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Active control of flexible marine risers

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Abstract

In this article, active control of flexible marine riser angle and the reduction of forced vibration under a time-varying distributed load are considered using boundary control approach. A torque actuator is introduced in the upper riser package and a boundary control law is designed to generate the required signal for riser angle control and vibration reduction with guaranteed closed-loop stability. Exponential stability can be achieved under the free vibration condition. The proposed control is simple, implementable with actual instrumentation, and is independent of system parameters, thus possessing stability robustness to variations in parameters. The design is based on the partial differential equations of the system, thus avoiding some drawbacks associated with the traditional truncated-model-based design approaches. Numerical simulations are provided to verify the effectiveness of the approach presented.

1. Introduction

The riser plays a very important role in oil drilling and production offshore [1]. As shown in Fig. 1, a marine riser is the connection between a platform on the water surface and the well head on the sea floor. A production riser is a pipe used for oil transportation, while a drilling riser is used for drilling pipe protection and transportation of the drilling mud. Tension is applied at the top of the riser which allows it to resist lateral loads, and its effects on natural frequencies, mode shapes and forced vibration have been studied in Refs. [2,3]. Both types of riser can be modelled as an extremely long and flexible tensioned prismatic tube, suspended from the ocean surface to the sea floor. With the trend towards oil and gas exploration in deeper waters and harsher environments, the response of the risers under various environmental conditions and sea states becomes increasingly complex. The dynamic response is nonlinear and governed by equations of motions dependent on both space and time. Idealized beam models characterized by partial differential equations (PDE) with various boundary conditions have been used to investigate and analyze the dynamic response of such structures subjected to different environmental loads (see e.g. Refs. [4–6]).

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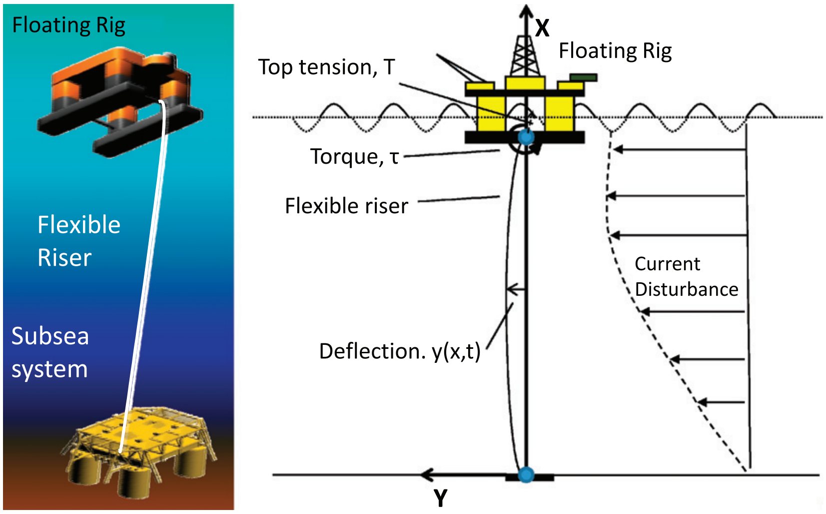

Fig. 1. (left) The marine riser. (right) Schematic and assigned frame of reference.

The riser is subjected to a time-varying distributed load due to the ocean current, resulting in undesirable transverse vibration. The vibration causes stresses in the slender body, which may result in fatigue problems from cyclic loads, damages due to wear and tear, propagation of cracks which requires inspections and costly repairs, and as a worst case, environmental pollution due to leakage from damaged areas. Another important consideration is the angle limit for the upper and lower end joints. The American Petroleum Institute requires that the mean lower and upper joint angles should be kept within 2° while drilling and the maximum non-drilling angles should be limited to 4°. Due to the motion of the surface vessel or the transverse vibrations of the riser, the upper or lower angle limit might be exceeded, resulting in damages to the riser end joints. For drilling and work-over operations, one objective is to minimize the bending stresses along the riser and the riser angle magnitudes at the platform and well head [7]. Hence, vibration reduction to reduce bending stresses and the control of the riser angle magnitude is desirable for preventing damage and improving lifespan.

In this article, we consider active control of a marine riser which is modelled as a tensioned beam, persistently perturbed by the environment. The control is being applied at the beam boundary through the introduction of a torque actuator at the upper riser package shown in Fig. 2. The objective is to reduce the riser angle deflection at the top joint and simultaneously reduce the vibrations of the riser. The control input to the actuator is designed via Lyapunov's synthesis and the required measurements for feedback are the inclination and its rate of change at the upper riser boundary. Although tensioned risers are being considered in this paper specifically, the analysis and control design can be extended and applied, without loss of generality, for vibration control for a class of tensioned beams exposed to undesirable distributed transverse loads. Other examples of practical application in the marine environment include free hanging underwater pipelines, drill strings and umbilicals.

The dynamics of the flexible riser is modelled by a set of PDE which possesses infinite number of dimensions which makes it difficult to control. In conventional approaches for control design, an approximate finite dimensional model is used. The approximate model can be obtained via spatial discretization to obtain a finite number of modes or by modal analysis and truncating the infinite number of modes to a finite number by neglecting the higher frequency modes. Based on a truncated model obtained from either the finite element method or assumed modes method (AMM), various control approaches have been applied to improve the performance of flexible systems (see e.g. Refs. [8–10]). However, spillover effects from the control to the residual modes, which results in instability has been observed in Refs. [11,12] when the control of the truncated system is restricted to a few critical modes. The control order needs to be increased with the number of flexible modes considered to achieve high accuracy of performance. The control may also be difficult to implement from the engineering point of view since full states measurements or observers are often required. To avoid the problems associated with the truncated-model-based design, control methodologies such as variable structure

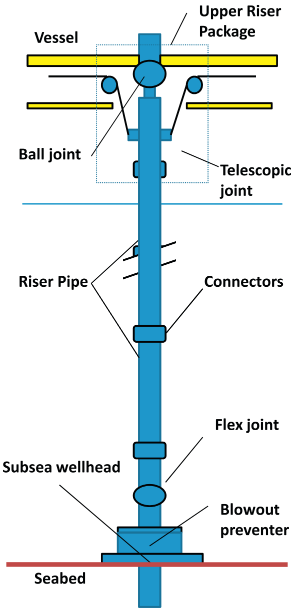

Fig. 2. Marine riser upper package and components.

control [13,14] and boundary control [15] can be used. In this paper, we design the boundary control based on the PDE directly to avoid the above mentioned problems.

Boundary control has been employed in a number of research fields such as vibration control of flexible structures and fluid dynamics. Boundary control of a nonlinear string has been investigated in Refs. [16,17] where feedback from the velocity at the boundary of a string has been shown to stabilize the vibrations. An active boundary control system was introduced in Ref. [18] to damp undesirable vibrations in a cable. In Ref. [19], the asymptotic and exponential stability of an axially moving string is proved by using a linear and nonlinear state feedback. Boundary control has been applied to beams in Refs. [20,21] where boundary feedback was used to stabilize the wave equations and design active constrained layer damping. Active boundary control of an Euler–Bernoulli beam which enables the generation of a desired boundary condition at any designators position of a beam structure has been investigated in Ref. [22]. In Ref. [15], the coupled model for longitudinal and transverse beam was derived, and the exponential stabilization of a beam in free transverse vibration, i.e. with external disturbance set to zero, via boundary control was shown with a riser example.

This is the first application of boundary control to a marine riser, for riser angle and forced vibration reduction, through a torque actuator at the upper riser end. The contributions of this paper are (i) the modelling of a torque actuator at the upper riser package for the control of a transversely vibrating marine riser subjected to an unknown time-varying distributed load due to the ocean current; (ii) design of a boundary control law to minimize the upper riser angle and simultaneously reduce the vibration of the riser; (iii) rigorous stability analysis of the designed control via Lyapunov synthesis which shows that uniform boundedness of the riser deflection can be guaranteed when excited by the transverse load, and exponential stability can be achieved under free vibration condition, and (iv) numerical simulations on a riser subjected to a mean current with worst case oscillating components which excites the riser natural modes, to verify the applicability and performance of the proposed approach.

The remainder of this article is organized as follows: In Section 2, the dynamic equation (PDE) of the flexible structure and boundary conditions are obtained, where the input torque is modelled into the boundary condition. Following that in Section 3, the boundary control design is presented via Lyapunov synthesis, where it is shown that uniform boundedness of the closed-loop system can be guaranteed under the distributed perturbations, and exponential stability can be achieved under free vibration condition. Section 4 presents the numerical method, AMM, for solving for the governing PDE, required for simulations through mode shapes and generalized coordinates. Finally, the simulation study is presented in Section 5 to demonstrate the effectiveness of the control and concluding remarks are made in Section 6.

2. Problem formulation

2.1. Derivation of the governing equation

The reference frame for the riser is shown in Fig. 1 with the origin set at the seabed. Due to the symmetry of the cross section for the riser, we can derive the equations of motions for the flexible riser independently for each principal vertical plane. As such, only the planar dynamics of the riser system is considered in the following analysis. The dynamics of the riser system is idealized as a tensioned slender beam for small angles of deflection (see Appendix). The lateral displacement of a point along its length is represented by $y(x,t)$, a function of space $x \in [0, L]$ and time $t \in [0, \infty)$.

In this paper, we assume that the platform is directly above the subsea well head with no horizontal offset. The riser is filled with seawater and is neutrally buoyant. Horizontal offset and platform motions are not considered as these effects can be included through displacement influence functions or shifting functions by following the guidelines in Refs. [23,24].

The kinetic energy of the riser system E_k can be represented as

$$E_k = \frac{1}{2} m_z \int_0^L \left[\frac{\partial y(x,t)}{\partial t}\right]^2 \mathrm{d}x, \tag{1}$$

where $m_z > 0$ is the uniform mass per unit length of the riser. The potential energy for the flexible riser due to the bending strain [8,25], can be obtained from

$$E_p = \frac{1}{2} EI \int_0^L \left[\frac{\partial^2 y(x,t)}{\partial x^2}\right]^2 \mathrm{d}x, \tag{2}$$

where EI is the flexural rigidity of the riser. A torque actuator is introduced at the upper riser package to produce a concentrated moment $\tau(t)$ for vibration reduction. To determine the virtual work of the concentrated moment [23], we observe that it does work through the rotation of $y'(x,t)$, at $x = L$, its point of application. The work done by the applied torque can be written as

$$W_m = \tau(t) \frac{\partial y(L,t)}{\partial x}, \tag{3}$$

and the total work done on the system, W, is given by

$$\begin{aligned} W &= W_t + W_f + W_d + W_m \\ &= \int_0^L \left\{ -\frac{1}{2} T \left[\frac{\partial y(x,t)}{\partial x}\right]^2 + \left[f(x,t) - c\frac{\partial y(x,t)}{\partial t}\right] y(x,t) \right\} \mathrm{d}x + \tau(t) \frac{\partial y(L,t)}{\partial x}, \end{aligned} \tag{4}$$

where W_t is the work done by the internal tension $T(x,t)$ in elongating the riser, W_f is the work done by the distributed transverse load due to the hydrodynamic effects of the current $f(x,t)$ and W_d is the work done by linear structural damping with the structural damping coefficient, $c > 0$.

2.2. Variation principle and Hamilton's approach

The extended Hamilton's principle [26] is represented by

$$\int_{t_0}^{t_f} \delta(E_k - E_p + W)\, \mathrm{d}t = 0, \tag{5}$$

where $t_0 < t < t_f$ is the operating interval and $\delta(\,)$ denotes the variation operator, physically interpreted as nature trying to equalize the kinetic and potential energies of a system. Substituting Eqs. (1), (2) and (4) into Eq. (5), applying the variation operator with $\delta y(x,t) = 0$ at $t = t_1$ and $t = t_2$ and integrating by parts, we obtain,

$$-\int_{t_1}^{t_2}\int_0^L\left[m_z\frac{\partial^2 y}{\partial t^2}+\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2}\right)-\frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right)-f+c\frac{\partial y}{\partial t}\right]\delta y\,\mathrm{d}x\,\mathrm{d}t$$
$$-\int_{t_1}^{t_2}\left[EI\frac{\partial^2 y}{\partial x^2}\delta\left(\frac{\partial y}{\partial x}\right)\Big|_0^L-\tau(t)\delta\frac{\partial y(L,t)}{\partial x}\right]\mathrm{d}t+\int_{t_1}^{t_2}\left[\frac{\partial}{\partial x}\left(EI\frac{\partial^2 y}{\partial x^2}\right)-T\frac{\partial y}{\partial x}\right]\delta y\Big|_0^L\mathrm{d}t=0. \tag{6}$$

As $\delta y(x,t)$ is assumed to be a non-zero arbitrary variation in $0<x<L$, the expression under the double integral in Eq. (6) is set equal to zero. Hence, we obtain the equation of motion as

$$EI\frac{\partial^4 y(x,t)}{\partial x^4}-T\frac{\partial^2 y(x,t)}{\partial x^2}+m_z\frac{\partial^2 y(x,t)}{\partial t^2}+c\frac{\partial y(x,t)}{\partial t}-f(x,t)=0, \tag{7}$$

$\forall(x,t)\in(0,L)\times[0,\infty)$. Setting the terms with single integrals in Eq. (6) equal to zero, we obtain the boundary conditions

$$y(0,t)=0, \tag{8}$$

$$EI\frac{\partial^2 y(0,t)}{\partial x^2}=0, \tag{9}$$

$$y(L,t)=0, \tag{10}$$

$$EI\frac{\partial^2 y(L,t)}{\partial x^2}-\tau(t)=0, \tag{11}$$

where Eqs. (8) and (9) represent a simple support at $x=0$, Eqs. (10) and (11) represent that there is zero deflection and a torque being applied at $x=L$, respectively.

Remark 1. In the above derivations, we have shown that the input torque $\tau(t)$ at the upper riser end can be modelled as a boundary condition (11) in relation to the dynamics of the system. The flex joint at the wellhead is commonly modelled as a ball joint during analysis [27]. The governing equation (7) for the flexible marine riser, a fourth-order PDE with axial tension, structural damping and external disturbances terms, remains in the same form as considered in Refs. [3,28].

2.3. *Effects of time-varying current*

The effects of a time-varying surface current $U(t)$ on a riser can be modelled as a vortex excitation force [29,30]. The distributed load on a 3D riser structure, $f(x,z,t)$ can be expressed as a combination of the in-line drag force $f_D(x,t)$, comprising of a mean drag term and an oscillating drag about the mean modelled as

$$f_D(x,t)=\tfrac{1}{2}\rho_s C_D(x,t)U(x,t)^2D+A_D\cos(4\pi f_v t+\beta), \tag{12}$$

and an oscillating lift $f_L(z,t)$, perpendicular to $f_D(x,t)$, about a mean deflected profile,

$$f_L(z,t)=\tfrac{1}{2}\rho_s C_L(z,t)U(z,t)^2D\cos(2\pi f_v t+\alpha), \tag{13}$$

where z is an axis perpendicular to plane XOY show in Fig .1, $C_D(x,t)$ and $C_L(z,t)$ are the time and spatially varying drag and lift coefficient respectively, f_v is the shedding frequency, ρ_s is the sea water density, α and β are phase angles, and A_D is the amplitude of the oscillatory part of the drag force, typically 20% of the first term in $f_D(x,t)$ [30]. The non-dimensional vortex shedding frequency can be expressed as

$$f_v=\frac{S_tU}{D}, \tag{14}$$

where S_t is the Strouhal number and D is the pipe outer diameter.

In this paper, we consider the deflection of the riser in only one direction. Hence, the distributed excitation force is considered as the drag force Eq. (12), $f(x,t)=f_D(x,t)$. The current profile $U(x,t)$, similar to that shown in Fig. 1, is a function which relates the depth to the ocean surface current velocity $U(t)$. The transverse

VIV from the lift component is not considered in this article but the proposed method can be similarly applied without any loss of generality if only the lift component is considered.

Assumption 1. For the distributed disturbance $f(x,t)$, we assume that there exists a constant $\bar{f} \in R^+$, such that $\|f(x,t)\| \leqslant \bar{f}$, $\forall (x,t) \in [0,L] \times [0,\infty)$. This is a reasonable assumption as the effects of the time-varying current, $f(x,t)$, are exogenous, have finite energy and hence are bounded, i.e., $f(x,t) \in \mathscr{L}_\infty([0,L])$.

Remark 2. For control design in Section 3, only the assertion that there exist an upper bound on the disturbance in Assumption 1, $\|f(x,t)\| < \bar{f}$, is necessary. The knowledge of the exact value for $f(x,t)$ is not required $\forall (x,t) \in [0,L] \times [0,\infty)$. As such, different VIV models up to various levels of fidelity, such as those found in Refs. [29,31–34], can be applied without affecting the control design or analysis.

Remark 3. The VIV problem can beseparated into the drag and the lift components, perpendicular to each other. The vector sum results in a force with oscillating magnitude and direction, thereby producing of figure of "8" response in the riser. Under Assumption 1, it is possible that control applied to these two cases in separate axis may be sufficient for vibration reduction of the VIV problem. The combination of drag and oscillating lift will be treated in future analysis using a 3D riser model.

Remark 4. In the following sections, the notations $y'(x,t) = \partial y(x,t)/\partial x$, $y''(x,t) = \partial^2 y(x,t)/\partial x^2$ and $\dot{y}(x,t) = \partial y(x,t)/\partial t$, etc. are used and dependencies of terms are omitted where obvious for conciseness.

3. Control design

The control objective is to minimize the upper riser angle $y'(L,t)$, and simultaneously reduce the vibrations of the riser $y(x,t)$, subjected to the time-varying distributed transverse load from the ocean current $f(x,t)$. In this section, we use Lyapunov's synthesis to construct a boundary control law $\tau(t)$ for the above objective, and to rigourously show the closed-loop stability of the distributed system. Now, we present some lemmas and properties that will be used in subsequent developments.

Lemma 1 (*Dawson et al. [35], Ge and Wang [36]*). *For bounded initial conditions, if there exists a* C^1 *continuous and positive definite Lyapunov function* $V(x)$ *satisfying* $\kappa_1(\|x\|) \leqslant V(x) \leqslant \kappa_2(\|t\|)$, *such that* $\dot{V}(x) \leqslant -\lambda V(x) + c$, *where* $\kappa_1, \kappa_2 : R^n \to R$ *are class K functions and c is a positive constant, then the solution* $x = 0$ *is uniformly bounded.*

Lemma 2. *Let* $y_1(x,t), y_2(x,t) \in R$ *with* $x \in [0,L]$ *and* $t \in [0,\infty)$, *the following inequalities hold*:

$$y_1 y_2 \leqslant |y_1 y_2| \leqslant y_1^2 + y_2^2, \tag{15}$$

$$2y_1 y_2 \leqslant 2|y_1 y_2| \leqslant y_1^2 + y_2^2, \quad \forall y_1, y_2 \in \mathscr{R}. \tag{16}$$

From Lemma 2, we can obtain the inequality [37],

$$|y_1 y_2| = \left| \left(\frac{1}{\sqrt{\delta}} y_1 \right) (\sqrt{\delta} y_2) \right| \leqslant \frac{1}{\delta} y_1^2 + \delta y_2^2, \quad \forall y_1, y_2 \in \mathscr{R} \text{ and } \delta > 0. \tag{17}$$

Lemma 3 (*Hardy et al. [38], Queiroz et al. [39]*). *Let* $y(x,t) \in R$ *be a function defined on* $x \in [0,L]$ *and* $t \in [0,\infty)$ *that satisfies the boundary condition*

$$y(0,t) = 0, \quad \forall t \in [0,\infty), \tag{18}$$

then the following inequalities hold:

$$\int_0^L y^2 \, dx \leqslant L^2 \int_0^L [y']^2 \, dx, \tag{19}$$

$$y^2 \leqslant L \int_0^L [y']^2 \, \mathrm{d}x, \quad \forall x \in [0, L]. \tag{20}$$

Property 1 (*Queiroz et al. [39]*). *If the kinetic energy of system* (7)–(11), *given by Eq.* (1) *is bounded* $\forall (x,t) \in [0,L] \times [0,\infty)$, *then* $\dot{y}'(x,t)$ *and* $\dot{y}''(x,t)$ *are bounded* $\forall (x,t) \in [0,L] \times [0,\infty)$.

Property 2 (*Queiroz et al. [39]*). *If the potential energy of system* (7)–(11), *given by Eq.* (2) *is bounded* $\forall (x,t) \in [0,L] \times [0,\infty)$, *then* $y''(x,t)$, $y'''(x,t)$ *and* $y''''(x,t)$ *are bounded* $\forall (x,t) \in [0,L] \times [0,\infty)$.

3.1. Boundary control

Consider the Lyapunov function candidate,

$$V(t) = E_b(t) + E_c(t) + \tfrac{1}{2}(k_2 + \beta EIk_1)[y'(L,t)]^2, \tag{21}$$

where k_1, $k_2 > 0$ are control parameters, $E_b(t)$ and a small crossing term $E_c(t)$ are defined as

$$E_b = \frac{1}{2} m_z \int_0^L \dot{y}^2 \, \mathrm{d}x + \frac{1}{2} EI \int_0^L [y'']^2 \, \mathrm{d}x + \frac{1}{2} T \int_0^L [y']^2 \, \mathrm{d}x, \tag{22}$$

$$E_c = \beta m_z \int_0^L y\dot{y} \, \mathrm{d}x, \tag{23}$$

and $\beta > 0$ is a small positive weighting constant.

Lemma 4. *Function* (21), *can be upper and lower bounded as*

$$0 \leqslant \lambda_1 (E_b + [y'(L,t)]^2) \leqslant V(t) \leqslant \lambda_2 (E_b + [y'(L,t)]^2), \tag{24}$$

where λ_1 *and* λ_2 *are positive constants.*

Proof. Using Eqs. (15) and (19) on Eq. (23), we obtain

$$|E_c| \leqslant \beta m_z \int_0^L (\dot{y}^2 + y^2) \, \mathrm{d}x \tag{25}$$

$$\leqslant \beta m_z \int_0^L \dot{y}^2 \, \mathrm{d}x + \beta m_z L^2 \int_0^L [y']^2 \, \mathrm{d}x \tag{26}$$

$$\leqslant 2\beta m_z \frac{\max(1, L^2)}{\min(m_z, T, EI)} E_b \tag{27}$$

$$\leqslant \xi E_b, \tag{28}$$

where

$$\xi = 2\beta m_z \frac{\max(1, L^2)}{\min(m_z, T, EI)}. \tag{29}$$

Selecting β according to the following sufficient condition:

$$\beta \leqslant \frac{\min(m_z, T, EI)}{\max(1, L^2)}, \tag{30}$$

we have

$$-\xi E_b \leqslant E_c \leqslant \xi E_b, \tag{31}$$

$$0 \leqslant \xi_1 E_b \leqslant E_b + E_c \leqslant \xi_2 E_b, \tag{32}$$

where for some positive constants $\xi_1 = 1 - \xi$ and $\xi_2 = 1 + \xi$,

$$\xi_1 = 1 - 2\beta m_z \frac{\max(1, L^2)}{\min(m_z, T, EI)} > 0, \tag{33}$$

$$\xi_2 = 1 + 2\beta m_z \frac{\max(1, L^2)}{\min(m_z, T, EI)} > 1. \tag{34}$$

Given the Lyapunov functional candidate in Eq. (21), we obtain

$$0 \leqslant \lambda_1(E_b + [y'(L,t)]^2) \leqslant V(t) \leqslant \lambda_2(E_b + [y'(L,t)]^2), \tag{35}$$

where $\lambda_1 = \min[\xi_1, 0.5(k_2 + \beta EIk_1)]$ and $\lambda_2 = \max[\xi_2, 0.5(k_2 + \beta EIk_1)]$. □

Lemma 5. *The time derivative of the Lyapunov function in Eq.* (21) *can be upper bounded with*

$$\dot{V}(t) \leqslant -\lambda_3(E_b + [y'(L,t)]^2) + \varepsilon, \tag{36}$$

where $\lambda_3 > 0$.

Proof. Taking time derivative of $V(t)$, we obtain

$$\dot{V}(t) = \dot{E}_b + \dot{E}_c + (k_2 + \beta EIk_1)\dot{y}'(L,t)y'(L,t). \tag{37}$$

The first term of Eq. (37) yields

$$\begin{aligned}
\dot{E}_b &= \int_0^L m_z\dot{y}\ddot{y} + EIy''\dot{y}'' + Ty'\dot{y}'\,\mathrm{d}x \\
&= \int_0^L (-c\dot{y} - EIy'''' + Ty'' + f)\dot{y} + EIy''\dot{y}'' + Ty'\dot{y}'\,\mathrm{d}x \\
&= [EIy''\dot{y}' - EIy'''\dot{y} + Ty'\dot{y}]_0^L + \int_0^L [(-c\dot{y} - EIy'''' + Ty'' + f)\dot{y} + EIy''''\dot{y} - Ty''\dot{y}]\,\mathrm{d}x \\
&= [EIy''\dot{y}' - EIy'''\dot{y} + Ty'\dot{y}]_0^L + \int_0^L [-c\dot{y}^2 + f\dot{y}]\,\mathrm{d}x.
\end{aligned} \tag{38}$$

From Eq. (7) and performing integration by parts, we obtain

$$m_z\ddot{y} = -c\dot{y} - EIy'''' + Ty'' + f, \tag{39}$$

$$EI\int_0^L y''\dot{y}''\,\mathrm{d}x = EI[y''\dot{y}']_0^L - EI[y'''\dot{y}]_0^L + EI\int_0^L y''''\dot{y}\,\mathrm{d}x, \tag{40}$$

$$T\int_0^L y'\dot{y}'\,\mathrm{d}x = T[y'\dot{y}]_0^L - T\int_0^L y''\dot{y}\,\mathrm{d}x. \tag{41}$$

Substituting Eqs. (39)–(41) and boundary conditions (8)–(11) into Eq. (38), we arrive at

$$\begin{aligned}
\dot{E}_b &= [EIy''(L,t)\dot{y}'(L,t)] + \int_0^L [-c\dot{y}^2 + f\dot{y}]\,\mathrm{d}x \\
&= \tau\dot{y}'(L,t) + \int_0^L [-c\dot{y}^2 + f\dot{y}]\,\mathrm{d}x \\
&= \tau\dot{y}'(L,t) - c\int_0^L \dot{y}^2\,\mathrm{d}x + \int_0^L f\dot{y}\,\mathrm{d}x,
\end{aligned} \tag{42}$$

where $\dot{y}(0,t) = \dot{y}(L,t) = 0$ due to the boundary conditions. Using inequality (17), we obtain

$$\begin{aligned}
\dot{E}_b &\leqslant \tau\dot{y}'(L,t) - c\int_0^L \dot{y}^2\,\mathrm{d}x + \int_0^L \frac{1}{\delta_1}f^2\,\mathrm{d}x + \int_0^L \delta_1\dot{y}^2\,\mathrm{d}x \\
&\leqslant \tau\dot{y}'(L,t) - (c - \delta_1)\int_0^L \dot{y}^2\,\mathrm{d}x + \frac{1}{\delta_1}\int_0^L f^2\,\mathrm{d}x,
\end{aligned} \tag{43}$$

where $\delta_1 > 0$ is a positive constant. Taking the time derivative of the crossing term (23), we have

$$\begin{aligned}\dot{E}_c &= \beta m_z \int_0^L (\dot{y}^2 + y\ddot{y})\,\mathrm{d}x \\ &= \beta \int_0^L [m_z\dot{y}^2 + y(-c\dot{y} - EIy'''' + Ty'' + f)]\,\mathrm{d}x \\ &= \beta \int_0^L [-EIyy'''' - cy\dot{y} + fy + Tyy'' + m_z\dot{y}^2]\,\mathrm{d}x. \end{aligned} \tag{44}$$

The first term of Eq. (44) simplifies via integration by parts and boundary conditions to

$$\begin{aligned}-\beta \int_0^L EIyy''''\,\mathrm{d}x &= -\beta EI[yy''' - y'y'']_0^L - \beta EI \int_0^L [y'']^2\,\mathrm{d}x \\ &= \beta EIy'(L,t)\tau - \beta EI \int_0^L [y'']^2\,\mathrm{d}x. \end{aligned} \tag{45}$$

The second term using Eq. (17), gives

$$-\beta c \int_0^L y\dot{y}\,\mathrm{d}x \leqslant \beta \frac{c}{\delta_2} \int_0^L \dot{y}^2\,\mathrm{d}x + \beta c\delta_2 L^2 \int_0^L [y']^2\,\mathrm{d}x, \tag{46}$$

where $\delta_2 > 0$. The third term with Eq. (17) gives

$$\beta \int_0^L yf\,\mathrm{d}x \leqslant \frac{\beta}{\delta_3} \int_0^L f^2\,\mathrm{d}x + \beta\delta_3 L^2 \int_0^L [y']^2\,\mathrm{d}x, \tag{47}$$

where $\delta_3 > 0$, and the fourth term yields through integration by parts,

$$\begin{aligned}\beta T \int_0^L yy''\,\mathrm{d}x &= \beta T[yy']_0^L - \beta T \int_0^L y'y'\,\mathrm{d}x \\ &= -\beta T \int_0^L [y']^2\,\mathrm{d}x. \end{aligned} \tag{48}$$

From Eqs. (43) and (45)–(48), we arrive at the inequalities

$$\dot{E}_b \leqslant \tau\dot{y}'(L,t) - (c - \delta_1) \int_0^L \dot{y}^2\,\mathrm{d}x + \frac{1}{\delta_1} \int_0^L f^2\,\mathrm{d}x, \tag{49}$$

$$\begin{aligned}\dot{E}_c \leqslant{}& \beta EIy'(L,t)\tau - \beta EI \int_0^L [y'']^2\,\mathrm{d}x + \beta \frac{c}{\delta_2} \int_0^L \dot{y}^2\,\mathrm{d}x + \beta c\delta_2 L^2 \int_0^L [y'']^2\,\mathrm{d}x \\ &+ \frac{\beta}{\delta_3} \int_0^L f^2\,\mathrm{d}x + \beta\delta_3 L^2 \int_0^L [y'']^2\,\mathrm{d}x - \beta T \int_0^L [y']^2\,\mathrm{d}x + \beta \int_0^L m_z\dot{y}^2\,\mathrm{d}x. \end{aligned} \tag{50}$$

Substituting Eqs. (49) and (50) into Eq. (37), we arrive at

$$\begin{aligned}\dot{V} &= \dot{E}_b + \dot{E}_c + (k_2 + \beta EIk_1)\dot{y}'(L,t)y'(L,t) \\ &\leqslant (\dot{y}'(L,t) + \beta EIy'(L,t))\tau - \left(c - \beta m_z - \delta_1 - \beta\frac{c}{\delta_2}\right) \int_0^L \dot{y}^2\,\mathrm{d}x + \left(\frac{1}{\delta_1} + \frac{\beta}{\delta_3}\right) \int_0^L f^2\,\mathrm{d}x \\ &\quad - \beta EI \int_0^L [y'']^2\,\mathrm{d}x - \beta(T - c\delta_2 L^2 - \delta_3 L^2) \int_0^L [y']^2\,\mathrm{d}x + (k_2 + \beta EIk_1)\dot{y}'(L,t)y'(L,t) \end{aligned} \tag{51}$$

Consider the following boundary control law:

$$\tau = -[k_1\dot{y}'(L,t) + k_2 y'(L,t)], \tag{52}$$

and substituting the control law (52) into Eq. (51) under Assumption 1, we obtain

$$
\begin{aligned}
\dot{V} \leqslant & -(\dot{y}'(L,t)+\beta EIy'(L,t))[k_1\dot{y}'(L,t)+k_2y'(L,t)]+(k_2+\beta EIk_1)\dot{y}'(L,t)y'(L,t) \\
& -\left(c-\beta m_z-\delta_1-\beta\frac{c}{\delta_2}\right)\int_0^L \dot{y}^2\,\mathrm{d}x-\beta EI\int_0^L[y'']^2\,\mathrm{d}x+\left(\frac{1}{\delta_1}+\frac{\beta}{\delta_3}\right)\int_0^L f^2\,\mathrm{d}x \\
& -\beta(T-c\delta_2L^2-\delta_3L^2)\int_0^L[y']^2\,\mathrm{d}x \\
\leqslant & -k_1[\dot{y}'(L,t)]^2-k_2\beta EI[y'(L,t)]^2-\left(c-\beta m_z-\delta_1-\beta\frac{c}{\delta_2}\right)\int_0^L \dot{y}^2\,\mathrm{d}x \\
& -\beta EI\int_0^L[y'']^2\,\mathrm{d}x-\beta(T-c\delta_2L^2-\delta_3L^2)\int_0^L[y']^2\,\mathrm{d}x+\left(\frac{1}{\delta_1}+\frac{\beta}{\delta_3}\right)\int_0^L f^2\,\mathrm{d}x \\
\leqslant & -\lambda_3(E_b+[y'(L,t)]^2)+\varepsilon, \qquad (53)
\end{aligned}
$$

where

$$
\begin{aligned}
\lambda_3 &= \min\left(\frac{\varepsilon_1}{m_z},\beta,\frac{\varepsilon_2}{T},k_2\beta EI\right)>0, \\
\varepsilon &= \left(\frac{1}{\delta_1}+\frac{\beta}{\delta_3}\right)\max_{t=[0,\infty)}\int_0^L f^2\,\mathrm{d}x<\infty, \\
\varepsilon_1 &= c-\beta m_z-\delta_1-\beta\frac{c}{\delta_2}>0, \\
\varepsilon_2 &= T-c\delta_2L^2-\delta_3L^2>0.
\end{aligned}
$$

From Eqs. (24) and (53) , we have

$$\dot{V}(t)\leqslant -\lambda V(t)+\varepsilon, \qquad (54)$$

where $\lambda=\lambda_3/\lambda_2$. After obtaining Eq. (54), we are ready to present the following theorem, which contain the results for the boundary control of the flexible riser. □

Theorem 1. *Consider the system described by Eq.* (7) *and boundary conditions* (8)–(11), *under Assumption* 1, *and the control law* (52). *Given that the initial conditions are bounded, and that the required state information* $y'(L,t)$ *and* $\dot{y}'(L,t)$ *are available, the closed-loop system is uniformly bounded.*

Proof. From Eq. (54) and Lemma 1, it is straightforward to show the deflection $y(x,t)$ is uniformly bounded. For completeness, the details of the proof are provided here. Multiplying Eq. (54) by $\mathrm{e}^{\lambda t}$, we obtain

$$\frac{\partial}{\partial t}(V\mathrm{e}^{\lambda t})\leqslant \varepsilon\mathrm{e}^{\lambda t}. \qquad (55)$$

Integration of the above and applying Lemma 1 yields

$$V(t)\leqslant\left(V(0)-\frac{\varepsilon}{\lambda}\right)\mathrm{e}^{\lambda t}+\frac{\varepsilon}{\lambda}\leqslant V(0)+\frac{\varepsilon}{\lambda}\in\mathscr{L}_\infty. \qquad (56)$$

Utilizing Eqs. (20), (22) and (24), we have

$$\frac{1}{2L}Ty^2(x,t)\leqslant\frac{1}{2}T\int_0^L[y'(x,t)]^2\,\mathrm{d}x\leqslant E_b(t)\leqslant\frac{1}{\lambda_1}V(t)\in\mathscr{L}_\infty. \qquad (57)$$

Hence, we have $y(x,t)\in\mathscr{L}_\infty$. From Eqs. (24) and (53), we can state that $E_b(t)$ and $y'(L,t)$ are bounded $\forall t\in[0,\infty)$. Since $E_b(t)$ is bounded, $\dot{y}(x,t)$, $y'(x,t)$ and $y''(x,t)$ are bounded $\forall(x,t)\in[0,L]\times[0,\infty)$. From Eq. (1), the kinetic energy of the system is bounded and using Property 1, $\dot{y}'(x,t)$ is bounded $\forall(x,t)\in[0,L]\times[0,\infty)$. At this point, we have shown that all the signals in the control law (52) are bounded. From the boundedness of the potential energy (2), we can use Property 2 to conclude that $y''''(x,t)$ is bounded. Finally, using Assumption 1, Eqs. (7)–(11), and the above statements, we can conclude that $\ddot{y}(x,t)$ is bounded $\forall(x,t)\in[0,L]\times[0,\infty)$. □

Corollary 1. *For the system described by governing equation* (7), *and boundary conditions* (8)–(11), *if the free vibration case is considered, i.e.* $f(x,t)=0$, *the boundary control* (52) *ensures that the riser displacement is exponentially stabilized as follows*:

$$|y(x,t)| \leqslant \sqrt{\frac{2\lambda_2 L}{T\lambda_1}\exp\left(-\frac{\lambda_3}{\lambda_2}t\right)}, \quad \forall x \in [0,L], \tag{58}$$

where λ, λ_1, λ_2 *are positive constants.*

Proof. From Eq. (53), under the free vibration condition, we obtain the time derivation of the Lyapunov function candidate (21) as

$$\dot{V} \leqslant -\lambda_3(E_b + [y'(L,t)]^2), \tag{59}$$

where

$$\lambda_3 = \min\left(\frac{\varepsilon_1}{m_z}, \beta, \frac{\varepsilon_2}{T}, k_2\beta EI\right) > 0,$$
$$\varepsilon_1 = c - \beta m_z - \delta_1 - \beta\frac{c}{\delta_2} > 0,$$
$$\varepsilon_2 = T - c\delta_2 L^2 - \delta_3 L^2 > 0.$$

From Eqs. (24) and (59) , we obtain the upper bound as

$$\dot{V}(t) \leqslant -\frac{\lambda_3}{\lambda_2}V(t), \tag{60}$$

which has a solution of

$$\begin{aligned} V(t) &\leqslant V(0)\exp\left(-\frac{\lambda_3}{\lambda_2}t\right) \\ &\leqslant \lambda_2[E_b(0) + y'(L,0)^2]\exp\left(-\frac{\lambda_3}{\lambda_2}t\right). \end{aligned} \tag{61}$$

Similarly, utilizing Eqs. (20), (22) and (24), we have

$$\frac{1}{2L}Ty^2(x,t) \leqslant \frac{1}{2}T\int_0^L [y'(x,t)]^2\,\mathrm{d}x \leqslant E_b(t) \leqslant \frac{1}{\lambda_1}V(t), \tag{62}$$

$\forall (x,t) \in [0,L]\times[0,\infty)$ and Eq. (58) follows from combining Eqs. (61) and (62). The bounds for $y'(x,t)$, $y''(x,t)$, $y''''(x,t)$, $\dot{y}(x,t)$, $\dot{y}'(x,t)$ and $\ddot{y}(x,t)$ can be similarly shown as in Theorem 1. This concludes the proof. □

Remark 5. The proposed control is simple in structure and implementable as $y'(L,t)$, the top riser angle, can be measured directly using inclinometers and $\dot{y}'(L,t)$ can be obtained by time differentiating the measurement of the top riser angle. The problem of the observer spillover effect is avoided as all the required states are measurable or observed directly.

Remark 6. As the boundary control design is based on the governing PDE (7) without the use of a truncated model, the problem of control spillover is also avoided.

Remark 7. The control is independent of system parameters and thus possesses stability robustness to uncertainties in the system parameters.

4. Method of numerical solution

To verify the effectiveness of the proposed approach, the developed boundary control (52) is applied to the closed-loop system (7) with boundary conditions (8)–(11). As the governing equation for the flexible system derived in this study does not have an easily obtainable solution, numerical methods are required for solving the PDE for simulation purposes. Different approximate methods such as FEM, AMM, finite difference, and Galerkin methods can be used to discretize the system for simulations. AMM is selected in this article for its ability to produce accurate, low order simulations that are easy and fast to compute numerically.

4.1. Natural vibration modes and orthogonality conditions

The natural modes of vibration can be obtained by setting external forces in Eq. (7) to zero and solving the homogenous equation

$$EI\frac{\partial^4 y}{\partial x^4} - T\frac{\partial^2 y}{\partial x^2} + m_z\frac{\partial^2 y}{\partial t^2} + c\frac{\partial y}{\partial t} = 0. \tag{63}$$

From the method of separating variables [40], and using the AMM with constrained modes, the solution $y(x,t)$ is assumed to take the form

$$y(x,t) = \sum_{i=1}^{\infty} \phi_i(x)q_i(t), \tag{64}$$

where $\phi_i(x)$ are the mode shape functions or eigenfunctions and $q_i(t)$ are the generalized coordinates. The natural frequencies of the riser can be expressed as

$$\omega_i^2 = \frac{1}{m_z}\left(\frac{i\pi}{L}\right)^2\left[EI\left(\frac{i\pi}{L}\right)^2 + T\right], \tag{65}$$

where ω_i is the natural frequency of the i-mode. Rearranging Eq. (63) into two systems of differential equation with one dependant on x and the other purely time varying, and noting that each mode shape function $\phi_i(x)$ is the solution of the boundary value problem for the differential equation dependant on x, multiplying ϕ_j and integrating from $x = 0$ to L, we have

$$EI\int_0^L \phi_i''''\phi_j\,\mathrm{d}x - T\int_0^L \phi_i''\phi_j\,\mathrm{d}x - \int_0^L m_z\omega_i^2\phi_i\phi_j\,\mathrm{d}x = 0, \tag{66}$$

As $\phi_i(x)$ and $\phi_j(x)$ should satisfy the boundary conditions with associated natural frequencies ω_i and ω_j, and integrating Eq. (66) by parts, we obtain the orthogonality condition,

$$\int_0^L \phi_i\phi_j\,\mathrm{d}x = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases} \tag{67}$$

The mode shape functions are expressed as

$$\phi_i(x) = \frac{\sqrt{2L}}{L}\left(\sin s_{2i}x - \frac{\sin s_{2i}L}{\sinh s_{1i}L}\sinh s_{1i}x\right), \tag{68}$$

where

$$s_{1i} = \left\{\frac{1}{2EI}[T + (T^2 + 4EIm_z\omega_i^2)^{1/2}]\right\}^{1/2}, \tag{69}$$

$$s_{2i} = \left\{\frac{1}{2EI}[-T + (T^2 + 4EIm_z\omega_i^2)^{1/2}]\right\}^{1/2}. \tag{70}$$

4.2. Forced vibration response

As the moment does not correspond to a generic translation, it must be handled indirectly using the method of virtual work. We can model the system as a simply supported structure with a moment at the boundary [23].

From Eq. (6), using pinned boundary conditions with a torque generated at $x = L$ leads to

$$\int_0^L m_z \ddot{y} \delta y \, \mathrm{d}x + \int_0^L (EIy'''' - Ty'')\delta y \, \mathrm{d}x + \int_0^L c\dot{y}\delta y \, \mathrm{d}x = \int_0^L f \delta y \, \mathrm{d}x - \tau \delta y'(L,t). \tag{71}$$

Substituting Eq. (64) into Eq. (71), and using Eq. (66), we obtain

$$\begin{aligned} &m_z \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \ddot{q}_i \int_0^L \phi_i\phi_j \, \mathrm{d}x + m_z\omega_i^2 \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} q_i \int_0^L \phi_i\phi_j \, \mathrm{d}x \\ &\quad + c\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \dot{q}_i \int_0^L \phi_i\phi_j \, \mathrm{d}x = \sum_{j=1}^{\infty}\int_0^L f\phi_j \, \mathrm{d}x - \sum_{j=1}^{\infty} \tau\phi_j'(L). \end{aligned} \tag{72}$$

In view of orthogonality condition (67), every term in the summation vanishes except when $i = j$. Hence Eq. (72) reduces to

$$\sum_{i=1}^{\infty}[m_z\ddot{q}_i(t) + c\dot{q}_i(t) + m_z\omega_i^2 q_i(t)] = \sum_{i=1}^{\infty}\int_0^L f(x,t)\phi_i(x)\, \mathrm{d}x - \sum_{i=1}^{\infty}\tau(t)\phi_i'(L). \tag{73}$$

From Eq. (64), we know that each $q_i(t)$ corresponds to a DOF of the system. It is also well known that the first several modes corresponds to lower frequencies are dominant in describing the system dynamics. The infinite series in Eq. (64) can be truncated into a finite one as follows:

$$y(x,t) = \sum_{i=1}^{N} \phi_i(x) q_i(t), \tag{74}$$

where N is the number of modes taken into consideration. Hence, we arrive at ordinary differential equation (ODE) of the AMM model,

$$\sum_{i=1}^{N}[m_z\ddot{q}_i(t) + c\dot{q}_i(t) + m_z\omega_i^2 q_i(t)] = \sum_{i=1}^{N}\int_0^L f(x,t)\phi_i(x)\, \mathrm{d}x - \sum_{i=1}^{N}\tau(t)\phi_i'(L). \tag{75}$$

The solution $y(x,t)$ can then be obtained by solving for the generalized coordinates, $q_i(t)$ in Eq. (75) and substituting mode shapes, $\phi_i(x)$ from Eq. (68) into Eq. (74).

5. Simulation

The closed-loop system (7) is simulated to investigate the performance of control law (52) with system parameters given in Table 1. The system is simulated using the AMM model (75) developed in the previous section where the first four modes, $N = 4$ are considered. A fourth-order Runge–Kutta–Merson program with adaptive step size [41] is used to numerically solve the ODE for the generalized coordinates.

The riser, initially at rest, is excited by a distributed transverse load. Large vibrational stresses are normally associated with a resonance that exists when the frequency of the imposed force is tuned to one of the natural frequencies [2]. Hence, the ocean surface current velocity $U(t)$ is modelled as a mean flow with worst case

Table 1
Numerical values of the system parameters

Parameters of the physical system	Value
Flexural rigidity (EI)	4.0×10^9 N/m^2
Length of riser (L)	1000 m
Mass per unit length (m_z)	15 kg/m
Outer diameter (D)	152.4×10^{-3} m
Sea water density (ρ_s)	1024 kg/m^3
Structural damping (c)	5.0 (–)
Tension (T)	1.11×10^6 N

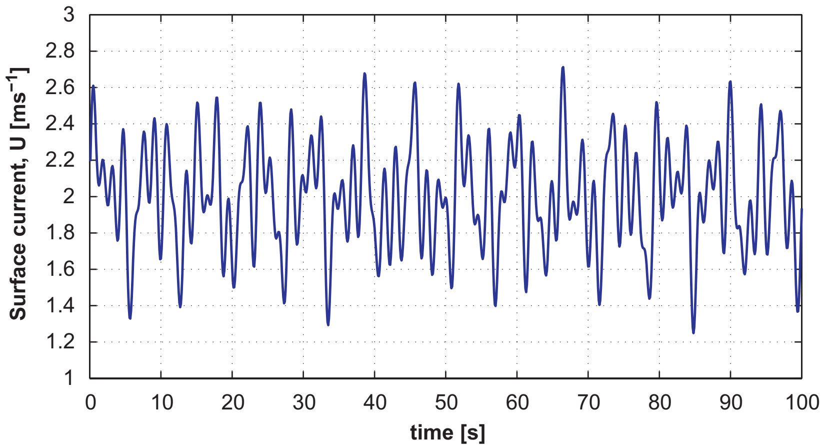

Fig. 3. Ocean current velocity modelled as a mean current with worst case sinusoids.

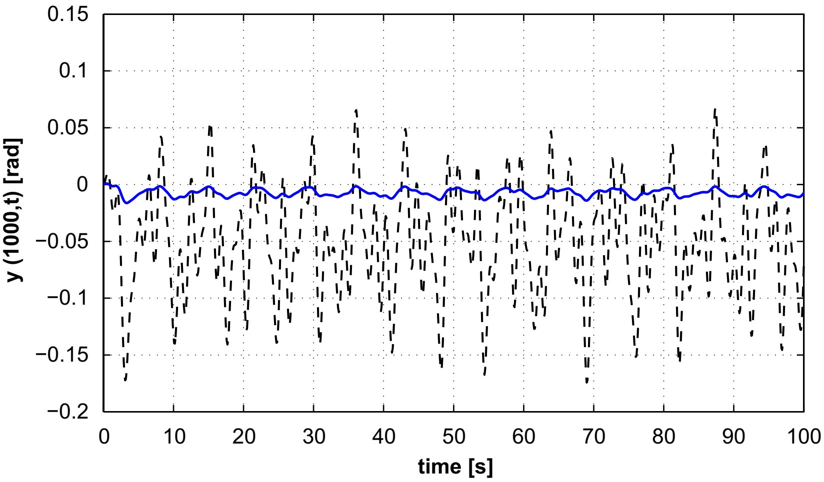

Fig. 4. Riser top angle $y'(1000, t)$ with control (solid) and without control (dashed).

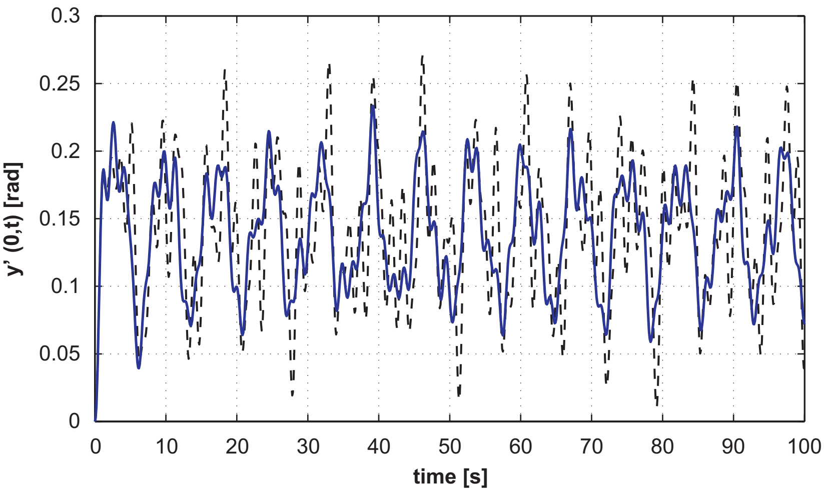

Fig. 5. Riser bottom angle $y'(0, t)$ with control (solid) and without control (dashed).

sinusoidal components to simulate the riser with a mean deflected profile. The sinusoids have frequencies of $\omega_i = \{0.867, 1.827, 2.946, 4.282\}$, for $i = 1$ to 4, corresponding to the four natural modes of vibration of the riser. The current $U(t)$ can be expressed as

$$U(t) = \bar{U} + U_m \sum_{i=1}^{N} \sin(\omega_i t), \quad i = 1, 2, \ldots, N, \tag{76}$$

where $\bar{U} = 2\,\mathrm{m\,s^{-1}}$ is the mean flow current and $U_m = 0.2$ is the amplitude of the oscillating flow. The surface current generated by Eq. (76) is shown in Fig. 3. The full current load is applied from $x = 1000$ to 700 m and thereafter linearly decline to zero at the ocean floor, $x = 0$, to obtain a depth dependent ocean current profile $U(x, t)$.

The vortex excitation $f(x, t)$ is simulated using Eq. (12) with $C_D = 1.361$ [32] and $\beta = 0$. From (14), a reasonable vale of $S_t = 0.2$ is adopted for subcritical flow [30], resulting in a vortex shedding frequency of $f_v = 2.625$. The control parameters are set as $k_1 = k_2 = 1 \times 10^9$.

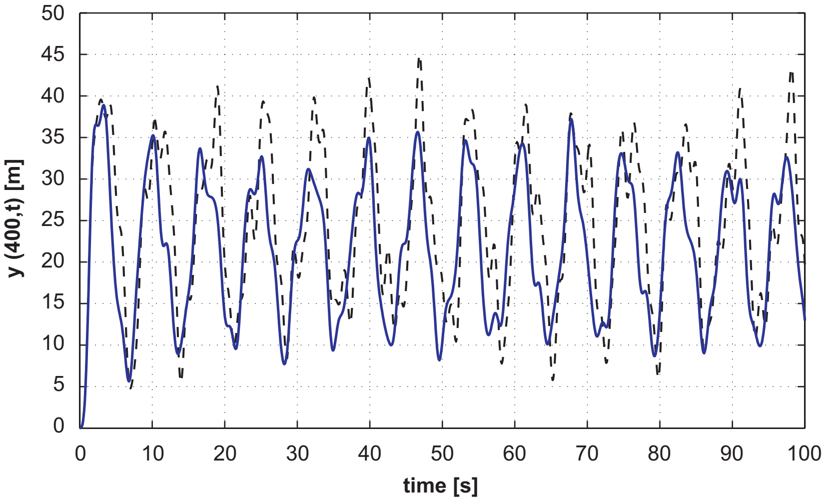

Fig. 6. Riser displacement at $x = 400$ m, with control (solid) and without control (dashed).

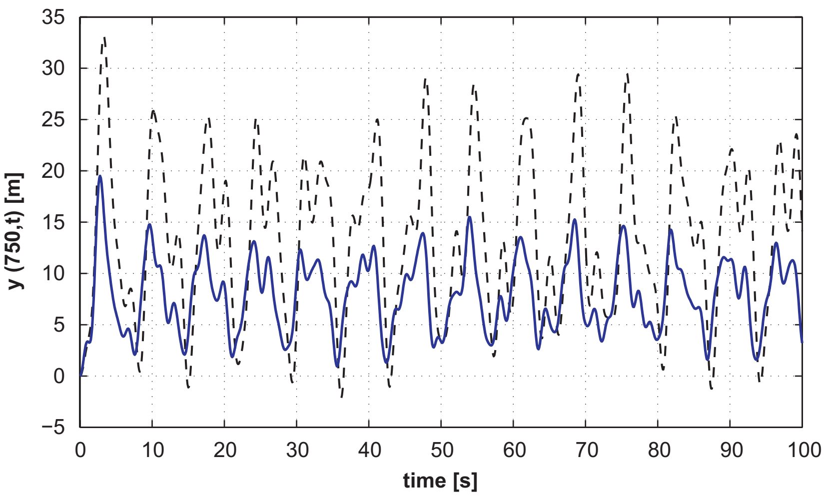

Fig. 7. Riser displacement at $x = 750$ m, with control (solid) and without control (dashed).

The controlled and uncontrolled upper and lower riser angles are shown in Figs. 4 and 5, respectively. It is observed that there are significant improvements in the top riser angle bringing the magnitude near zero when the control is applied. There are also some peak angle reduction in the bottom angle though the actuator is not located at that position.

Transverse vibration magnitude of the riser is examined at $x = 400$ and 750 m. The results for controlled and uncontrolled responses are shown in Figs. 6 and 7. It can be observed that the peak vibration magnitudes are reduced at both locations. The riser profiles for controlled and uncontrolled responses under excitation

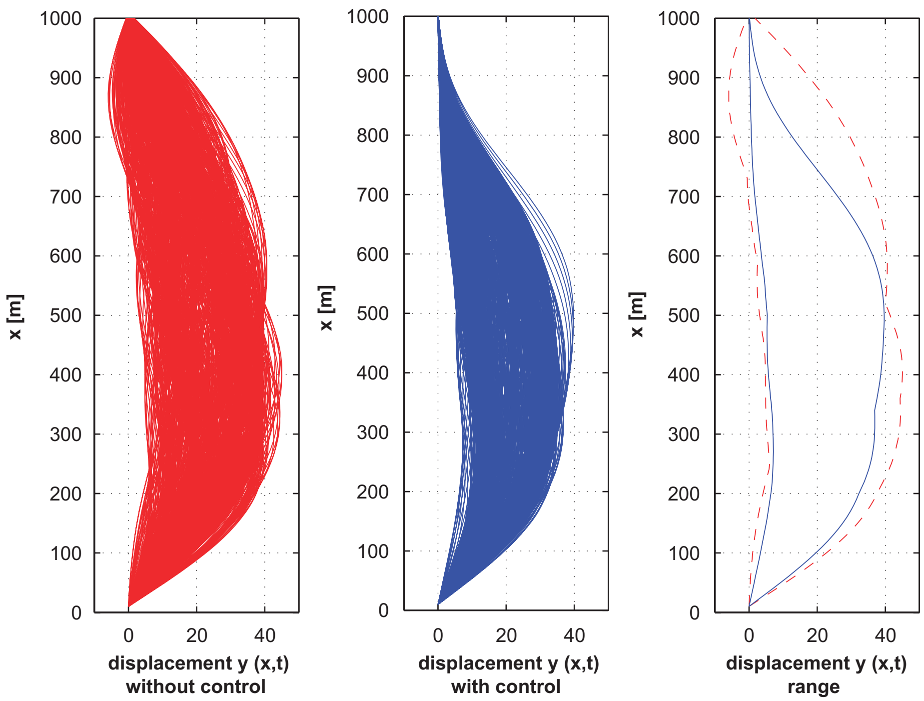

Fig. 8. Overlay of riser profiles with control, without control and displacement range.

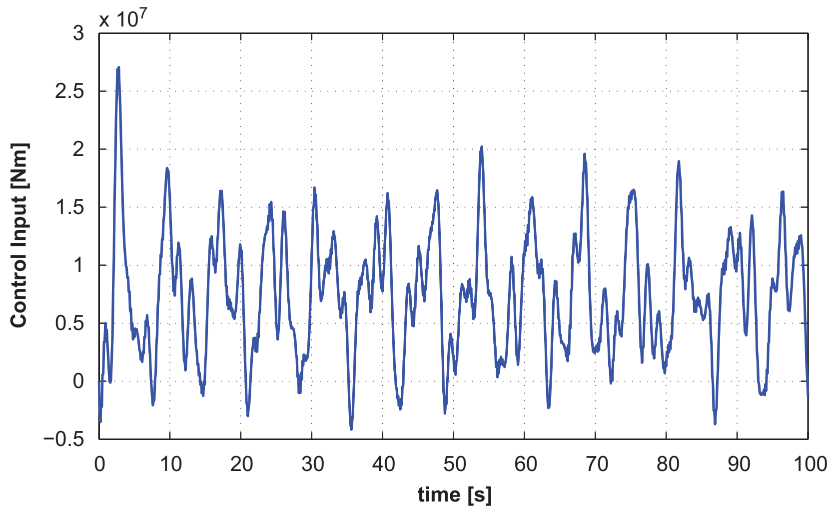

Fig. 9. Control input at the boundary.

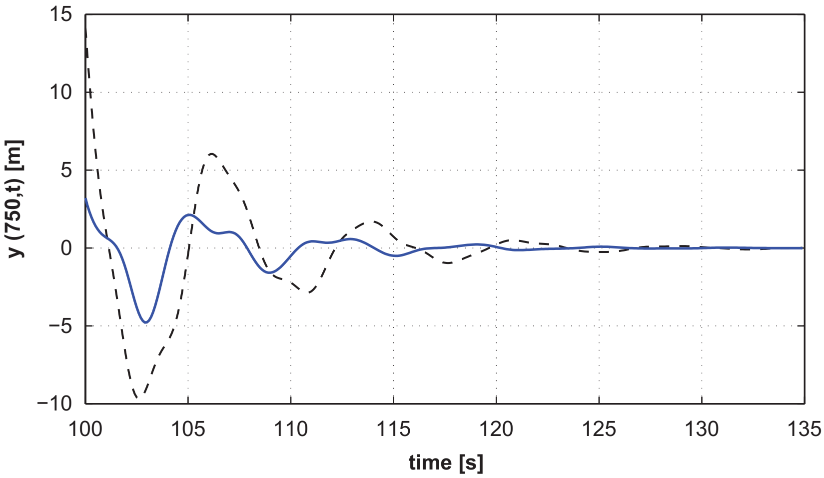

Fig. 10. Displacement at $x = 750$ without disturbance, with control (solid) and without control (dashed).

were overlaid for different time instances and the displacement range are shown in Fig. 8. The riser angle and deflection magnitudes are reduced when the control is active with control input shown in Fig. 9.

The ocean current disturbance was set to zero at $t = 100\,\text{s}$ to simulate a free vibrating case similar to that carried out in Ref. [15]. In Fig. 10, it is shown that the riser deflection at $x = 750\,\text{m}$ approaches the equilibrium exponentially with the control law activated.

6. Conclusion

In this paper, the model of a flexible marine riser with a torque actuator at the upper riser package has been derived. Boundary control has been introduced to reduce the upper riser angle magnitude and the transverse vibration of a riser subjected to a distributed load. Closed-loop stability has been proven directly from the PDE of the system and the problems of traditional truncated-model-based design have been avoided. The control is implementable as the required signals for the control law are generated using measurements which can be obtained from the upper riser boundary. When the disturbance is persistent as in the case of the marine environment, the magnitude of deflection has been shown to be reduced under the control action. The riser has also been shown to be exponentially stabilized in the absence of external disturbance. From the numerical simulations, we observe that there is significant improvement in the upper riser angle magnitude and the vibration reduction of the riser has been achieved.

Appendix

Nomenclature

Symbol	Description	Units
$y(x,t)$	Displacement in the transverse direction of the riser	m
$y'(x,t)$	Slope of the riser	m
L	Length of riser	m
T	Tension of riser	N
t	Time	s
m_z	Uniform mass per unit length of the flexible riser	kg/m
ρ_s	Density of seawater	kg/m^3
ω	Frequency	rad/s

E_k	Kinetic energy	J
E_p	Potential energy	J
W	Work done	J
EI	Uniform flexural rigidity of beam	$N\,m^2$
$f(x,t)$	Time dependent distributed load	N
$U(t)$	Velocity of ocean current	m/s^2
$U(x,t)$	Current profile	m/s^2
D	Riser external diameter	m
C_D	Drag coefficient	Dimensionless
C_M	Inertia coefficient	Dimensionless
$(\dot{*})$	Derivative with respect to time	Dimensionless
$(*')$	Derivative with respect to x	Dimensionless
δ	Variation operator	Dimensionless
c	Linear structural damping coefficient	Dimensionless

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