of vibration is

$$\omega = \frac{\sqrt{3\pi}c_1|A|^{1/2}}{\sqrt{2}} \frac{\Gamma(\frac{5}{6})}{\Gamma(\frac{1}{3})} = 0.91468 c_1|A|^{1/2}, \tag{40}$$

and the corresponding solution

$$x = A\cos(0.91468 c_1 t|A|^{1/2}). \tag{41}$$

In Fig. 4 we plot the time evolution diagrams obtained numerically ($x_N - t$), applying the Runge–Kutta procedure for Eq. (6) when $\alpha = 2$, and analytically ($x - t$) (41). The parameter of the system is $c_1 = 1$ and the initial amplitude $A = 0.5$. The difference between the curves is negligible. This indicates that higher-order odd harmonics do not play any important role in the solution.

4. For $\alpha = 3$ the exact period is

$$T_{\mathrm{ex}} = \frac{1}{c_1|A|}\left(\frac{\sqrt{2\pi}\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}\right) = \frac{1}{c_1|A|}\left(\frac{3.62561\sqrt{2\pi}}{1.2254167}\right) = \frac{7.4163}{c_1|A|}. \tag{42}$$

The exact solution of the pure cubic equation exists in the form of the Jacobi elliptic function [25]

$$x_{\mathrm{ex}} = Acn(\Omega t, k^2), \tag{43}$$

where the frequency Ω and the modulus k^2 of the elliptic function are

$$\Omega = c_1|A|, \quad k^2 = \tfrac{1}{2}. \tag{44}$$

Due to the fact that the cn elliptic function is periodical with period $4K(k^2)$, where $K(k^2)$ is the complete elliptic integral of the first kind [5], the period of vibration is

$$T = \frac{4K(\frac{1}{2})}{c_1|A|} = \frac{7.4163}{c_1|A|}, \tag{45}$$

where $K(\frac{1}{2}) = 1.854075$. Comparing Eqs. (42) and (45), it is obvious that the obtained values are equal.

For Eq. (42) the angular frequency is

$$\omega = \frac{2\pi}{T_{\mathrm{ex}}} = 0.84721c_1|A|, \tag{46}$$

and the solution

$$x = A\cos(0.84721c_1t|A|). \tag{47}$$

In Fig. 5 the exact x_{ex} (43) and approximate x (47) solutions are compared. The approximate solution is on the top of the exact one for $A = 0.5$ and $c_1 = 1$.

5. For the pure fifth-order differential equation

$$\ddot{x} + c_1^2x|x|^4 = 0, \tag{48}$$

the approximate analytical solution is

$$x = A\cos(0.74683c_1t|A|^2). \tag{49}$$

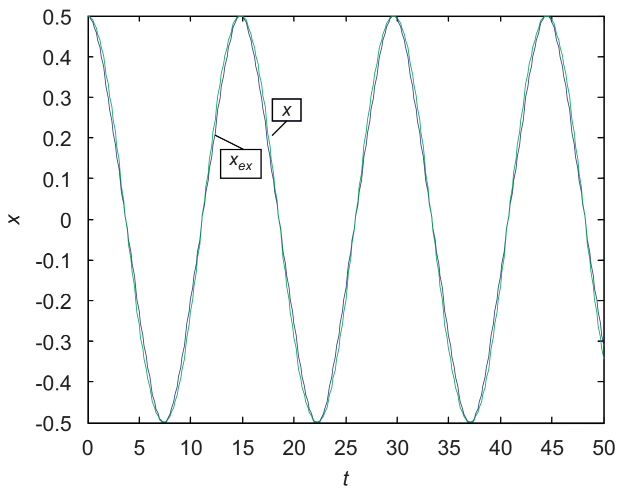

Fig. 5. Plots of Eq. (47) and the corresponding exact solution, x_{ex}, (43) versus t for $\alpha = 3$.

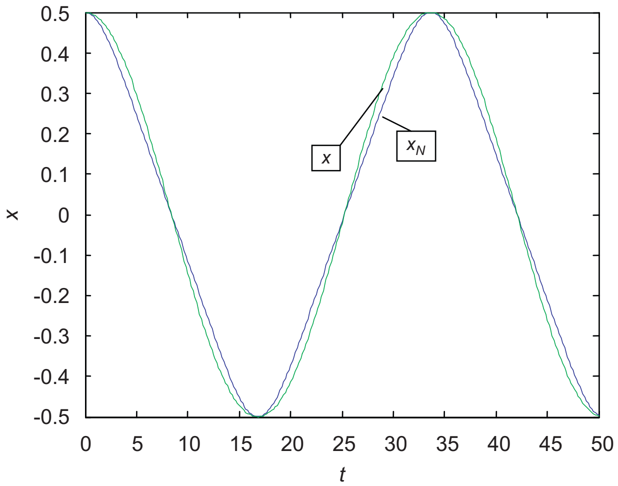

Fig. 6. Plots of Eq. (49) and the corresponding numerical solution, x_N, versus t for $\alpha = 5$.

In spite of the fact that the amplitude and the period of vibration of the oscillator are exactly determined by the suggested analytical procedure, the shape of the approximate solution (49) differs from the exact numerical one of Eq. (48) (see Fig. 6 for $A = 0.5$ and $c_1 = 1$). This clearly implies that higher-order odd harmonics do play a significant role in the solution for this nonlinear oscillator. (Namely, for odd parity oscillators, only odd harmonics appear.)

4. Solution of the fraction order equation with small linear and nonlinear terms

The mathematical model of the oscillator with fraction order restoring elastic and other small forces is a second-order nonlinear ordinary differential equation (18). The approximate solving procedure based on the Krylov–Bogolubov method of time variable amplitude and phase [22] is extended for solving Eq. (18). For the generating equation (6), when $\varepsilon = 0$, the approximate solution is Eq. (27) with Eq. (29). The trial solution of Eq. (18) and its first time derivative are assumed in the form of the generating solution with variable amplitude $A(t)$, frequency $\omega(A)$ and phase $\beta(t)$

$$x = A(t)\cos\psi(t) \equiv a\cos\psi, \tag{50}$$

$$\dot{x} = -A(t)\omega(A)\sin\psi(t) \equiv -a\omega(a)\sin\psi, \tag{51}$$

with

$$\dot{a}\cos\psi(t) - a\dot{\beta}\sin\psi = 0 \tag{52}$$

and

$$\psi \equiv \psi(t) = \int_t \omega(a)\,\mathrm{d}t + \beta(t), \tag{53}$$

where $a \equiv A(t)$, $\psi \equiv \psi(t)$, $\beta \equiv \beta(t)$ and $\omega \equiv \omega(a)$. Substituting the time derivative of Eq. (51) and the solution (50) into Eq. (18) we obtain

$$\dot{a}\omega\sin\psi + a\dot{a}\omega'\sin\psi + a\omega\dot{\beta}\cos\psi = -\varepsilon f(a\cos\psi, -a\omega\sin\psi), \tag{54}$$

where $\omega' \equiv \mathrm{d}\omega/\mathrm{d}a$ as the frequency $\omega(a)$ depends on the amplitude. The transformation of Eqs. (52) and (54) gives the two first-order differential equations corresponding to Eq. (18)

$$\dot{a}(\omega + a\omega'\sin^2\psi) = -\varepsilon f(a\cos\psi, -a\omega\sin\psi)\sin\psi, \tag{55}$$

$$a\omega\dot{\beta} + a\dot{a}\omega'\sin\psi\cos\psi = -\varepsilon f(a\cos\psi, -a\omega\sin\psi)\cos\psi, \tag{56}$$

i.e., using Eq. (29) and the notation $A(t) \equiv a$ (see Eq. (50))

$$\dot{a}\left(1+\frac{\alpha-1}{2}\sin^2\psi\right) = -\frac{1}{q|a|^{(\alpha-1)/2}}\varepsilon f(a\cos\psi, -q|a|^{(\alpha+1)/2}\sin\psi)\sin\psi, \tag{57}$$

$$a\dot{\beta}+\dot{a}\frac{\alpha-1}{4}\sin 2\psi = -\frac{1}{q|a|^{(\alpha-1)/2}}\varepsilon f(a\cos\psi, -aq|a|^{(\alpha-1)/2}\sin\psi)\cos\psi, \tag{58}$$

where

$$\omega \equiv \omega(a) = q|a|^{(\alpha-1)/2}, \quad q = \frac{\sqrt{(\alpha+1)}}{\sqrt{2}}\frac{\sqrt{\pi}\Gamma\left(\frac{3+\alpha}{2(\alpha+1)}\right)}{\Gamma\left(\frac{1}{\alpha+1}\right)}c_1. \tag{59}$$

This is the point where the averaging for the period 2π is introduced

$$\dot{a} = -\frac{2\varepsilon}{\pi q(\alpha+3)|a|^{(\alpha-1)/2}}\int_0^{2\pi} f(a\cos\psi, -q|a|^{(\alpha+1)/2}\sin\psi)\sin\psi\,d\psi, \tag{60}$$

$$\dot{\beta} = -\frac{\varepsilon}{2\pi aq|a|^{(\alpha-1)/2}}\int_0^{2\pi} f(a\cos\psi, -aq|a|^{(\alpha-1)/2}\sin\psi)\cos\psi\,d\psi, \tag{61}$$

i.e.,

$$\dot{a} = -\frac{2\varepsilon}{\pi(\alpha+3)\omega(a)}\int_0^{2\pi} f(a\cos\psi, -a\omega(a)\sin\psi)\sin\psi\,d\psi, \tag{62}$$

$$\dot{\psi} = \omega(a) - \frac{\varepsilon}{2\pi a\omega(a)}\int_0^{2\pi} f(a\cos\psi, -a\omega(a)\sin\psi)\cos\psi\,d\psi. \tag{63}$$

The differential equations (62) and (63) are solved for the initial values

$$a(0) = A, \quad \psi(0) = 0. \tag{64}$$

4.1. Oscillator with linear damping

The differential equation of the oscillator with linear damping is

$$\ddot{x} + c_1^2 x|x|^{\alpha-1} = -\varepsilon\dot{x}, \tag{65}$$

where $\varepsilon \ll 1$ is the damping coefficient. The corresponding averaged first-order equations (62) and (63) become

$$\dot{a} = -\frac{2\varepsilon a}{\pi(\alpha+3)}\int_0^{2\pi}\sin^2\psi\,d\psi = -\frac{2\varepsilon a}{\alpha+3}, \tag{66}$$

$$\dot{\psi} = q|a|^{(\alpha-1)/2} - \frac{\varepsilon}{4\pi}\int_0^{2\pi}\sin 2\psi\,d\psi = q|a|^{(\alpha-1)/2}. \tag{67}$$

Integrating Eqs. (66) and (67) for the initial values (64) we obtain

$$a = A\exp\left(-\frac{2\varepsilon t}{\alpha+3}\right), \quad \psi = -\frac{\alpha+3}{\varepsilon(\alpha-1)}q|A|^{(\alpha-1)/2}\left[\exp\left(-\frac{\varepsilon(\alpha-1)t}{\alpha+3}\right)-1\right], \tag{68}$$

and the approximate solution of Eq. (65) yields

$$x = A\exp\left(-\frac{2\varepsilon t}{\alpha+3}\right)\cos\left(\frac{\alpha+3}{\varepsilon(\alpha-1)}q|A|^{(\alpha-1)/2}\left[1-\exp\left(-\frac{\varepsilon(\alpha-1)t}{\alpha+3}\right)\right]\right). \tag{69}$$

Analyzing Eq. (69), it can be concluded:

1. The fraction deflection function has the significant influence on the amplitude decrease: the higher the values of parameter α, the slower the amplitude decreases. For $\alpha = 1$ the amplitude-time decrease is $A\exp(-\varepsilon t/2)$ and for $\alpha \to \infty$ the amplitude tends to the initial constant value A independently on the value of the small parameter ε.
2. The frequency of vibration of the linear damped oscillator depends on the initial amplitude A, fraction value α and the damping coefficient ε (see Eq. (68)). For small damping coefficient ε the frequency (68) tends to Eq. (29) and the period of vibration to T_{ex} (26). Namely, using the series expansion of exp function and using only the first two terms the function ψ simplifies to

$$\psi \approx qt|A|^{(\alpha-1)/2}. \tag{70}$$

3. For $\varepsilon \to 0$, when the damping is omitted, the amplitude (68) is $a = A$, the angle $\psi \to \omega(A)t$ as

$$\lim_{\varepsilon\to 0}\frac{\alpha+3}{\varepsilon(\alpha-1)}\left[\exp\left(-\frac{\varepsilon(\alpha-1)t}{\alpha+3}\right)-1\right] = -t, \tag{71}$$

and solution (27).
4. For the linear oscillator, when $\alpha = 1$, the angle ψ is

$$\lim_{\alpha\to 1}\frac{\alpha+3}{\varepsilon(\alpha-1)}\left[\exp\left(-\frac{\varepsilon(\alpha-1)t}{\alpha+3}\right)-1\right] = -t, \tag{72}$$

and the solution in the first approximation becomes

$$x = A\exp\left(-\frac{\varepsilon t}{2}\right)\cos(\omega t). \tag{73}$$

Comparing Eq. (73) with the exact solution for the linear damped system, it is obvious that the approximation (73) is valid only for $\varepsilon \ll 1$ when the damping influence on the period of vibration can be neglected.
5. For $\alpha = \frac{5}{3}$, the first-order approximation is

$$x = A\exp\left(-\frac{3\varepsilon t}{7}\right)\cos\left(\frac{6.5857}{\varepsilon}q|A|^{1/3}\left[1-\exp\left(-\frac{\varepsilon t}{7}\right)\right]\right). \tag{74}$$

In Fig. 7 the analytical (74) and numerical solution of

$$\ddot{x} + c_1^2 x|x|^{2/3} = -\varepsilon\dot{x}, \tag{75}$$

for $c_1 = 3$, $\varepsilon = 0.1$, $A = 0.5$ are plotted. In spite of the long time interval the difference between approximate x and the exact numerical solution x_N is negligible.
6. For $\alpha = 2$ and

$$\ddot{x} + c_1^2 x|x| = -\varepsilon\dot{x}, \tag{76}$$

the vibration of the linear damped pure quadratic oscillator is approximately

$$x = A\exp\left(-\frac{2\varepsilon t}{5}\right)\cos\left(\frac{4.5734}{\varepsilon}c_1|A|^{1/2}\left[1-\exp\left(-\frac{\varepsilon t}{5}\right)\right]\right). \tag{77}$$

The approximate solution x (77) is compared with numerical one x_N for Eq. (76) and plotted in Fig. 8. The parameters are $c_1 = 3$, $\varepsilon = 0.1$ and the initial amplitude $A = 0.5$.
7. For the linear damped pure cubic oscillator where $\alpha = 3$ the mathematical model is

$$\ddot{x} + c_1^2 x|x|^2 = -\varepsilon\dot{x}, \tag{78}$$

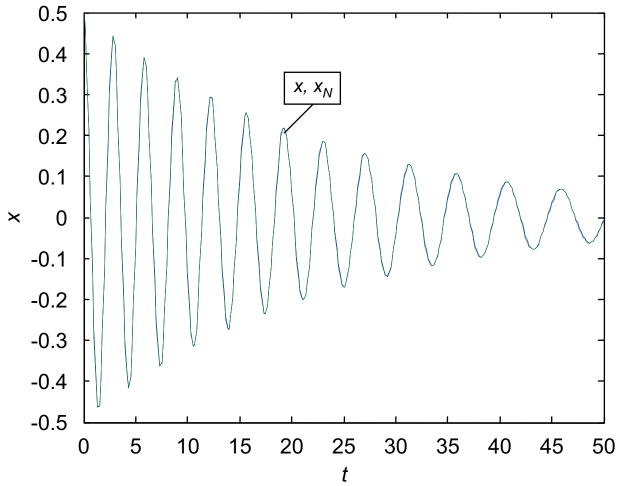

Fig. 7. Plots of Eq. (74) and the corresponding numerical solution, x_N, versus t for $\alpha = \frac{5}{3}$.

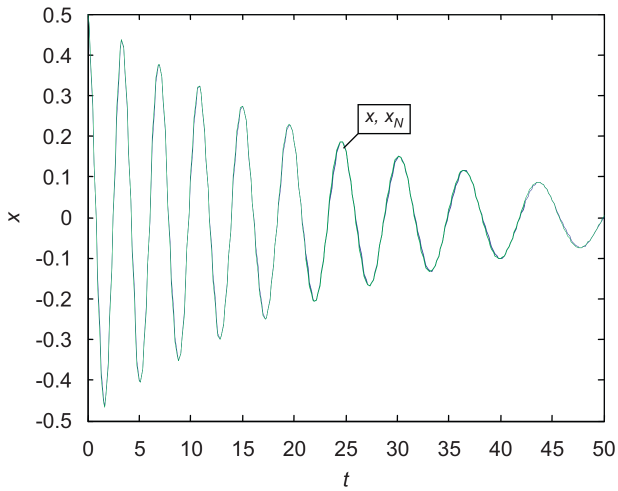

Fig. 8. Plots of Eq. (77) and the corresponding numerical solution, x_N, versus t for $\alpha = 2$.

which was considered by Yuste and Bejarano [6]. Using the elliptic-Krylov–Bogolubov method, based on the perturbation of the Jacobi elliptic function, the approximate solution is obtained

$$x_J = A\exp\left(-\frac{\varepsilon t}{3}\right) cn\left(\frac{3c_1|A|}{\varepsilon}\left[1-\exp\left(-\frac{\varepsilon t}{3}\right)\right], 1/2\right). \tag{79}$$

In this paper the solution of Eq. (78) in the first approximation is

$$x = A\exp\left(-\frac{\varepsilon t}{3}\right)\cos\left(\frac{2.54163c_1|A|}{\varepsilon}\left[1-\exp\left(-\frac{\varepsilon t}{3}\right)\right]\right). \tag{80}$$

Comparing the obtained result x (80), the solution with Jacobi elliptic function x_J (79) and the 'exact' numeric solution of Eq. (78) calculated by Runge–Kutta method for $A = 0.5$, $c_1 = 2$ and $\varepsilon = 0.1$ (see Fig. 9), it can be concluded that the both approximations are with high accuracy. The approximation with the Jacobi elliptic function is more appropriate, but the calculating procedure is more complex than for circular function.

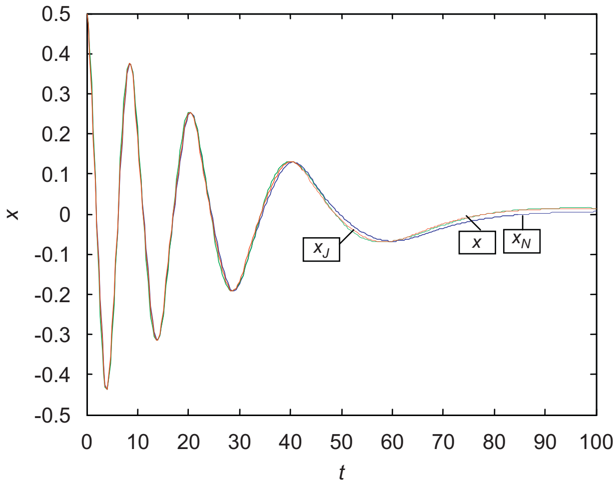

Fig. 9. Plots of Eq. (80) and the corresponding numerical solution, x_N, and analytical solution, x_J, versus t for $\alpha = 3$.

5. Conclusion

The following is concluded:

1. The oscillator with fraction order deflection (6) oscillates with constant amplitude A and period whose exact analytical value is determined (26). The period of vibration is given in the form of the Euler beta function i.e., Euler gamma function and is the function of fraction value α, coefficient of the rigidity c_1 and initial amplitude A. For higher value of α, the period of vibrations is longer and for larger values of A and c_1, the period T_{ex} is shorter.
2. The approximate solution of Eq. (6)

$$x = A\cos\left(\frac{\sqrt{(\alpha+1)}}{\sqrt{2}}\frac{\sqrt{\pi}\Gamma\left(\dfrac{3+\alpha}{2(\alpha+1)}\right)}{\Gamma\left(\dfrac{1}{\alpha+1}\right)}c_1 t|A|^{(\alpha-1)/2}\right) \tag{81}$$

   is valid for all values of $\alpha \geqslant 1$ as is based on the exact value of the amplitude A and period of vibration T_{ex}.
3. The main advantage of the suggested method is that the accuracy of the approximate solution (81) is independent on time. Namely, as it is based on the exact amplitude and period of vibration the solution is correct with the same accuracy for all vibration periods and the time from zero to indefinite. Usually, it is not the case for other approximate solutions where the accuracy of the solution is good only for short time intervals and decreases with time.
4. The approximation of the oscillations (81) in the form of the circular function is very good for $1 \leqslant \alpha \leqslant 3$. Namely, difference between the shapes of the approximate and exact the time evolution diagrams is negligible.
5. In the paper the exact period of vibration and the approximate solution for the pure quadratic oscillator ($\alpha = 2$) with strong nonlinearity is obtained.
6. The approximate solving procedure of the fraction order deflection oscillator with small nonlinearity (18), based on the Krylov–Bogolubov method of variable amplitude and phase, is extended by introducing the variable frequency, as it is the function of amplitude of vibration. The approximate solution of Eqs. (62) and (63) is on the top of the exact solution of Eq. (18).
7. In the future the more general oscillator with the linear and fraction order deflection function

$$\ddot{x} + \omega_0^2 x + c_1^2 x|x|^{\alpha-1} = \varepsilon f(x, \dot{x}), \quad x(0) = A, \quad \dot{x}(0) = 0,$$

   will be investigated.

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