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Transverse vibrations of a clamped elliptical plate carrying a concentrated mass at an arbitrary position

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Abstract

Free vibrations of clamped, thin elliptical plates carrying a concentrated mass at an arbitrary position are studied in order to analyze the mass effect on the natural frequencies and mode shapes of the plate. A variational approach, the well-known Ritz method, is used, where the displacement amplitude is approximated by polynomial expressions in the Cartesian coordinates. The present proposal exhibits excellent agreement with particular cases of the problem available in the literature and also with an independent solution obtained by means of the finite element method. Some new results are presented for the natural frequencies and modes of clamped elliptical plates with a concentrated mass.

1. Introduction

Elastic circular and elliptical plates are very commonly encountered structural elements and are found in a diverse set of engineered systems. They are extensively used in pressure vessels, ship and aircraft structures and even for optical lens and printed circuit boards. In most of these technological applications the plate performs its function in a dynamic fashion and the design engineer needs to know its natural transverse vibratory behavior (natural frequencies and mode shapes). In many situations, these plates carry loads at eccentric positions, for example when a centrifugal pump is attached to a cover plate of a water tank.

There is a great amount of published work in the case of vibrating solid and annular circular plates. For isotropic plates of uniform thickness many basic dynamic problems can be solved using Bessel functions. A survey of the literature on the subject and results for several cases are provided in the excellent monograph by Leissa [1]. There are many papers which consider the effect of additional complexities on the dynamic behavior of circular plates. Among them, one can mention the following works: Avalos et al. [2] obtained the first six axisymmetric natural frequencies of vibration of a circular plate carrying an elastically mounted

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centered mass. Nallim et al. [3] and also Laura et al. [4] studied a circular plate of rectangular orthotropy with a central concentrated mass. Bambill et al. [5] obtained analytically, by means of the Ritz method, and experimentally the fundamental frequency of vibration of solid and annular circular plates carrying a concentrated mass at an arbitrary position. Ranjan and Ghosh [6] studied the same problem using the finite element method.

In the case of elliptical plates, the amount of published work is significatively lesser. It is worthy of note the contributions of Sato [7–10] who investigated the transverse vibrations of solid elliptical plates supported by different boundary conditions. Rajalingham and Bhat [11] and Rajalingham et al. [12] studied the elliptical plate vibration basing on its analogy with the circular plate vibration. Additional complexities are considered by Irie and Yamada [13] who studied orthotropic elliptical annular plates by means of the Ritz method, Nallin and Grossi [14] studied laminated elliptical plates, Narita [15] analyzed the free orthotropic elliptical plate. Singh and Tyagi [16], Singh and Chakraverty [17] and Bayer et al. [18], Hassan and Makary [19] considered variable thickness of the vibrating elliptical plate. Kaplunov et al. [20] studied moderately thick elliptic plates. Hassan [21] analyzed elliptical plates with discontinuous boundary conditions.

The present study deals with the natural frequencies and mode shapes of transverse vibration of a clamped elliptical plate carrying a concentrated mass at an arbitrary position. To the best of the authors' knowledge, no results on the matter have been previously published in the literature. The variational Ritz method with polynomial approximation for the displacements is employed to perform the analysis. The proposed approach exhibits an excellent accuracy for particular cases available in the literature, and in some cases an independent solution is also obtained by the finite element method to show the agreement with the analytical predictions.

2. The Ritz method

Fig. 1 shows the elliptical plate in the x–y plane of the Cartesian coordinate system. In the case of normal modes of vibration it is assumed that the plate executes a simple harmonic motion, which is characterized by the deflection of the middle surface $w(x,y,t)$, represented in the form of a product

$$w(x,y,t) = W(x,y)e^{i\omega t}, \tag{1}$$

where W is a given continuous function which represents the shape of the deflected middle surface of the vibrating plate, t the time coordinate and ω the natural circular frequency.

An approximate solution of the problem can be obtained by means of the Ritz method, using an approximation for the transverse displacement amplitude W, any expression which satisfies at least the essential boundary conditions at the plate edge. Here the expression for W will be defined as a summation with

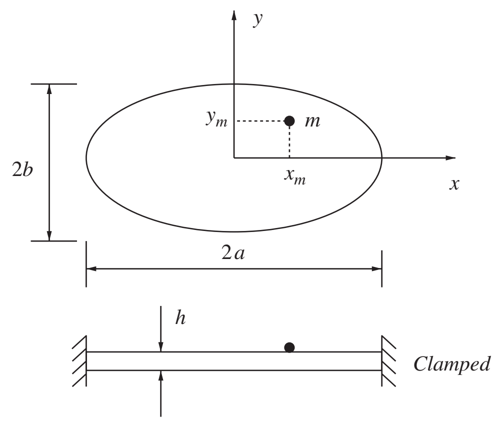

Fig. 1. Clamped elliptical plate with an attached mass.

0	1								
1	x	y							
2	x^2	xy	y^2						
.	.	.	.	.					
.	.	.	.	.	.				
.	.	.	.	.	.	.			
q	x^q	$x^{q-1}y$	$x^{q-2}y^2$......	$x^{q-p}y^p$	· ·	y^q			
.	.								
k	x^k	$x^{k-1}y$	$x^{k-2}y^2$......	$x^{k-p}y^p$	· · · · ·		xy^{k-1}	y^k	
	0	1	2	p	·	q · ·	· $k-1$	k	

Fig. 2. Set of selected monomials.

undetermined coefficients C_i

$$W(x,y) \cong W_a(x,y) = \sum_{i=1}^{N} C_i f_i(x,y), \tag{2}$$

where f_i represents continuous functions:

$$f_i(x,y) = \left[\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - 1\right]^2 \phi_i(x,y) \tag{3}$$

the expression $(x/a)^2 + (y/b)^2 - 1$ is the curve in the x–y plane that defines the contour of the plate, a the semi-major axis and b the semi-minor axis of the ellipse and ϕ_i's are adopted as monomials functions selected from a set of monomials [22], Fig. 2, of the form

$$x^{q-p}y^p \tag{4}$$

then, the approximate solution, Eq. (2), becomes the expression

$$W_a(x,y) = \sum_{i=1}^{N} C_i f_i(x,y) = \left[\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - 1\right]^2 \sum_{q=0}^{s}\sum_{p=0}^{q} C_i x^{q-p} y^p, \tag{5}$$

where $i = 0.5q(q+1)+p+1$ and $N = 0.5(s+1)(s+2)$ which, obviously, satisfies the boundary conditions at the clamped edge.

The energy functional for the vibrating elliptical plate of Fig. 1, is given by the expression

$$J(W) = \int\int_A \left\{ \frac{D}{2}\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2}\right)^2 - 2(1-v)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\right\} \mathrm{d}x\,\mathrm{d}y$$
$$-\frac{1}{2}\rho\omega^2 h \int\int_A W^2\,\mathrm{d}x\,\mathrm{d}y - \frac{1}{2}m\omega^2[W(x_m,y_m)]^2, \tag{6}$$

where A is the plate domain, v the Poisson's ratio and ρ the density of the plate material, h the uniform thickness and m the concentrated mass attached to the plate at x_m, y_m location

$$D = \frac{Eh^3}{12(1-v^2)}, \tag{7}$$

where D is the flexural rigidity of the plate and E the Young's modulus.

For generality and convenience, the coordinates are normalized by equations

$$\xi = \frac{x}{a}, \quad \eta = \frac{y}{b} \quad \text{and} \quad \xi_m = \frac{x_m}{a}, \quad \eta_m = \frac{y_m}{b}, \tag{8}$$

then

$$W_a(\xi,\eta)=\sum_{q=0}^{s}\sum_{p=0}^{q}\bar{C}_i[\xi^2+\eta^2-1]^2\xi^{q-p}\eta^p, \tag{9}$$

where $\bar{C}_i=C_i a^{q-p}b^p$. According to the Ritz method, after introducing the approximate expression $W_a(\xi,\eta)$, Eq. (9), into the Eq. (6), the integration of $J(W_a)$ will appear in the form of an homogeneous quadratic function of coefficients $\bar{C}_i$. The minimum of this function will lead to a system of homogeneous equations of the first order for the unknown displacement coefficients $\bar{C}_i$:

$$\frac{\partial J(W_a)}{\partial \bar{C}_i}=0;\quad i=1,2,3,\ldots,N. \tag{10}$$

The obtained homogeneous set of equations can be written as

$$[\mathbf{K}-\Omega^2\mathbf{L}]\bar{\mathbf{C}}=\mathbf{0}, \tag{11}$$

where $\Omega=\omega ab\sqrt{\rho h/D}$ are the natural frequency parameters and $\mathbf{K}$ and $\mathbf{L}$ are symmetric matrices whose elements are given by

$$k_{ij}=\int\int_{\bar{A}}\left[\left(\frac{b}{a}\right)^2\varphi_{i,\xi\xi}\varphi_{j,\xi\xi}+\left(\frac{a}{b}\right)^2\varphi_{i,\eta\eta}\varphi_{j,\eta\eta}+v(\varphi_{i,\xi\xi}\varphi_{j,\eta\eta}+\varphi_{j,\xi\xi}\varphi_{i,\eta\eta})+2(1-v)\varphi_{i,\xi\eta}\varphi_{j,\xi\eta}\right]\mathrm{d}\xi\,\mathrm{d}\eta,$$

$$l_{ij}=\int\int_{\bar{A}}\varphi_i\varphi_j\,\mathrm{d}\xi\,\mathrm{d}\eta+M\pi\varphi_i(\xi_m,\eta_m)\varphi_j(\xi_m,\eta_m),$$

where $\bar{A}$ is the normalized domain, i, j are integers (1, 2, ..., N), $\varphi_i(\xi,\eta)=[\xi^2+\eta^2-1]^2\xi^{q-p}\eta^p$ and $M=m/m_p$ the mass ratio, which refers the concentrated mass m to the mass of the elliptical plate, $m_p=\rho abh\pi$.

The natural frequency parameters Ω are obtained by setting the determinant of Eq. (11) to be equal to zero. The non-triviality condition yields to the transcendental equation in Ω. As it is known, the roots of this equation constitute upper bounds on the frequency parameters.

Table 1
Comparison and convergence analysis of frequency coefficients $\Omega_i=\omega_i ab\sqrt{\rho h/D}$ for a clamped elliptical plate $a/b=3$, $v=0.3$

Ref. [12]	FEM	$k=10$	$k=11, p\leqslant 7$	$k=12, p\leqslant 7$	$k=13$	$k=15$	$k=20$
18.934	19.022	18.933	18.933	18.933	18.933	18.933	18.933
23.865	24.033	23.863	23.863	23.863	23.863	23.863	23.863
30.083	30.276	30.077	30.077	30.077	30.077	30.077	30.077
37.670	37.817	37.654	37.650	37.650	37.650	37.650	37.650
46.699	46.792	46.666	46.666	46.653	46.653	46.653	46.653
50.031	50.226	50.030	50.030	50.030	50.030	50.030	50.030
57.248	57.310	57.714	57.182	57.182	57.143	57.142	57.142
58.146	58.499	58.142	58.142	58.141	58.141	58.141	58.141
67.420	67.887	67.428	67.397	67.397	67.395	67.395	67.395
69.393	69.329	70.304	70.304	69.260	69.260	69.163	69.159
77.917	78.405	77.937	77.937	77.850	77.850	77.844	77.844
83.137	83.001	91.252	84.859	84.859	82.967	82.744	82.730
89.695	89.977	94.169	89.755	89.755	89.554	89.539	89.538
96.057	96.398	96.056	96.058	96.057	96.056	96.056	96.056
98.625	98.216	105.52	105.52	101.59	101.59	98.382	97.873
102.80	103.06	107.42	107.36	102.97	102.97	102.56	102.52
107.37	107.96	115.17	115.17	107.36	107.36	107.36	107.36
116.50	114.97	119.99	120.00	119.79	117.69	115.61	114.60

3. The finite element approach

A well-known finite element code, Algor [23], is used to obtain an independent solution. The Veubeke's thin plate elements have been used. The calculations have been performed for a clamped elliptical plate with aspect ratio $a/b = 1.5, 2, 3$, $\nu = 0.30$ and different locations of the concentrated mass.

Table 2
Frequency coefficients $\Omega_i = \omega_i a^2 \sqrt{\rho h/D}$ for a clamped circular plate with a concentrated mass at different positions, $\nu = 0.3$

(ξ_m, η_m)	M	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
	0	10.216 (10.216)	21.260 (21.26)	34.877 (34.88)	39.771 (39.771)	51.030 (51.04)	60.829 (60.82)
(0,0)	0.05	9.0133 [9.01]	21.260	33.130 {33.080}	34.877	51.030	60.829
	0.1	8.1173 [8.11]	21.260	30.079 {29.866}	34.877	51.030	60.829
	0.2	6.8837 [6.87]	21.260	27.386 {27.070}	34.877	51.030	60.829
	0.5	5.0410 [5.02]	21.260	25.082 {24.720}	34.877	51.030	60.829
	1	3.7773 [3.76]	21.260	24.162 {23.794}	34.877	51.030	60.829
	∞	21.260	23.160	34.877	51.030	60.829	63.969
(0.1,0)	0.05	9.0518 [9.07]	20.898	21.260	33.926	34.877	34.987
	0.1	8.1689 [8.21]	20.622	21.260	31.290	34.877	34.919
	0.2	6.9397 [7.01]	20.253	21.260	28.998	34.877	34.909
	0.5	5.0880 [5.19]	19.775	21.260	27.183	34.877	34.905
	1	3.8136 [3.91]	19.522	21.260	26.519	34.877	34.904
	∞	19.197	21.260	25.848	34.877	34.903	51.027
(0.2,0)	0.05	9.1658 [9.22]	20.112	21.260	34.150	34.877	36.935
	0.1	8.3265 [8.45]	19.379	21.260	33.072	34.877	36.084
	0.2	7.1156 [7.34]	18.573	21.260	31.701	34.877	35.734
	0.5	5.2394 [5.56]	17.756	21.260	30.443	34.877	35.580
	1	3.9318 [4.25]	17.398	21.260	29.948	34.877	35.538
	∞	16.992	21.260	29.421	34.877	35.499	48.483
(0.5,0)	0.05	9.8082 [9.84]	18.870	21.260	31.627	34.877	39.278
	0.1	9.3694 [9.48]	17.160	21.260	30.176	34.877	39.136
	0.2	8.4964 [8.82]	15.305	21.260	29.062	34.877	39.034
	0.5	6.5858 [7.33]	13.701	21.260	28.255	34.877	38.960
	1	5.0107 [5.88]	13.158	21.260	27.968	34.877	38.932
	∞	12.668	21.260	27.675	34.877	38.903	45.550

Values between round brackets are from Leissa [1], square brackets are from Bambill et al. [5] and curly brackets from Laura et al. [4].

The mesh employed for the elliptical plate of $a/b = 1.5$ has 20,432 quadrilateral elements with 20,537 nodes and 60,987 degrees of freedom, the plate's mesh of relation $a/b = 2$ has 19,184 quadrilateral elements with 19,289 nodes and 57,243 degrees of freedom and the plate's mesh of relation $a/b = 3$ has 25,424 quadrilateral elements with 25,529 nodes and 75,951 degrees of freedom.

Table 3
Frequency coefficients $\Omega_i = \omega_i ab\sqrt{\rho h/D}$ for a clamped elliptical plate of relation $a/b = 1.5$ with a concentrated mass at different positions, $\nu = 0.3$

(ξ_m,η_m)	M	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
	0	11.420	18.981	27.658	29.593	38.023	43.221
		(11.443)	(19.042)	(27.703)	(29.702)	(38.123)	(43.384)
		11.350[a]					
		11.469[b]					44.650[b]
		11.420[c]	18.981[c]	27.658[c]	29.594[c]	38.023[c]	43.223[c]
(0,0)	0.05	10.022	18.981	27.495	27.658	38.023	43.221
	0.1	8.9828	18.981	26.175	27.658	38.023	42.669
	0.2	7.5667	18.981	24.790	27.658	38.023	40.788
	0.5	5.4936	18.981	23.455	27.658	38.023	39.448[d]
		(5.4727)	(19.042)	(23.282)	(27.703)	(38.123)	(38.930)[d]
	1	4.0982	18.981	22.889	27.658	38.023	38.974[d]
		(4.0773)	(19.042)	(22.698)	(27.703)	(38.123)	(38.463)[d]
	∞	18.981	22.256	27.658	38.023	38.491	43.221
		(19.042)	(22.050)	(27.703)	(37.993)	(38.123)	(43.384)
(0.1,0)	0.05	10.068	18.672	27.658	28.160	38.023	41.062
	0.1	9.0389	18.476	27.203	27.658	38.023	39.379
	0.2	7.6218	18.253	26.163	27.658	37.890	38.023
	0.5	5.5348	18.007	25.151	27.658	36.763	38.023
		(5.5178)	(18.007)	(25.096)	(27.703)	(36.397)	(38.123)
	1	4.1284	17.890	24.726	27.658	36.361	38.023
		(4.1107)	(17.882)	(24.659)	(27.703)	(35.998)	(38.123)
	∞	17.747	24.256	27.658	35.953	38.023	45.188
		(17.704)	(24.169)	(27.703)	(35.536)	(38.123)	(45.159)
(0.2,0)	0.05	10.204	18.011	27.658	29.378	38.023	39.591
	0.1	9.2137	17.486	27.658	29.209	37.281	38.023
	0.2	7.7994	16.976	27.658	28.974	35.257	38.023
	0.5	5.6725	16.513	27.658	28.651	33.698	38.023
	1	4.2309	16.323	27.658	28.473	33.147	38.023
	∞	16.115	27.658	28.238	32.606	38.023	46.751
(0.5,0)	0.05	10.985	16.998	26.887	27.658	38.023	41.593
	0.1	10.445	15.622	25.792	27.658	38.023	40.940
	0.2	9.2835	14.300	24.966	27.658	38.023	40.408
	0.5	6.9127	13.398	24.371	27.658	38.023	39.980
		(6.8728)	(13.408)	(24.397)	(27.703)	(38.123)	(39.996)
	1	5.1625	13.139	24.159	27.658	38.023	39.815
		(5.1239)	(13.153)	(24.185)	(27.703)	(38.123)	(39.819)
	∞	12.914	23.944	27.658	38.023	39.637	50.972
		(12.932)	(23.969)	(27.703)	(38.123)	(39.629)	(51.130)

[a] Leissa [1].
[b] Rajalingham and Bhat [11].
[c] Rajalingham et al. [12], results between round brackets were calculated with ALGOR [23].
[d] Biggest percentage deviation with ALGOR: 1.31%.

4. Numerical results

Results are presented here for clamped circular and elliptical plates ($a/b = 1$, 1.1, 1.2, 1.5, 2, 3) with and without a concentrated mass attached. A convergence analysis is made for the elliptical plate of $a/b = 3$ and a comparison of the present results with existing results (Refs. [1,4,5,9,11,12]) is made when it is possible.

Table 1 depicts a comparison of the obtained results for the first 18 natural frequencies of vibration of a clamped elliptical plate of aspect ratio $a/b = 3$ ($\nu = 0.3$) and no mass attached. The effect of increasing the number of monomials ($k = 10$, 11, 12, 13, 15 and 20) has more influence on the higher frequencies. For $k = 11$ and 12 the corresponding numerical approximation are truncated after $p = 7$. There is a good agreement with values of Ref. [12] and those of the finite element approach. As it can be seen in Table 1, in view that the Ritz method gives upper bound on the frequency, the column with $k = 20$, has the best approximation to the exact values.

Tables 2–5 show the first six natural frequency parameters obtained for the plate under study using the proposed approach with $k = 15$. For these calculations, the approximation, Eq. (5), was generated using a complete set of monomials of 136 terms ($N = 136$). The mass was considered attached at four different

Table 4
Frequency coefficients $\Omega_i = \omega_i ab\sqrt{\rho h/D}$ for a clamped elliptical plate of relation $a/b = 2$ with a concentrated mass at different positions, $\nu = 0.3$

(ξ_m,η_m)	M	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
	0	13.689	19.749	27.988	34.929	38.498	44.024
		13.689[a]					
		13.872[b]					
		13.689[c]	19.749[c]	27.989[c]	34.929[c]	38.505[c]	44.025[c]
		13.687[d]					
(0,0)	0.05	11.878	19.749	26.088	34.929	38.498	44.024
	0.1	10.540	19.749	25.078	34.929	38.498	44.024
	0.2	8.7636	19.749	24.116	34.929	38.498	43.210
	0.5	6.2669	19.749	23.254	34.929	38.498	41.709
	1	4.6415	19.749	22.903	34.929	38.498	41.146
	∞	19.749	22.518	34.929	38.498	40.553	44.024
(0.1,0)	0.05	11.938	19.405	26.731	34.929	37.150	44.024
	0.1	10.606	19.218	26.032	34.929	36.419	44.024
	0.2	8.8218	19.031	25.353	34.929	35.735	44.024
	0.5	6.3057	18.850	24.742	34.929	35.141	44.024
	1	4.6685	18.772	24.493	34.904	34.929	44.024
	∞	18.683	24.222	34.647	34.929	44.024	45.657
(0.2,0)	0.05	12.121	18.708	27.839	34.929	35.888	44.024
	0.1	10.820	18.238	27.743	34.535	34.929	44.024
	0.2	9.0169	17.843	27.629	33.315	34.929	44.024
	0.5	6.4438	17.529	27.495	32.297	34.929	44.024
	1	4.7681	17.411	27.429	31.905	34.929	44.024
	∞	17.286	27.344	31.492	34.929	44.024	48.611
(0.5,0)	0.05	13.197	17.713	25.455	34.929	37.041	44.024
	0.1	12.431	16.362	24.557	34.929	36.549	44.024
	0.2	10.698	15.369	23.950	34.929	36.177	44.024
	0.5	7.6617	14.881	23.546	34.929	35.893	44.024
	1	5.6452	14.756	23.409	34.929	35.786	44.024
	∞	14.661	23.316	34.929	35.757	44.024	51.258

[a] Leissa [1].
[b] Rajalingham and Bhat [11].
[c] Rajalingham et al. [12].
[d] Sato [9].

Table 5
Frequency coefficients $\Omega_i = \omega_i ab\sqrt{\rho h/D}$ for a clamped elliptical plate of relation $a/b = 3$ with a concentrated mass at different positions, $\nu = 0.3$

(ξ_m,η_m)	M	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
	0	18.933	23.863	30.077	37.650	46.653	50.030
		19.564[a]					
		18.934[b]	23.865[b]	30.083[b]	37.670[b]	46.699[b]	50.031[b]
		18.930[c]					
(0,0)	0.05	15.958	23.863	28.193	37.650	44.054	50.030
	0.1	13.834	23.863	27.422	37.650	42.991	50.030
	0.2	11.207	23.863	26.805	37.650	42.128	50.030
	0.5	7.8100	23.863	26.325	37.650	41.441	50.030
	1	5.7208	23.863	26.145	37.650	41.178	50.030
	∞	23.863	25.955	37.650	40.896	50.030	57.142
(0.1,0)	0.05	16.054	23.366	28.979	36.365	45.920	50.030
	0.1	13.921	23.172	28.508	35.866	45.569	50.030
	0.2	11.270	23.018	28.122	35.472	45.253	50.030
	0.5	7.8454	22.896	27.818	35.166	44.979	50.030
	1	5.7438	22.849	27.703	35.051	44.868	50.030
	∞	22.799	27.580	34.928	44.746	50.030	51.610
(0.2,0)	0.05	16.362	22.486	30.053	35.442	46.041	50.030
	0.1	14.223	22.081	30.040	34.538	45.790	50.030
	0.2	11.511	21.818	30.027	33.811	45.577	50.030
	0.5	8.0038	21.644	30.013	33.240	45.399	50.030
	1	5.8561	21.583	30.007	33.024	45.328	50.030
	∞	21.522	29.999	32.794	45.250	50.030	53.020
(0.5,0)	0.05	18.342	21.379	27.194	35.684	46.229	50.030
	0.1	16.718	20.042	26.480	35.239	46.095	50.030
	0.2	13.550	19.599	26.113	34.959	45.991	50.030
	0.5	9.3143	19.448	25.907	34.772	45.910	50.030
	1	6.7760	19.410	25.841	34.706	45.879	50.030
	∞	19.377	25.778	34.640	45.845	50.030	55.122

[a]Leissa [1].
[b]Rajalingham et al. [12].
[c]Sato [9].

positions: $(0,0)$, $(0.10,0)$, $(0.20,0)$ and $(0.50,0)$ and its mass ratio's M: 0; 0.05; 0.10; 0.20; 0.50; 1; and $M\rightarrow\infty$. The situation $M\rightarrow\infty$ models a punctual support at the mass position. Table 2 presents results obtained for the circular plate, $a/b = 1$, by the proposed approach and those available in the open literature. It can be seen that when a finite mass is located at the center of the plate the frequency parameters, corresponding to Ω_2, Ω_4, Ω_5 and Ω_6, remain constant[1] and independent of the mass ratio. This phenomenon occurs because the mass is attached at a nodal line.

A similar situation happens for other positions of the mass when its coordinates, ξ_m, η_m, fit in with nodal lines, and consequently, the concentrated mass does not move during these vibrations. As it is mentioned in Ref. [24] the relation between pairs of spatially orthogonal vibration patterns that occurs at each of the normal-mode frequencies and the phenomenon of doublet splitting is not always taken into account. It is interesting to point out that when the mass is at the center of the plate more than one mode may exist with the same frequency (For example $\Omega_2 = 21.260$). When the mass is located out of the center a new second frequency appears. According to Rayleigh's principle [25], the vibration of the plate will always reorient such that the point mass is situated on either a radial node or antinode. When the mass is situated on a radial node, the addition of the mass causes no change in the physical situation and the plate resonates at the natural

[1]Obviously, the rotatory inertia of the mass is not taken into account.

frequency of the unperturbed plate, $\Omega = 21.260$. When the mass is not located on a radial node a different frequency is required to induce resonance. For example $\Omega = 20.892$ ($M = 0.05$ at $\xi_m = 0.10$; $\eta_m = 0$).

A tabulation of the first six natural frequencies for elliptical clamped plates of three different aspect ratios and a concentrated mass are presented in Tables 3–5. The position of the mass changes from the center $\xi_m = 0$ to $\xi_m = 0.10$, 0.20 and 0.50 with $\eta_m = 0$ and different values of the mass relation M. The situation $M \to \infty$ is also considered. The results are compared in some cases with those obtained by the finite element method. The agreement between finite element results and analytical predictions is excellent, from an engineering point of view. The frequency coefficients decrease or remain constant as the mass ratio increases. It is obvious that for modes where the point mass falls on a nodal line, it will not affect the plate vibration and the corresponding natural frequency will be unperturbed. The influence of the presence of the mass is in general higher on the fundamental frequency than on the other frequencies (see Tables 3–5).

The first 23 parameters for the clamped circular plate ($a/b = 1$) having a concentrated mass m attached at the coordinates $\xi_m = 0.50$; $\eta_m = 0$ are listed in Table 6, with $m/m_p = 0$; 0.10; 0.20; 0.50; 1.00 and ∞. The first two columns present the coefficients when there is no mass attached. The agreement between the present results and those available in Ref. [12] is excellent. It may be noted that these results include all the 23 first modes of vibration of the clamped plate with an eccentric mass. Figs. 3 and 4 present the mode shapes for the first six natural frequencies for the circular clamped plate. Fig. 4 corresponds to the circular plate with a point

Table 6
Frequency coefficients $\Omega_i = \omega_i a^2 \sqrt{\rho h/D}$ for a clamped circular plate with a concentrated mass at $\xi_m = 1/2$, $\nu = 0.3$

$M = 0$		M				
		0.10	0.20	0.50	1.00	∞
Ref. [12]	Present study	Present study				
10.216	10.216	9.3695	8.4964	6.5858	5.0108	
21.260	21.260	17.160	15.305	13.701	13.158	12.668
		21.260	21.260	21.260	21.260	21.260
34.877	34.877	30.176	29.062	28.255	27.968	27.675
		34.877	34.877	34.877	34.877	34.877
39.771	39.771	39.136	39.034	38.960	38.932	38.903
51.030	51.030	47.095	46.406	45.912	45.734	45.550
		51.030	51.030	51.030	51.030	51.030
60.829	60.829	60.786	60.770	60.753	60.745	60.735
		60.829	60.829	60.829	60.829	60.829
69.666	69.666	65.252	64.388	63.757	63.528	63.289
		69.666	69.666	69.666	69.666	69.666
84.583	84.583	80.882	80.004	79.391	79.175	78.952
		84.583	84.583	84.583	84.583	84.583
89.104	89.105	86.544	86.342	86.227	86.190	86.154
90.739	90.739	89.759	89.735	89.720	89.715	89.710
		90.739	90.739	90.739	90.739	90.739
111.02	111.02	101.27	100.27	99.648	99.440	99.233
		111.02	111.02	111.02	111.02	111.02
114.21	114.21	113.21	113.17	113.14	113.13	113.12
		114.21	114.21	114.21	114.21	114.21
120.08	120.08	116.74	116.61	116.54	116.51	116.49
		120.08	120.08	120.08	120.08	120.08

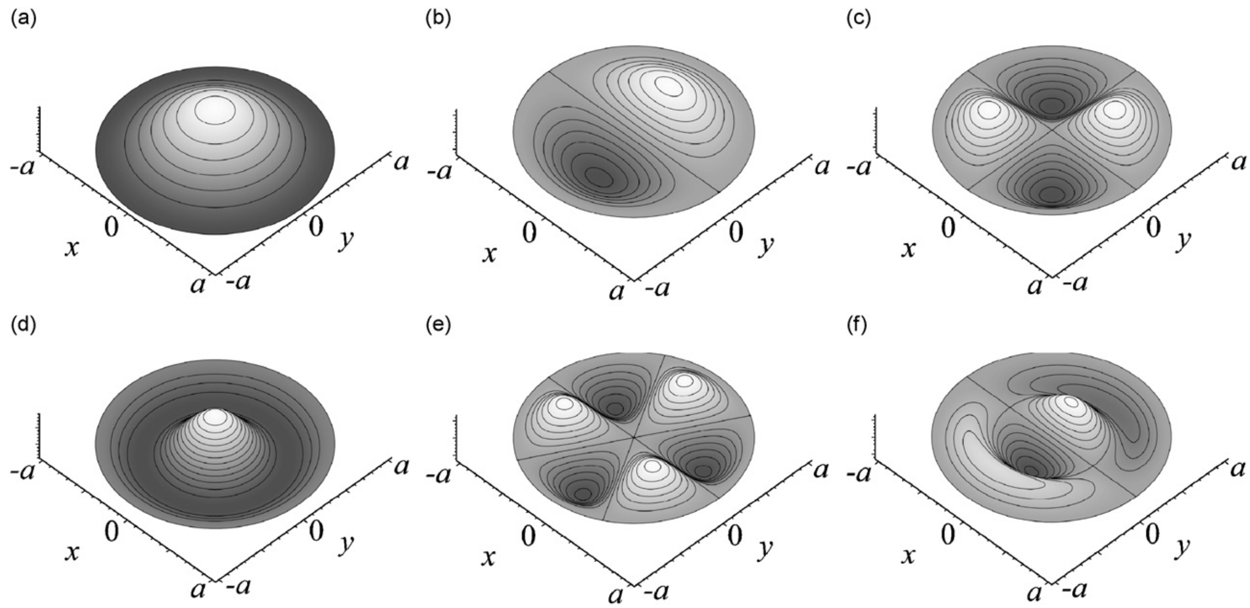

Fig. 3. Vibrating modes of a clamped circular plate: (a) $\Omega_1 = 10.216$, (b) $\Omega_2 = 21.260$, (c) $\Omega_3 = 34.877$, (d) $\Omega_4 = 39.771$, (e) $\Omega_5 = 51.030$ and (f) $\Omega_6 = 60.829$.

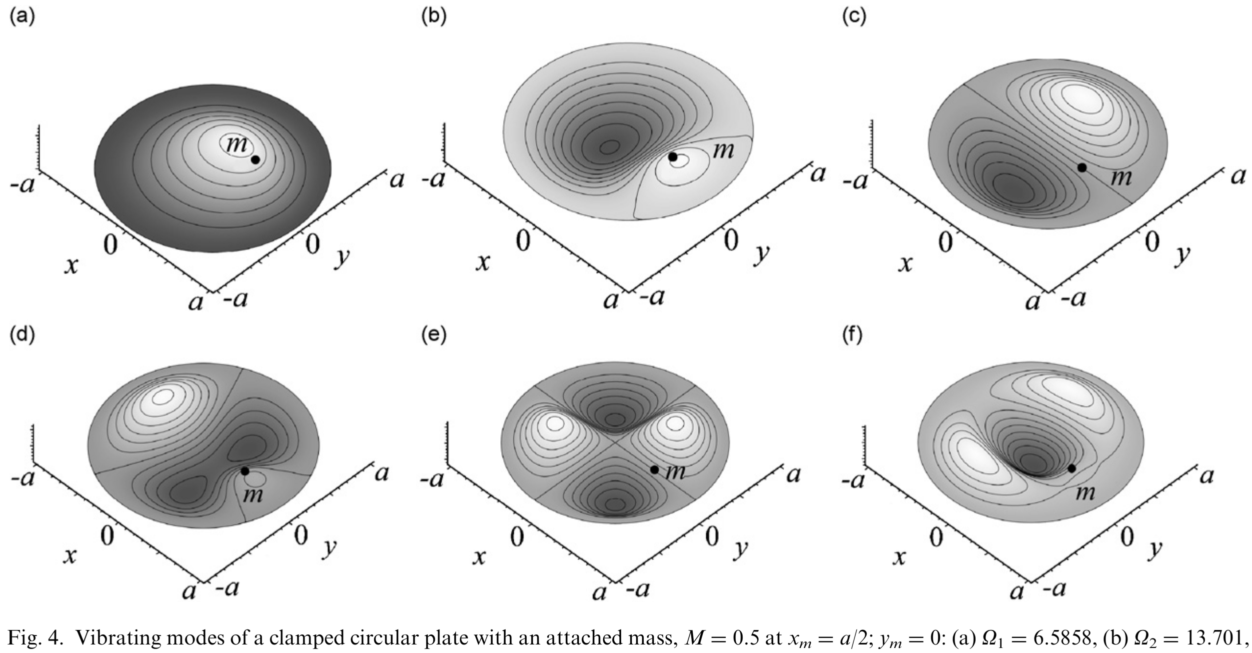

Fig. 4. Vibrating modes of a clamped circular plate with an attached mass, $M = 0.5$ at $x_m = a/2$; $y_m = 0$: (a) $\Omega_1 = 6.5858$, (b) $\Omega_2 = 13.701$, (c) $\Omega_3 = 21.260$, (d) $\Omega_4 = 28.255$, (e) $\Omega_5 = 34.877$ and (f) $\Omega_6 = 38.960$.

mass attached at a distance $a/2$ from the center and $M = 0.5$. The difference in the number of frequency parameters for $M = 0$, also 13, and the other values of M (0.10; 0.20; 0.50; 1), obeys to the phenomenon of doubled splitting mentioned previously (23 frequency parameters).

The lowest 20 natural frequency parameters for plates of aspect ratios $a/b = 1.1$, 1.2 and 1.5 having a concentrated mass attached at three different locations are given in Tables 7–15. For the case $M \to \infty$ (punctual support) the fundamental frequency coefficient increases significatively because of the rigidization

Table 7
Frequency coefficients $\Omega_i = \omega_i ab\sqrt{\rho h/D}$ for a clamped elliptical plate of relation $a/b = 1.1$ with a concentrated mass at $\xi_m = 1/2$, $\eta_m = 0$, $\nu = 0.3$

$M = 0$		M				
		0.10	0.20	0.50	1.00	∞
Ref. [12]	Present study	Present study				
10.283	10.283	9.4314	8.5318	6.5729	4.9844	12.501
20.360	20.360	16.539	14.833	13.398	12.925	22.387
22.387	22.387	22.387	22.387	22.387	22.387	26.646
34.017	34.017	29.145	28.035	27.227	26.940	35.052
35.052	35.052	35.052	35.052	35.052	35.052	40.970
40.987	40.987	40.977	40.975	40.972	40.971	44.611
50.431	50.431	46.450	45.654	45.061	44.842	50.778
50.778	50.778	50.778	50.778	50.778	50.778	58.600
59.038	59.038	58.785	58.717	58.655	58.630	63.540
64.496	64.496	64.496	64.496	64.035	63.793	64.496
69.150	69.151	65.566	64.693	64.496	64.496	69.233
69.234	69.233	69.233	69.233	69.233	69.233	74.688
81.643	81.644	76.349	75.554	75.040	74.865	85.692
85.692	85.692	85.692	85.692	85.692	85.692	87.700
90.187	90.188	88.436	88.106	87.871	87.787	90.202
90.204	90.202	90.202	90.202	90.202	90.202	93.458
93.640	93.640	93.580	93.548	93.507	93.486	94.313
108.22	108.22	95.915	95.059	94.590	94.447	110.60
110.60	110.60	110.60	110.60	110.60	110.60	113.33
113.59	113.58	113.36	113.35	113.34	113.33	113.58

Table 8
Frequency coefficients $\Omega_i = \omega_i ab\sqrt{\rho h/D}$ for a clamped elliptical plate of relation $a/b = 1.1$ with a concentrated mass at $\xi_m = 0$, $\eta_m = 1/2$, $\nu = 0.3$

M				
0.10	0.20	0.50	1.00	∞
9.4269	8.5593	6.6596	5.0782	12.963
17.922	15.898	14.127	13.517	20.360
20.360	20.360	20.360	20.360	28.604
30.721	29.794	29.105	28.858	35.052
35.052	35.052	35.052	35.052	37.429
38.310	37.893	37.619	37.524	46.444
47.481	47.005	46.678	46.563	50.431
50.431	50.431	50.431	50.431	59.038
59.038	59.038	59.038	59.038	61.914
63.318	62.752	62.284	62.105	65.127
65.577	65.322	65.198	65.162	69.233
69.233	69.233	69.233	69.233	80.867
81.535	81.417	81.185	81.045	81.975
83.674	82.702	82.189	82.067	85.692
85.692	85.692	85.692	85.692	90.186
90.188	90.188	90.188	90.188	92.399
92.522	92.462	92.425	92.412	103.36
105.29	104.37	103.77	103.57	108.22
108.22	108.22	108.22	108.22	111.30
111.41	111.35	111.32	111.31	113.58

Table 9
Frequency coefficients $\Omega_i = \omega_i ab\sqrt{\rho h/D}$ for a clamped elliptical plate of relation $a/b = 1.1$ with a concentrated mass at $\xi_m = 1/2$, $\eta_m = 1/2$, $v = 0.3$

M				
0.10	0.20	0.50	1.00	∞
10.131	9.9352	9.0494	7.3995	11.133
18.907	16.751	13.196	11.857	21.134
21.715	21.439	21.250	21.190	24.289
28.877	26.382	24.968	24.599	34.094
34.110	34.102	34.097	34.096	38.027
39.103	38.586	38.249	38.137	43.098
44.658	43.801	43.356	43.223	50.498
50.501	50.500	50.499	50.499	54.193
55.069	54.624	54.363	54.277	61.916
62.457	62.189	62.025	61.970	65.555
65.823	65.682	65.604	65.580	69.228
69.228	69.228	69.228	69.228	75.786
76.943	76.373	76.022	75.905	82.686
82.842	82.761	82.716	82.701	87.524
87.667	87.595	87.552	87.538	90.198
90.198	90.198	90.198	90.198	93.351
93.378	93.365	93.357	93.354	102.30
103.50	102.91	102.55	102.42	108.22
108.22	108.22	108.22	108.22	111.62
111.67	111.65	111.63	111.63	113.58

Table 10
Frequency coefficients $\Omega_i = \omega_i ab\sqrt{\rho h/D}$ for a clamped elliptical plate of relation $a/b = 1.2$ with a concentrated mass at $\xi_m = 1/2$, $\eta_m = 0$, $v = 0.3$

$M = 0$		*M*				
		0.10	0.20	0.50	1.00	∞
Ref. [12]	Present study	Present study				
10.461	10.461	9.5918	8.6480	6.6122	4.9951	12.467
19.735	19.735	16.106	14.520	13.239	12.830	23.615
23.615	23.615	23.615	23.615	23.615	23.615	25.739
32.625	32.626	28.089	27.049	26.289	26.017	35.516
35.516	35.516	35.516	35.516	35.516	35.516	42.876
43.325	43.325	43.223	43.150	43.036	42.968	43.944
48.725	48.725	45.387	44.704	44.235	44.084	50.495
50.496	50.495	50.495	50.495	50.495	50.495	57.409
59.437	59.436	58.261	57.928	57.645	57.533	64.131
67.502	67.502	65.632	65.011	64.514	64.327	68.361
68.363	68.361	68.361	68.361	68.361	68.361	68.815
68.815	68.815	68.815	68.815	68.815	68.815	70.633
79.539	79.539	72.519	71.559	70.991	70.810	87.721
88.365	88.365	88.255	88.083	87.897	87.815	88.365
88.572	88.569	88.365	88.365	88.365	88.365	88.907
88.912	88.907	88.907	88.907	88.907	88.907	89.519
100.27	100.27	91.147	90.245	89.777	89.643	100.273
103.82	103.82	100.27	100.27	100.27	100.27	111.37
111.37	111.37	111.37	111.37	111.37	111.37	111.84
111.97	111.85	111.85	111.85	111.85	111.85	111.96

Table 11
Frequency coefficients $\Omega_i = \omega_i ab\sqrt{\rho h/D}$ for a clamped elliptical plate of relation $a/b = 1.2$ with a concentrated mass at $\xi_m = 0$, $\eta_m = 1/2$, $v = 0.3$

M				
0.10	0.20	0.50	1.00	∞
9.5828	8.7029	6.7813	5.1760	13.340
18.728	16.532	14.611	13.947	19.735
19.735	19.735	19.735	19.735	29.183
30.576	29.985	29.528	29.359	35.516
35.516	35.516	35.516	35.516	36.773
38.611	37.737	37.162	36.967	47.408
47.973	47.705	47.530	47.469	48.725
48.725	48.725	48.725	48.725	59.436
59.436	59.436	59.436	59.436	61.664
63.276	62.560	62.046	61.859	68.361
68.361	68.361	68.361	68.361	68.651
68.664	68.658	68.654	68.652	77.537
78.335	78.011	77.748	77.647	82.565
83.867	83.233	82.833	82.699	88.365
88.365	88.365	88.365	88.365	88.569
88.569	88.569	88.569	88.569	96.675
97.557	97.153	96.876	96.777	103.82
103.82	103.82	103.82	103.82	105.58
106.71	106.16	105.81	105.69	111.39
111.39	111.39	111.39	111.39	111.96

Table 12
Frequency coefficients $\Omega_i = \omega_i ab\sqrt{\rho h/D}$ for a clamped elliptical plate of relation $a/b = 1.2$ with a concentrated mass at $\xi_m = 1/2$, $\eta_m = 1/2$, $v = 0.3$

M				
0.10	0.20	0.50	1.00	∞
10.308	10.107	9.1862	7.4815	11.291
18.510	16.636	13.252	11.975	20.917
22.633	21.858	21.244	21.069	25.073
28.891	26.694	25.571	25.298	32.966
33.164	33.047	32.995	32.980	39.014
40.739	39.877	39.347	39.178	44.556
45.399	44.916	44.685	44.618	48.774
48.786	48.780	48.776	48.775	53.973
54.929	54.441	54.156	54.064	62.875
63.567	63.219	63.011	62.943	68.127
68.141	68.135	68.130	68.129	68.665
68.672	68.668	68.666	68.665	73.479
74.472	73.979	73.679	73.579	82.719
83.186	82.949	82.810	82.764	88.446
88.447	88.446	88.446	88.446	88.870
88.870	88.870	88.870	88.870	98.814
99.363	99.126	98.948	98.882	101.28
101.74	101.50	101.37	101.32	104.83
105.11	104.96	104.88	104.86	111.60
111.60	111.60	111.60	111.60	111.86

Table 13
Frequency coefficients $\Omega_i = \omega_i ab\sqrt{\rho h/D}$ for a clamped elliptical plate of relation $a/b = 1.5$ with a concentrated mass at $\xi_m = 1/2$, $\eta_m = 0$, $v = 0.3$

$M = 0$		M				
		0.10	0.20	0.50	1.00	∞
Ref. [12]	Present study	Present study				
11.420	11.420	10.445	9.2835	6.9127	5.1625	12.914
18.981	18.981	15.622	14.300	13.398	13.139	23.944
27.658	27.658	25.792	24.966	24.371	24.159	27.658
29.594	29.593	27.658	27.658	27.658	27.658	38.023
38.023	38.023	38.023	38.023	38.023	38.023	39.637
43.223	43.222	40.940	40.408	39.980	39.815	50.972
50.974	50.972	50.972	50.972	50.972	50.972	51.135
51.570	51.570	51.345	51.272	51.202	51.171	57.524
59.793	59.789	59.663	59.286	58.379	57.969	59.917
65.332	65.331	61.410	60.279	59.982	59.943	66.546
66.553	66.546	66.546	66.546	66.546	66.546	68.905
79.192	79.176	70.941	69.859	69.265	69.081	79.675
81.694	81.688	79.723	79.698	79.684	79.680	82.883
82.883	82.883	82.883	82.883	82.883	82.883	84.738
84.764	84.738	84.738	84.738	84.738	84.738	89.107
100.03	100.03	90.410	89.757	89.366	89.236	100.03
100.82	100.80	100.03	100.03	100.03	100.03	101.12
101.25	101.22	101.12	101.12	101.12	101.12	105.51
105.55	105.51	105.51	105.51	105.51	105.51	114.81
119.69	119.68	115.76	115.30	115.01	114.91	119.68

Table 14
Frequency coefficients $\Omega_i = \omega_i ab\sqrt{\rho h/D}$ for a clamped elliptical plate of relation $a/b = 1.5$ with a concentrated mass at $\xi_m = 0$, $\eta_m = 1/2$, $v = 0.3$

M				
0.10	0.20	0.50	1.00	∞
10.420	9.4346	7.3214	5.5779	14.748
18.981	18.456	16.212	15.446	18.981
21.123	18.981	18.981	18.981	29.155
29.216	29.187	29.168	29.162	38.023
38.023	38.023	38.023	38.023	38.356
41.810	40.205	39.116	38.738	43.222
43.222	43.222	43.222	43.222	51.219
51.223	51.221	51.220	51.219	58.327
58.598	58.474	58.389	58.359	65.331
65.331	65.331	65.331	65.331	66.546
66.546	66.546	66.546	66.546	72.835
75.095	74.062	73.350	73.097	79.176
79.176	79.176	79.176	79.176	82.433
82.461	82.447	82.439	82.436	83.504
83.553	83.528	83.513	83.508	97.068
97.964	97.557	97.274	97.173	100.03
100.03	100.03	100.03	100.03	100.80
100.80	100.80	100.80	100.80	105.51
105.51	105.51	105.51	105.51	113.47
115.38	114.49	113.90	113.69	119.74

Table 15
Frequency coefficients $\Omega_i = \omega_i ab\sqrt{\rho h/D}$ for a clamped elliptical plate of relation $a/b = 1.5$ with a concentrated mass at $\xi_m = 1/2$, $\eta_m = 1/2$, $\nu = 0.3$

M				
0.10	0.20	0.50	1.00	∞
11.258	11.034	9.9021	7.9071	12.147
17.954	16.505	13.650	12.658	20.680
25.546	23.196	21.456	21.021	28.187
28.601	28.369	28.252	28.218	31.058
33.256	31.970	31.363	31.201	41.312
42.803	42.148	41.637	41.471	43.376
44.012	43.520	43.415	43.393	51.487
51.491	51.489	51.487	51.487	56.073
57.364	56.726	56.332	56.202	60.405
60.709	60.535	60.452	60.428	66.025
66.049	66.037	66.030	66.027	71.671
72.813	72.237	71.895	71.783	79.835
79.882	79.858	79.844	79.840	82.822
82.825	82.824	82.823	82.822	84.738
84.738	84.738	84.738	84.738	90.820
92.031	91.434	91.067	90.943	100.21
100.22	100.22	100.21	100.21	101.12
101.12	101.12	101.12	101.12	104.36
104.45	104.41	104.38	104.37	113.88
114.66	114.28	114.05	113.96	121.54

Table 16
Frequency coefficients $\Omega_i = \omega_i ab\sqrt{\rho h/D}$ for a clamped elliptical plate of relation $a/b = 2$ with a concentrated mass at $\xi_m = 1/2$, $\eta_m = 0$, $\nu = 0.3$

$M = 0$			*M*									
			0.1		0.2		0.5		1		∞	
Ref. [12]	Ritz	FEM	Ritz	FEM	Ritz	FEM	Ritz	FEM	Ritz	FEM	Ritz	FEM
13.689	13.689	13.688	12.431	12.365	10.698	10.589	7.6617	7.5522	5.6452	5.5560	14.648	14.666
19.749	19.749	19.752	16.362	16.334	15.369	15.369	14.881	14.895	14.756	14.772	23.271	23.323
27.989	27.988	27.999	24.557	24.600	23.950	23.997	23.546	23.596	23.409	23.456	34.929	34.932
34.929	34.929	34.932	34.929	34.932	34.929	34.932	34.929	34.932	34.929	34.932	35.672	35.764
38.505	38.498	38.521	36.549	36.660	36.177	36.283	35.893	35.992	35.786	35.882	44.024	44.037
44.025	44.024	44.037	44.024	44.037	44.024	44.037	44.024	44.037	44.024	44.037	51.240	51.358
51.338	51.323	51.365	51.288	51.363	51.274	51.362	51.258	51.361	51.250	51.360	54.969	54.297
54.979	54.969	55.002	54.969	55.002	54.969	55.002	54.969	55.002	54.969	54.939	56.440	55.002
66.179	66.178	66.196	60.953	59.493	59.077	57.245	57.591	55.556	57.031	55.002	66.198	66.218
66.516	66.476	66.544	66.199	66.219	66.199	66.219	66.198	66.218	66.198	66.218	67.857	67.923
67.879	67.857	67.923	67.857	67.923	67.857	67.923	67.857	67.923	67.857	67.923	70.748	70.512
78.517	78.513	78.568	72.601	72.130	71.640	71.273	71.092	70.802	70.917	70.654	79.530	79.595
82.813	82.754	82.874	79.761	79.812	79.638	79.969	79.572	79.663	79.551	79.614	82.754	82.874
84.031	83.952	84.052	82.754	82.874	82.754	82.874	82.754	82.874	82.754	82.874	88.269	88.007
92.644	92.625	92.718	88.782	88.509	88.528	88.257	88.373	88.106	88.321	88.056	98.414	98.381
99.846	99.710	99.908	99.151	99.147	98.787	98.766	98.564	98.535	98.489	98.458	99.710	99.908
103.92	103.74	103.88	99.710	99.908	99.710	99.908	99.710	99.908	99.710	99.908	105.23	105.06
107.30	107.30	107.33	105.37	105.18	105.30	105.12	105.26	105.09	105.25	105.07	107.30	107.33

Table 17
Frequency coefficients $\Omega_i = \omega_i ab\sqrt{\rho h/D}$ for a clamped elliptical plate of relation $a/b = 3$ with a concentrated mass at $\xi_m = 1/2$, $\eta_m = 0$, $v = 0.3$

$M = 0$			M									
			0.1		0.2		0.5		1		∞	
Ref. [12]	Ritz	FEM	Ritz	FEM	Ritz	FEM	Ritz	FEM	Ritz	FEM	Ritz	FEM
18.934	18.933	19.022	16.718	16.665	13.550	13.416	9.3143	9.1703	6.7760	6.6562	19.377	19.472
23.865	23.863	24.033	20.042	20.122	19.599	19.690	19.448	19.543	19.410	19.505	25.778	25.934
30.083	30.077	30.276	26.480	26.627	26.113	26.264	25.907	26.061	25.841	25.996	34.640	34.733
37.670	37.650	37.817	35.239	35.342	34.959	35.057	34.772	34.687	34.706	34.801	45.845	45.835
46.699	46.653	46.792	46.095	46.140	45.991	46.014	45.910	45.915	45.879	45.876	50.030	50.226
50.031	50.030	50.226	50.030	50.226	50.030	50.226	50.030	50.226	50.030	50.226	55.122	55.038
57.248	57.142	57.310	56.044	56.179	55.699	55.762	55.391	55.377	55.264	55.216	58.141	58.498
58.146	58.141	58.499	58.141	58.499	58.141	58.499	58.141	58.498	58.141	58.498	61.664	60.881
67.420	67.395	67.888	63.727	63.168	62.782	62.075	62.127	61.359	61.898	61.119	67.395	67.888
69.393	69.163	69.329	67.395	67.888	67.395	67.888	67.395	67.888	67.395	67.888	77.657	76.786
77.917	77.844	78.405	77.844	78.091	77.844	77.482	77.844	77.072	77.782	76.930	77.844	78.411
83.137	82.744	83.001	78.751	78.431	78.248	78.415	77.904	78.412	77.844	78.411	89.539	89.970
89.695	89.539	89.977	89.539	89.927	89.539	89.972	89.539	89.971	89.539	89.971	95.854	94.588
96.057	96.056	96.398	95.958	95.920	95.923	95.436	95.888	94.970	95.873	94.786	97.51	96.579
98.625	98.382	98.216	97.92	96.851	97.77	96.677	97.63	96.611	97.58	96.594	102.11	100.83
102.80	102.56	103.06	102.56	102.87	102.56	101.78	102.56	101.18	102.39	101.00	102.56	103.07
107.37	107.36	107.96	104.49	103.10	103.42	103.07	102.65	103.07	102.56	103.07	108.64	108.81
116.50	115.61	114.97	109.42	109.20	108.98	108.98	108.76	108.87	108.70	108.84	116.92	115.89

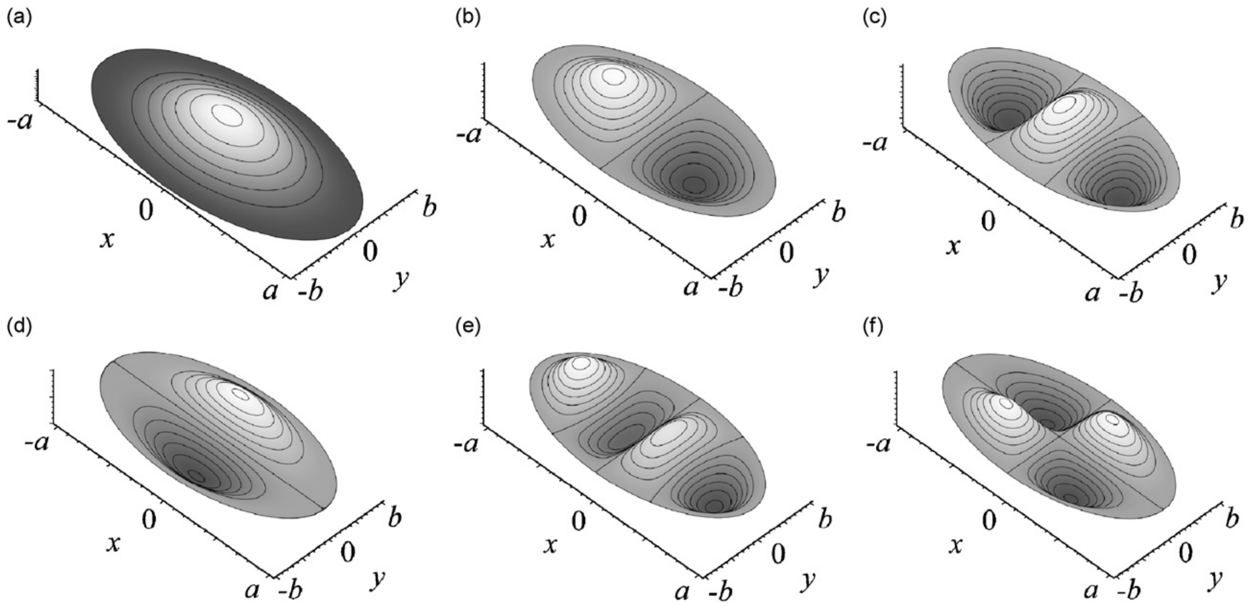

Fig. 5. Vibrating modes of a clamped elliptical plate ($a/b = 2$): (a) $\Omega_1 = 13.689$, (b) $\Omega_2 = 19.749$, (c) $\Omega_3 = 27.988$, (d) $\Omega_4 = 34.929$, (e) $\Omega_5 = 38.498$ and (f) $\Omega_6 = 44.024$.

effect, and differs slightly from the second frequency coefficient of the case $M = 1$. Obviously the fundamental mode shape of the clamped plate with a finite mass is not possible when a fixed point is imposed. It may be seen from the table that the n frequency coefficient of the case $M \to \infty$ fits in or is rather similar to the $(n+1)$ frequency coefficient of the case $M = 1$. The agreement increases with for higher frequencies.

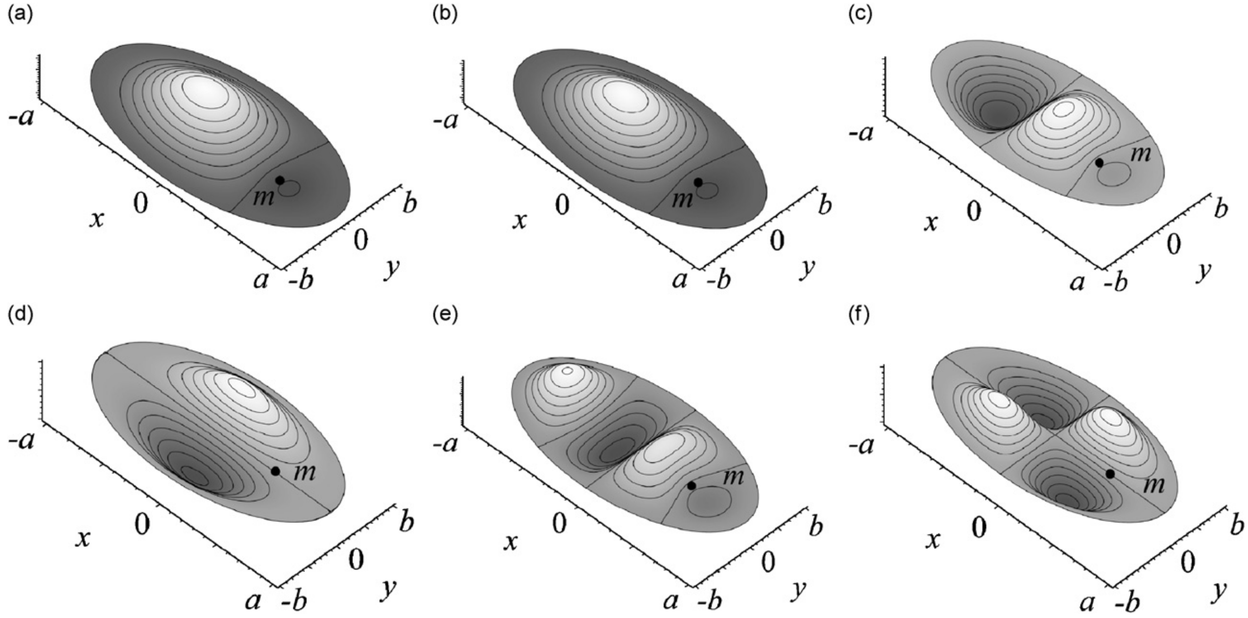

Fig. 6. Vibrating modes of a clamped elliptical plate ($a/b = 2$) with an attached mass $M = 0.5$ at $x_m = a/2$; $y_m = 0$: (a) $\Omega_1 = 7.6617$, (b) $\Omega_2 = 14.881$, (c) $\Omega_3 = 23.546$, (d) $\Omega_4 = 34.929$, (e) $\Omega_5 = 35.893$ and (f) $\Omega_6 = 44.024$.

Tables 16 and 17 present the results for plates of aspect ratios $a/b = 2$ and 3 obtained by the Ritz method with the proposed approach and by the finite element method. The agreement between both sets of results is very good as it is shown in the tables. Figs. 5 and 6 show the mode shapes for the elliptical clamped plates of $a/b = 2$. In Fig. 6, the mass ($M = 0.5$) is attached at $x_m = 0.5a$; $y_m = 0$. It can be seen that new lower frequencies of the plate appear as a consequence of the presence of the mass.

5. Concluding remarks

It has been shown that the present approach is applicable to the vibration problem of elliptical plates with concentrated masses attached at an arbitrary position. The numerical technique converges faster for small aspect ratios and lower frequencies. The rate of convergence decreases with the increasing aspect ratio and higher frequencies. The accuracy is feasible to be improved by increasing the numbers of terms taken in the deflection approximation. The approach presented here may be adapted to include different boundary conditions for the plate or certain additional complexities such as the existence of holes, more than one mass attached, orthotropicity and others.

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