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Vibration analysis of multiple-cracked non-uniform beams

K. Mazanoglu[a], I. Yesilyurt[b], M. Sabuncu[a,*]

[a]Department of Mechanical Engineering, Dokuz Eylul University, Bornova, İzmir 35100, Turkey
[b]Department of Mechanical Engineering, Usak University, 64300 Uşak, Turkey

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Abstract

This paper presents the energy-based method for the vibration identification of non-uniform Euler–Bernoulli beams having multiple open cracks. The method includes significant modifications for the energy-based method presented by Yang et al. [Crack identification in vibrating beams using the energy method, *Journal of Sound and Vibration* 244 (2) (2001) 339–357.] The distribution of the energy consumed is determined by taking into account not only the strain change at the cracked beam surface as in general but also the considerable effect of the stress field caused by the angular displacement of the beam due to bending. The Rayleigh–Ritz approximation method is used in the analysis. The method is adapted to the cases of multiple cracks with an approach based on the definition of strain disturbance variation along the beam. Examples are presented on cantilever beams having different truncation factors. When the results are compared with a commercial finite element program and with the results of Zheng and Fan [Natural frequencies of a non-uniform beam with multiple cracks via modified Fourier series, *Journal of Sound and Vibration* 242 (4) (2001) 701–717], good agreements are obtained. The effects of truncation factors and positions of cracks on the natural frequency ratios are presented in graphics.

1. Introduction

Flaws in the components of a structure can influence the dynamic behaviour of the whole structure. It is well known from the literature that one form of damage that can lead to catastrophic failure if undetected is fatigue cracking of the structure elements. The recognition of the vibration effects of cracks is important in practise since vibration monitoring has revealed a great potential for investigation of cracks in the last three decades. Detailed review on the vibration of cracked structure was given by Dimarogonas [1].

In the literature, cracks were assumed always open or breathing in time. The nonlinear effect of a breathing crack on the flexural vibration of cracked structures was discussed in some papers [2–4]. They concluded that the difference of solutions between the open and breathing crack models was quite small when the amplitude was not so large, and the difference became large as the amplitude increased. Thus, most researchers assume the crack remains open in their models to simplify the problem by lifting the nonlinear influences. The effect of crack on dynamic behaviour of the beam was simulated by the definition of local flexibility in several models

*Corresponding author. Tel.: +90232 3887868; fax: +90232 3887868.
E-mail address: mustafa.sabuncu@deu.edu.tr (M. Sabuncu).

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Nomenclature

a	crack depth
A	cross-section area
b	width of a beam
b_1	width of a beam at the root
b_2	width of a beam at the tip
BE	energy balance equation
CE	the energy consumed
E	modulus of elasticity
G	strain energy release rate
h	height of a beam
h_1	height of a beam at the root
h_2	height of a beam at the tip
I	second moment of inertia
k	stiffness of which type is specified by the superscripts
K_1	stress intensity factor for the first mode crack
KE	maximum kinetic energy
L	length of the beam
m	total number of terms of polynomial mode shape function
M	bending moment
n	total number of cracks
PE	maximum potential energy
r	ratio between crack depth and height of a beam at the crack location
w	coordinate axis along the beams height
W	the transverse vibration mode shape of the beam
y	coordinate axis along the beam's width
z	coordinate axis along the beam's length
α	truncation factor of beam's height or width
Γ	distribution of the energy
Δu	linear displacement at the crack-edge
ΔU	change in strain energy
Δv	inverse linear displacement at the crack tip with the effect of material stress
ΔV	change in stress energy
$\Delta\theta$	angular displacement at the crack tip
$\Delta\phi$	angular displacement of the beam due to the bending at the crack location
κ	coefficient of the term of polynomial mode shape function.
v	Poisson ratio
ρ	mass density
φ	term of polynomial mode shape function
ω	circular frequency
ω_0	natural frequency of the uncracked beam
c	subscript abbreviation for the word "crack" to relate the parameters with the cracks
δ	difference between the numerators of crack and part
i	crack and part numerator
j	numerator of the mode shape terms
p	abbreviation for the word "part"

that use massless rotational spring or reduced cross-section. Magnitudes of the flexibility changes were estimated by the fracture mechanics methods [5,6] or by experimental works. Changes in the dynamic characteristics defined by the local flexibilities were obtained by several ways.

In many works including crack models with rotational springs, the solution of equation or equation set was generally obtained by means of compatibility and continuity conditions at the crack locations. Either numerical [7–9] or analytical [10,11] approaches could be straightforwardly used in continuous models for uniform beams. However, analytical solution is very difficult for non-uniform beams because of the nonlinear equations resulting from the geometric nonlinearities. These approaches also suffer from the lack of the fact that the modification of the stress field induced by the crack decays with the distance from the crack. Chondros discussed the effects of decaying stress fields on crack models evaluated with rotational springs [12].

The methods, including exponentially decaying stress/strain functions based on a variational principle, were proposed to develop vibration equations for continuous models [13–16]. Christides and Barr [13] firstly presented an exponential-type crack disturbance function to model the stress/strain variation around the crack zone for one or more pairs of symmetric cracks. Shen and Pierre [14] proposed a similar approach for single-cracked beams by using many termed Galerkin's method. Chondros et al. [15] developed another crack disturbance function for the vibration of simply supported beams having one or two cracks. Another approach based on the stiffness definition of cracked beams using strain energy variation around the crack was proposed by Yang et al. [16], for single- and double-cracked beams. The case where two or more cracks lie in close proximity to each other was not analysed in this study. These approaches suffer from the overlap of exponential functions when the multiple cracks interact with each other.

Many of the other energy-based approaches were used in the finite element models. Finite element models may be preferable since they can be applied to any structural member. However, there are so many parameters that can be varied in the flexural vibration of structural members with cracks that it would be very difficult to present and compare results for all cases. Parameters may vary mainly with modelling of the crack and meshing properties. Indiscriminate application of the frequencies calculated using the finite element methods, without consideration of the assumptions under which the crack models were derived, might lead to gross errors. On the other hand, careful observation of the behaviour of these damage models can lead to the extension of their utility in practical engineering. Behaviour of the damages could be observed by the special element or connection models [17–22]. If the FEM includes no special models for the cracks, the method should be supported by extremely refined meshes near the cracks for an accurate solution even though the computation time will increase.

A few studies were presented on numerical and analytical solution of non-uniform beams. An approach was presented by Li [23,24] for determining the natural frequencies and mode shapes of cracked stepped beams having varying cross-sections and cracked non-uniform beams having concentrated masses, respectively. However, only some specific forms of non-uniformities could be solved in these papers. Coudhari and Maiti [25,26] proposed a method for defining transverse vibrations of tapered beams and geometrically segmented slender beams with a single crack using the Frobenius technique. Even though the beam had a single crack, their results were quite coarse. Energy-based numerical approaches were also presented for non-uniform beams. Zheng and Fan [27] determined the approximate natural frequencies of multiple-cracked non-uniform beams using the modified Fourier series. El Bikri et al. [28] presented a semi-analytical model based on an extension of the Rayleigh–Ritz method to nonlinear vibrations, which is mainly influenced by the choice of the admissible functions. This study was also restricted with a single crack and fundamental frequency.

This paper presents the vibration analysis of multiple-cracked non-uniform beams using the distributions of the energies consumed caused by the open cracks. The energy consumed is obtained by the change of the strain energy distribution given by Yang et al. [16] for the cracked surface of the beam, together with the effect of stress field due to the angular displacement of the beam. The energy consumed is also determined by arranging the variation of the strain disturbances for defining the vibration of the multiple-cracked non-uniform beams. Results obtained by the present method are compared with the results of Zheng and Fan [27] and a commercial finite element program (ANSYS©) for several non-uniform cantilever beams.

2. Vibration of the beams with a crack

According to the fracture mechanics theory, structural strain energy increases with crack growth. Increase in strain energy, which is assumed equal to the energy consumed, under the constant external bending moment is defined as follows [5,6]:

$$\Delta U = \mathrm{CE} = \int_0^a G b_c \, \mathrm{d}a. \tag{1}$$

G is called the strain energy release rate, which can be written as $G = K_1^2/E'$ for the transverse vibration of the beam by taking only the effects of bending stresses into account and neglecting the effects of shear stresses on the crack. E' is equal to E for plain stress, or $E/(1-v^2)$ for plain strain. The stress intensity factor for the first mode crack (K_1) is given as

$$K_1 = \frac{6M(z)\sqrt{\pi a}}{b_c h_c^2} F(r), \tag{2}$$

$M(z)$ is the bending moment that can be defined as

$$M(z) = E' I(z) \frac{\mathrm{d}^2 W(z)}{\mathrm{d}z^2}, \tag{3}$$

$F(r)$ is the function that is valid for $r = a/h_c < 0.6$, and defined as follows:

$$F(r) = 1.12 - 1.4r + 7.33r^2 - 13.8r^3 + 14r^4. \tag{4}$$

Finally, the energy consumed can be written using Eq. (1) [16] as

$$\mathrm{CE} = D(r)[M(z)]^2, \tag{5}$$

where

$$D(r) = \frac{18\pi F(r)^2 a^2}{E b_c h_c^4}. \tag{6}$$

The expressions given above for the energy consumed [16] are valid only when the increase in strain energy through the cracked side of the beam is taken into account. Increase in strain energy through the cracked beam surface can correspond to the energy of linear springs located along the crack-edge, which can be transformed into the energy of rotational springs placed along the crack tip:

$$\Delta U = \frac{1}{2b_c}\int_{\tilde{y}=0}^{b_c} k_c^{(u)}(\Delta u_c)^2\,\mathrm{d}\tilde{y}, \quad \Delta U = \frac{1}{2b_c}\int_{\tilde{y}=0}^{b_c} k_c^{(\theta)}(\Delta\theta_c)^2\,\mathrm{d}\tilde{y}. \tag{7,8}$$

However, when slope change at the crack location of the beam is considered, angular displacement of the crack ($\Delta\theta_c$) also results with the angular displacement of the beam ($\Delta\phi_c$) at the crack location as shown in Fig. 1. Angular displacement of the beam causes the additional stress field in the vicinity of crack tip. Similar to the additional strain energy definitions, stress energy change (ΔV) can also be defined by using linear or rotational spring models $k_c^{(v)}, k_c^{(\phi)}$ seen in Fig. 1. As the strain caused by the crack decreases the potential energy, additional stress field increases it. Thus, angular displacement of the beam due to the bending decreases the energy consumed. Here, it should be noted that negative compressive strain field required to be considered under the neutral layer in the vicinity of crack is assumed to be approximately equal to strain at the crack tip. These minor effects neutralise each other and thus can be neglected in the model. Consequently, the energy consumed can be written as follows:

$$\mathrm{CE} = \frac{1}{2b_c}\int_{\tilde{y}=0}^{b_c} (k_c^{(\theta)}(\Delta\theta_c)^2 - k_c^{(\phi)}(\Delta\phi_c)^2)\,\mathrm{d}\tilde{y}, \tag{9}$$

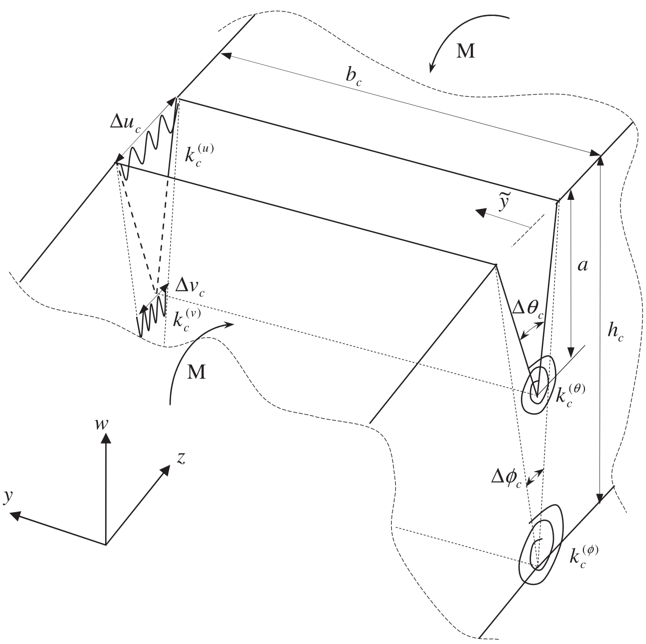

Fig. 1. Angular displacement of the beam caused by a crack.

where $\Delta\phi_c = (a/h_c)\Delta\theta_c$. The stiffness relation can also be established by providing bending moment equivalence as

$$\frac{1}{b_c}\int_{\tilde{y}=0}^{b_c}(k_c^{(\theta)}\Delta\theta_c - k_c^{(\phi)}\Delta\phi_c)\,\mathrm{d}\tilde{y} = 0, \tag{10}$$

which results in $k_c^{(\phi)} = (h_c/a)k_c^{(\theta)}$.

Thus, Eq. (6) can be redefined for $r \leqslant 0.5$, to include the effects of stress field caused by the angular displacement of the beam:

$$D(r) = \frac{18\pi F(r)^2 a^2}{E b_c h_c^4}(1 - r). \tag{11}$$

The energy consumed is distributed along the beam as follows [16]:

$$\Gamma^{\mathrm{CE}} = \frac{Q(r, z_c)}{1 + [(z - z_c)/(q(r)a)]^2}, \tag{12}$$

where $Q(r, z_c)$ and $q(r)$ are the terms that can be defined as follows [16]:

$$Q(r, z_c) = \frac{D(r)[M(z)]^2}{q(r)a\{\arctan[(L - z_c)/(q(r)a)] + \arctan[z_c/(q(r)a)]\}}, \tag{13}$$

$$q(r) = \frac{3\pi[F(r)]^2(h_c - a)^3 a}{(h_c^3 - (h_c - a)^3)h_c}. \tag{14}$$

According to the principle of conservation of energy, maximum potential and kinetic energies should be equal along the beam when there is no crack. If a crack exists on a beam, the energy consumed results with the decrease in maximum potential energy with the assumption of no mass loss at the crack location. As a consequence, the balance of maximum energies can be obtained as follows:

$$\int_{z=0}^{L}((\Gamma^{\mathrm{PE}} - \Gamma^{\mathrm{CE}}) - \Gamma^{\mathrm{KE}})\,\mathrm{d}z = 0, \tag{15}$$

where Γ^{PE} and Γ^{KE} represent the distributions of the maximum potential and kinetic energies as

$$\Gamma^{\mathrm{PE}} = \frac{1}{2}EI(z)\left(\frac{\mathrm{d}^2 W(z)}{\mathrm{d}z^2}\right)^2, \quad \Gamma^{\mathrm{KE}} = \frac{1}{2}\rho A(z)\omega^2(W(z))^2. \tag{16,17}$$

Eq. (15) can be approximated to zero using the Rayleigh–Ritz method. In this method, approximation to zero is provided by differentiating Eq. (15) with the coefficients of the admissible mode shape function as

$$\partial\left(\int_{z=0}^{L}((\Gamma^{\mathrm{PE}} - \Gamma^{\mathrm{CE}}) - \Gamma^{\mathrm{KE}})\mathrm{d}z\right)\Big/\partial\kappa_j = 0. \tag{18}$$

If $\varphi_j(z)$ are a series of functions satisfying end conditions, the mode shape function can be written as $W(z) = \sum_{j=1}^{m}\kappa_j\varphi_j(z)$.

3. Energy balance in a multiple-cracked beam

In the case of multiple cracks, parameters in Eqs. from (1) to (12) can be modified as r_i, a_i, $z_{c(i)}$, $h_{c(i)}$, $b_{c(i)}$ where $i = 1$ to n. The effect of interference of cracks on the distribution of the energy consumed for a multiple-cracked beam is considered throughout the beam length. Typical distributions are shown in Fig. 2 for the case of three cracks as an example. It can be noticed that the distributions cannot be directly superposed, because the overlap of the distributions is considerably influential on the result, especially when the cracks approach each other. Therefore, the contribution of each crack to the maximum potential energy can be arranged according to the change of strain disturbance at other crack locations. In this respect, although the energy consumed caused by crack 1 results in the decrease of maximum potential energy in parts 1 and 2, strain

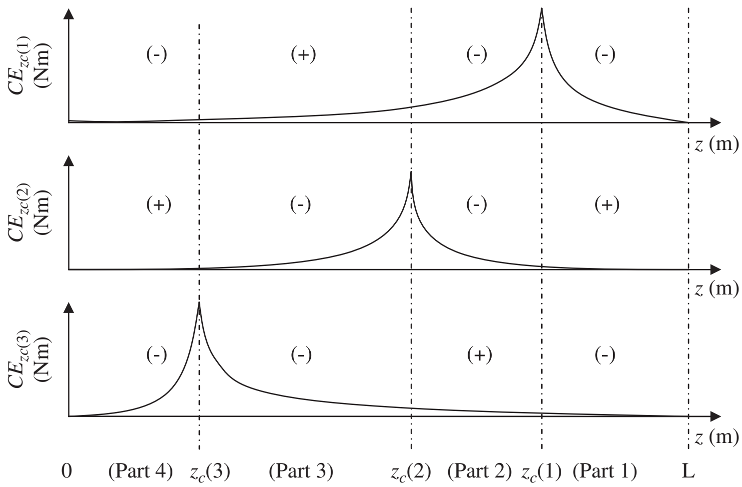

Fig. 2. Example distributions of the energies consumed caused by three cracks and contributions of these distributions to the maximum potential energy.

disturbance as a result of crack 1 changes the phase at $z_{c(2)}$, and the energy consumed caused by crack 1 results in the increase of maximum potential energy in part 3. Strain disturbance changes the phase again at $z_{c(3)}$, and crack 1 negatively affects the maximum potential energy in part 4. Similarly, contributions of other cracks on maximum potential energies are seen in Fig. 2.

As a consequence, if n cracks exist on the beam surface, the following equations can be written for $n+1$ parts:

$$
\begin{aligned}
\mathrm{BE}_1 &= \int_{z_{c(1)}}^{L} ((\Gamma^{\mathrm{PE}} - \Gamma^{\mathrm{CE}}_{c(1)} + \Gamma^{\mathrm{CE}}_{c(2)} - \cdots \pm \Gamma^{\mathrm{CE}}_{c(n)}) - \Gamma^{\mathrm{KE}})\,\mathrm{d}z,\\
\mathrm{BE}_2 &= \int_{z_{c(2)}}^{z_{c(1)}} ((\Gamma^{\mathrm{PE}} - \Gamma^{\mathrm{CE}}_{c(1)} - \Gamma^{\mathrm{CE}}_{c(2)} + \Gamma^{\mathrm{CE}}_{c(3)} - \cdots \mp \Gamma^{\mathrm{CE}}_{c(n)}) - \Gamma^{\mathrm{KE}})\,\mathrm{d}z,\\
\mathrm{BE}_3 &= \int_{z_{c(3)}}^{z_{c(2)}} ((\Gamma^{\mathrm{PE}} + \Gamma^{\mathrm{CE}}_{c(1)} - \Gamma^{\mathrm{CE}}_{c(2)} - \Gamma^{\mathrm{CE}}_{c(3)} + \Gamma^{\mathrm{CE}}_{c(4)} - \cdots \pm \Gamma^{\mathrm{CE}}_{c(n)}) - \Gamma^{\mathrm{KE}})\,\mathrm{d}z,\\
&\vdots\\
\mathrm{BE}_{n-1} &= \int_{z_{c(n-1)}}^{z_{c(n-2)}} ((\Gamma^{\mathrm{PE}} \pm \Gamma^{\mathrm{CE}}_{c(1)} \mp \cdots + \Gamma^{\mathrm{CE}}_{c(n-3)} - \Gamma^{\mathrm{CE}}_{c(n-2)} - \Gamma^{\mathrm{CE}}_{c(n-1)} + \Gamma^{\mathrm{CE}}_{c(n)}) - \Gamma^{\mathrm{KE}})\,\mathrm{d}z,\\
\mathrm{BE}_{n} &= \int_{z_{c(n)}}^{z_{c(n-1)}} ((\Gamma^{\mathrm{PE}} \mp \Gamma^{\mathrm{CE}}_{c(1)} \pm \cdots + \Gamma^{\mathrm{CE}}_{c(n-2)} - \Gamma^{\mathrm{CE}}_{c(n-1)} - \Gamma^{\mathrm{CE}}_{c(n)}) - \Gamma^{\mathrm{KE}})\,\mathrm{d}z,\\
\mathrm{BE}_{n+1} &= \int_{0}^{z_{c(n)}} ((\Gamma^{\mathrm{PE}} \pm \Gamma^{\mathrm{CE}}_{c(1)} \mp \cdots + \Gamma^{\mathrm{CE}}_{c(n-1)} - \Gamma^{\mathrm{CE}}_{c(n)}) - \Gamma^{\mathrm{KE}})\,\mathrm{d}z.
\end{aligned} \tag{19}
$$

Thus, the energy balance can be obtained by satisfying the following equation:

$$\sum_{i=1}^{n+1} \mathrm{BE}_i = 0. \tag{20}$$

Eq. (20) can also be approximated to zero using the Rayleigh–Ritz method.

4. Results and discussion

Results are represented by applying the method on several non-uniform cantilever beams that are dimensioned as seen in Fig. 3. Relations between heights and length, or widths and length for tapered beams

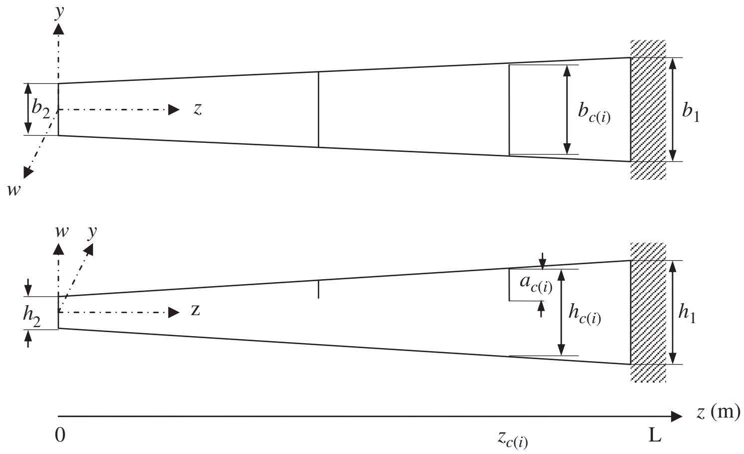

Fig. 3. Geometry of a beam.

can be defined as follows:

$$h(z) = h_2 + (h_1 - h_2)z/L,$$
$$b(z) = b_2 + (b_1 - b_2)z/L, \tag{21, 22}$$

Four different cantilever beams have the same density, $\rho = 7800\,\mathrm{kg/m^3}$, and modulus of elasticity, $E = 210\,\mathrm{GPa}$. The beams have also the following geometric and material properties:

$$\begin{aligned}
&\text{Beam1}: \quad L = 0.6\,\mathrm{m}, \quad h_1 = b_1 = 0.02\,\mathrm{m}, \quad \alpha_h = h_2/h_1 = 0.25, \quad \alpha_b = b_2/b_1 = 1, \quad v = 0.3,\\
&\text{Beam2}: \quad L = 0.6\,\mathrm{m}, \quad h_1 = b_1 = 0.02\,\mathrm{m}, \quad \alpha_h = 2, \quad \alpha_b = 1, \quad v = 0.3,\\
&\text{Beam3}: \quad L = 0.6\,\mathrm{m}, \quad h_1 = b_1 = 0.02\,\mathrm{m}, \quad \alpha_h = 0.25, \quad \alpha_b = 0.5, \quad v = 0.3,\\
&\text{Beam4}: \quad L = 0.8\,\mathrm{m}, \quad h_1 = b_1 = 0.02\,\mathrm{m}, \quad \alpha_h = 0.5, \quad \alpha_b = 1, \quad v = 0.
\end{aligned}$$

The mode shape function of the beams can be assumed as

$$W(z) = \sum_{j=1}^{m} \kappa_j (z/L)^{j-1}(1 - z/L)^2. \tag{23}$$

Results of the method are compared with the results of the commercial finite element program (ANSYS$^{©}$) for Beam1, Beam2, and Beam3. Cracks are considered as the slots causing discontinuities on the beams. They are formed by subtracting thin transverse blocks from "solid95" beams in the program. Element size is set to 0.009 m with the "esize" command, and crack widths are chosen as 0.0004 m. Much smaller-sized elements are unavoidable in the vicinity of cracks to observe the effects of discontinuities. Smaller sizes are automatically provided by the use of the "smrtsize,1" command in the free meshing procedures. Resultantly, modal frequencies are obtained by using "modal analysis" as the analysis type. It should be noted that changes in the element number caused by variation of crack location and crack size have negligible effects on the results. Natural frequencies of the uncracked beams obtained by the Rayleigh–Ritz approximations and the finite element program can be seen in Table 1.

Vibrations of the beams defined above are inspected in the cases of single, double, and multiple cracks as represented in the following examples.

Example 1. Tapered beams with a crack

Beam1, Beam2, and Beam3 are examined by following crack properties:

$$a_c = 0.15h_1,\ 0.3h_1,\ z_c(\text{variable}).$$

Results of the method are in good agreement with the results of the finite element program for single-crack cases of different beams as shown in Figs. 4–6. The method is valid for the crack depth ratio $r \leqslant 0.5$ as defined

Table 1
Natural frequencies of the uncracked beams (ω_o)

Beams	Vibration modes	Frequencies (Hz) obtained by Rayleigh–Ritz (4 terms)	Frequencies (Hz) obtained by Rayleigh–Ritz (5 terms)	Frequencies (Hz) obtained by Rayleigh–Ritz (6 terms)	Frequencies (Hz) obtained by finite element program
Beam1	1	55.3163	55.3157	55.3153	55.350
	2	215.4652	214.4026	214.4007	214.183
	3	–	520.6647	514.4896	511.814
Beam2	1	43.4178	43.3889	43.3870	43.4305
	2	374.8819	373.7616	373.7518	369.933
	3	–	1146.4491	1146.4276	1114.86
Beam3	1	66.0191	66.0146	66.0144	66.036
	2	230.7768	228.8314	228.8170	228.550
	3	–	540.6372	529.9834	527.355
Beam4	1	28.4894	28.4866	28.4863	–
	2	136.7583	136.6345	136.4713	–
	3	–	355.5734	354.0987	–

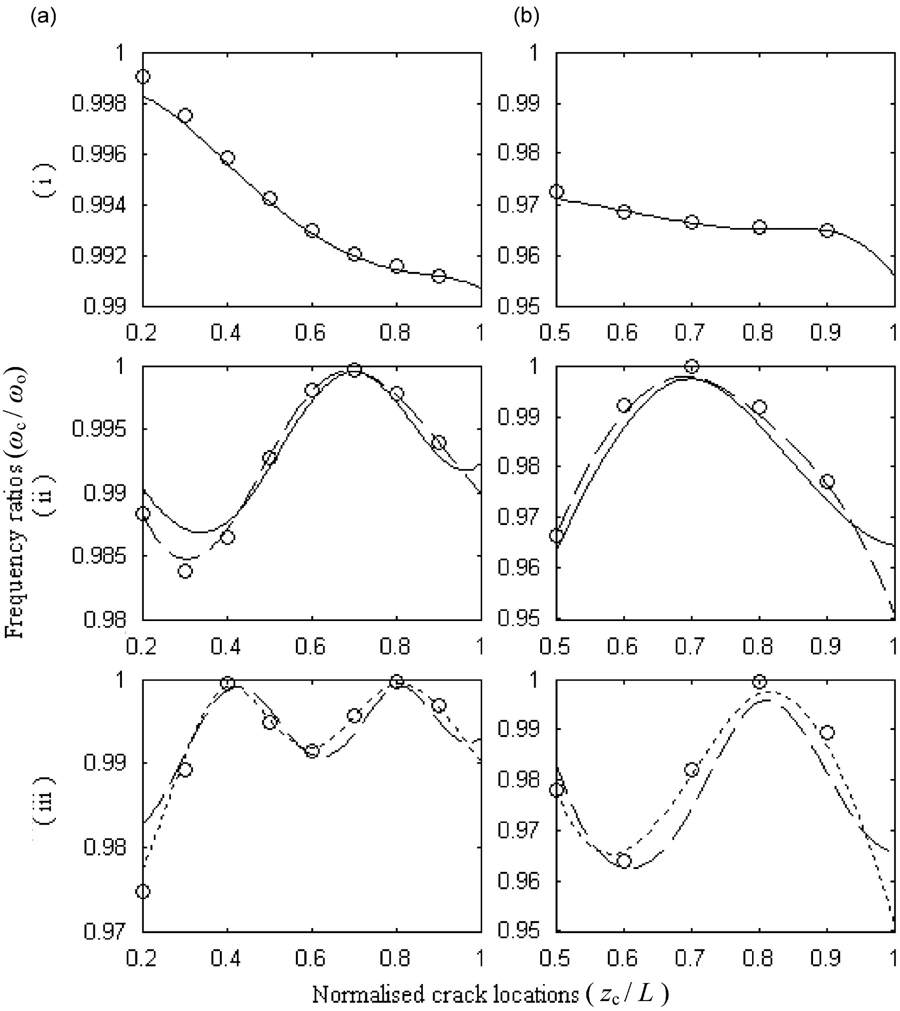

Fig. 4. Natural frequency ratios for the (i) first, (ii) second, and (iii) third mode vibration of Beam1 with variably located crack having depths (a) $a = 0.15h_1$, and (b) $a = 0.3h_1$. (○) Ansys results, (——) approximation with 4 terms, (– –) approximation with 5 terms, (- - -) approximation with 6 terms.

before. It is for this reason that normalised crack locations of Beam1 and Beam3 are considered between 0.2 and 1 for $a = 0.15h_1$, and between 0.5 and 1 for $a = 0.3h_1$. Application of the Rayleigh–Ritz approximation with 4, 5, and 6 terms is sufficient to obtain the best agreement with the first, second, and third mode of vibrations, respectively. It is clear that higher vibration modes require the use of a larger number of terms.

If the trends of the natural frequency ratios are comparatively examined for the cracks on Beam1 and Beam2, some distinctions can be obtained. Natural frequency reductions of Beam1 is lower than that of Beam2 when non-dimensional crack locations are lower than 0.8. Besides, node points, where no natural frequency reduction is obtained, are shifted from the root to the tip with the decrease in truncation factor. On the other hand, relatively minor influences of second taper on natural frequency ratios can be observed when Fig. 4 is compared with Fig. 6. Variation of the mass and inertia moment together with the variation of crack depth ratio along the beam are all influential on the observation of the natural frequency ratios seen in the figures.

Example 2. Tapered beam with two cracks

Beam3 is examined by the following crack properties:

$$a_1 = 0.3h_1,\ a_2 = 0.15h_1,\ 0.3h_1,\ z_{c(1)} = 0.91L,\ z_{c(2)}(\text{variable}).$$

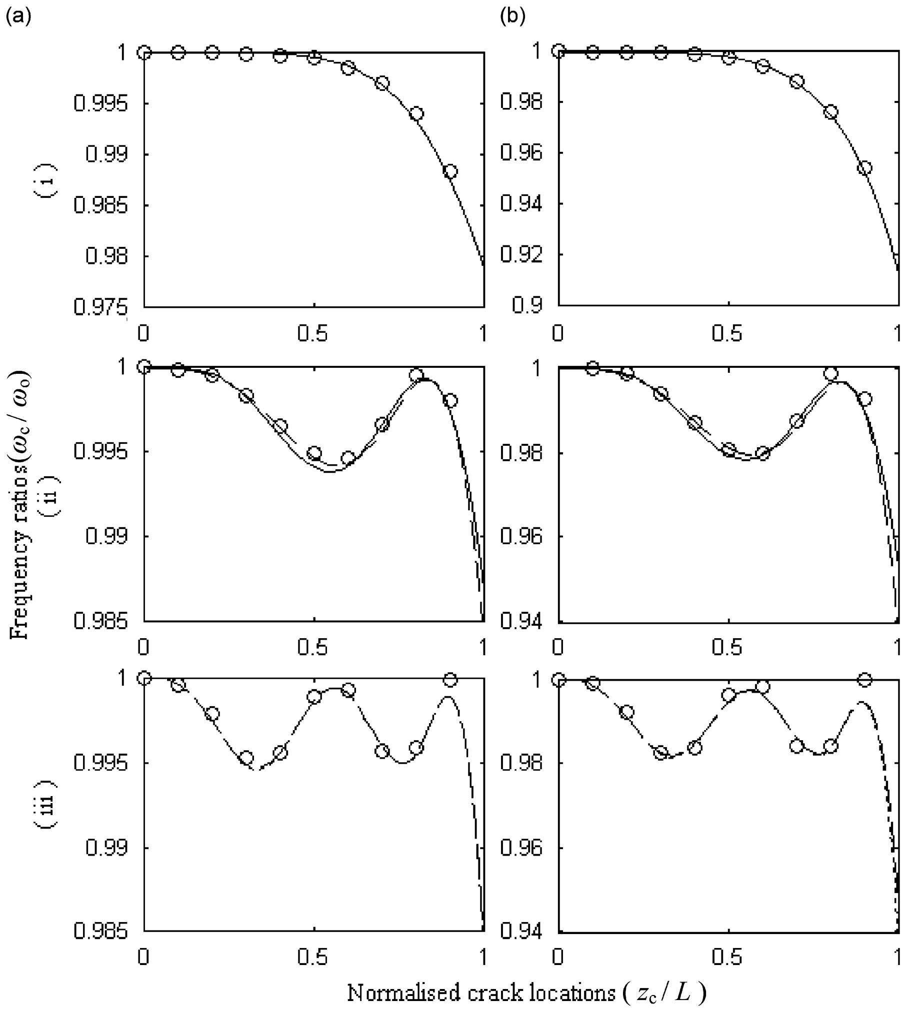

Fig. 5. Natural frequency ratios for the (i) first, (ii) second, and (iii) third mode vibration of Beam2 with variably located crack having depths (a) $a = 0.15h_1$, and (b) $a = 0.3h_1$. (○) Ansys results, (——) approximation with 4 terms, (– –) approximation with 5 terms, (- - -) approximation with 6 terms.

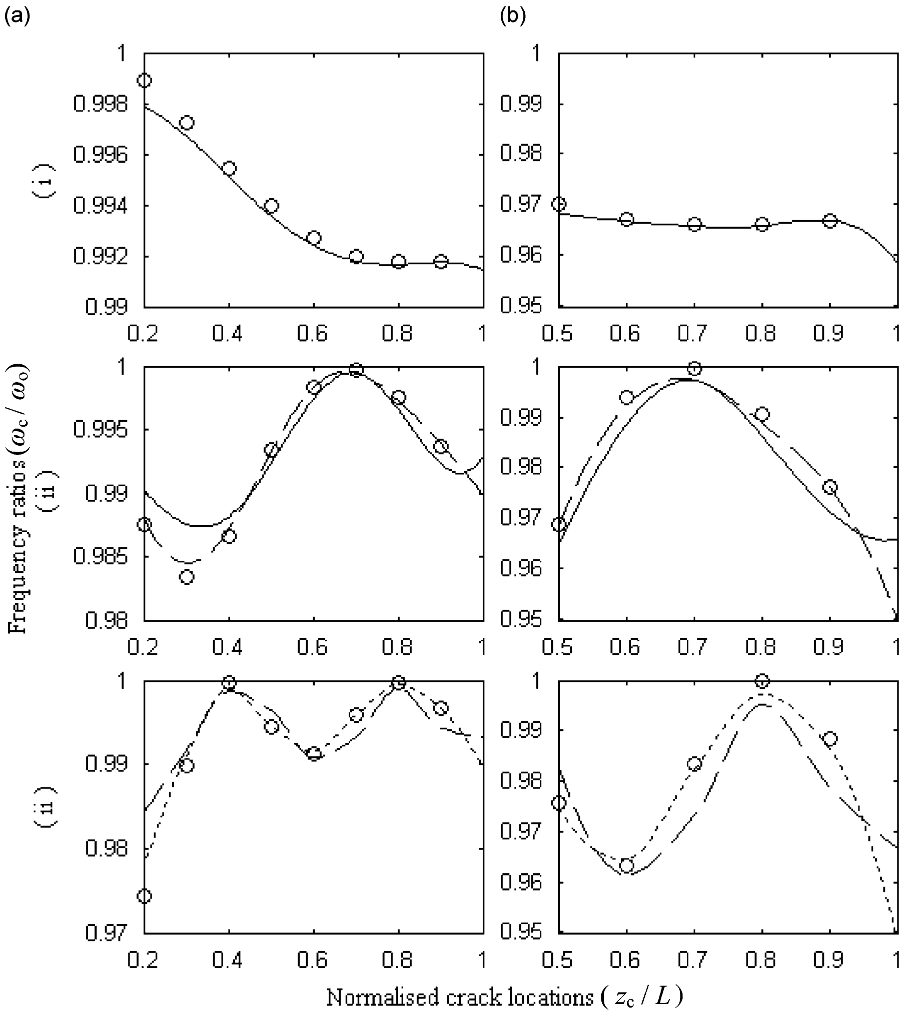

Fig. 6. Natural frequency ratios for the (i) first, (ii) second, and (iii) third mode vibration of Beam3 with variably located crack having depths (a) $a = 0.15h_1$, and (b) $a = 0.3h_1$. (○) Ansys results, (——) approximation with 4 terms, (– –) approximation with 5 terms, (- - -) approximation with 6 terms.

Natural frequency ratios obtained by the method are also quite agreeable with those obtained by the finite element program for the double-cracked Beam3 as shown in Fig. 7. It can be observed that as crack 2 comes closer to crack 1, the natural frequency ratios of the double-cracked beams have a tendency of approaching the natural frequency ratio of beams having a single crack at $z_{c(1)}$.

Example 3. Tapered beam with four cracks

Beam4 has the following crack properties:

$$a_1 = 0.3h_1,\ a_2 = 0.2h_1,\ a_3 = 0.1h_1,\ a_4 = 0.1h_1,\ 0.2h_1,\ 0.3h_1(\text{variable}),$$
$$z_{c(1)} = 0.95L,\ z_{c(2)} = 0.9L,\ z_{c(3)} = 0.85L,\ z_{c(4)}(\text{variable}).$$

Natural frequency ratios of Beam4 having four cracks are seen in Figs. 8 and 9 for the first and second mode of vibration, respectively. Good agreement with the results of Zheng and Fan [27] is obtained by using four-term approximation for the first mode of vibration as seen in Fig. 8. However, small differences between the results of two methods for a beam with four cracks can be seen in Fig. 9, which depicts the second mode

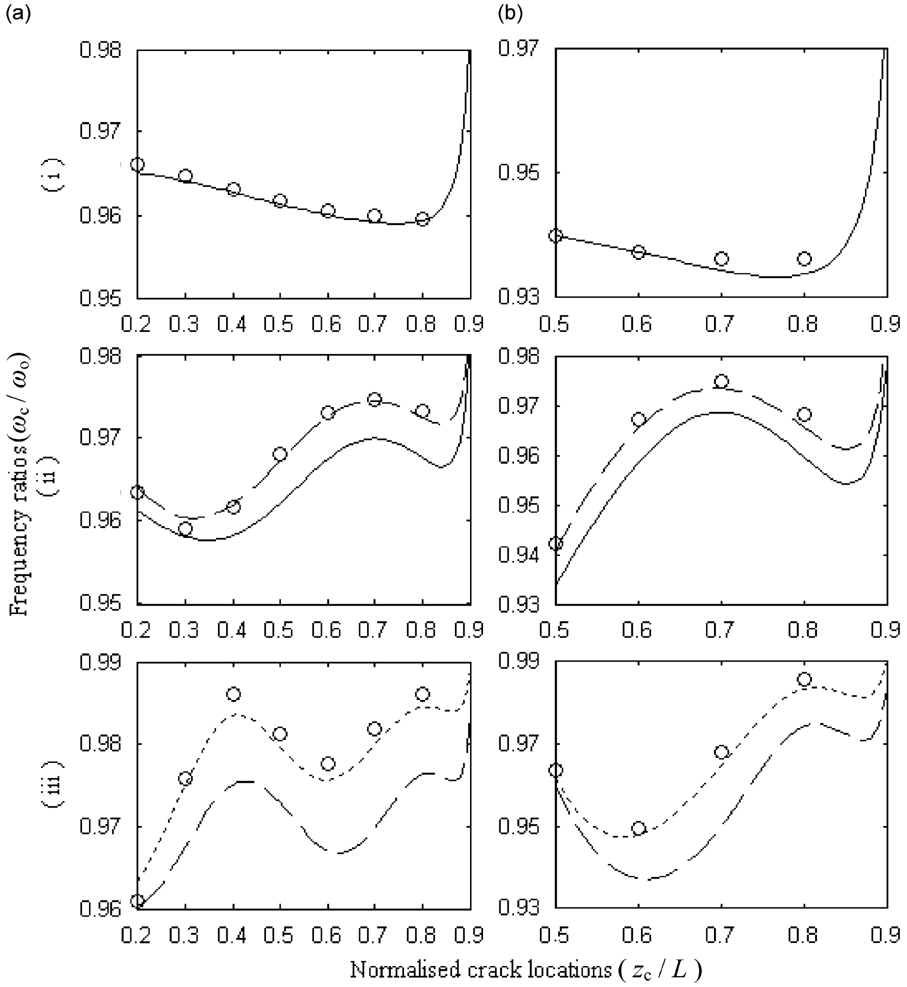

Fig. 7. Natural frequency ratios for the (i) first, (ii) second, and (iii) third mode vibration of double-cracked Beam3 with variably located second crack having depths (a) $a_2/h_1 = 0.15$, and (b) $a_2/h_1 = 0.3$. (○) Ansys results, (——) approximation with 4 terms, (– –) approximation with 5 terms, (- - -) approximation with 6 terms.

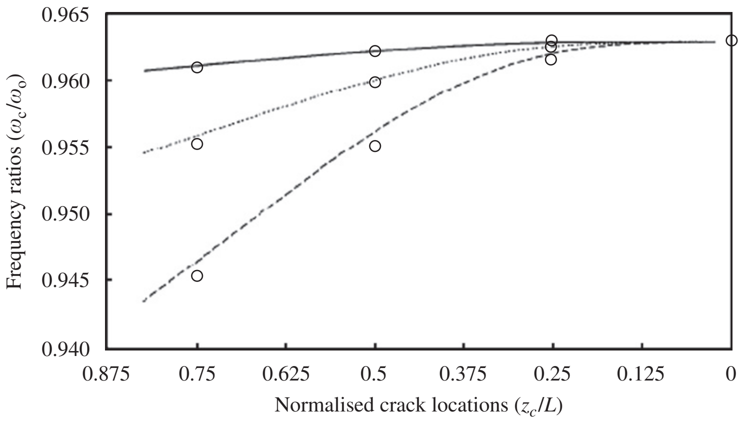

Fig. 8. Natural frequency ratios of the first mode vibration of Beam4 with variably located fourth crack having depths (——) $a_4 = 0.1h_1$, (…) $a_4 = 0.2h_1$, (- - -) $a_4 = 0.3h_1$ as given in Ref. [27], and findings of the method with 4 terms (○).

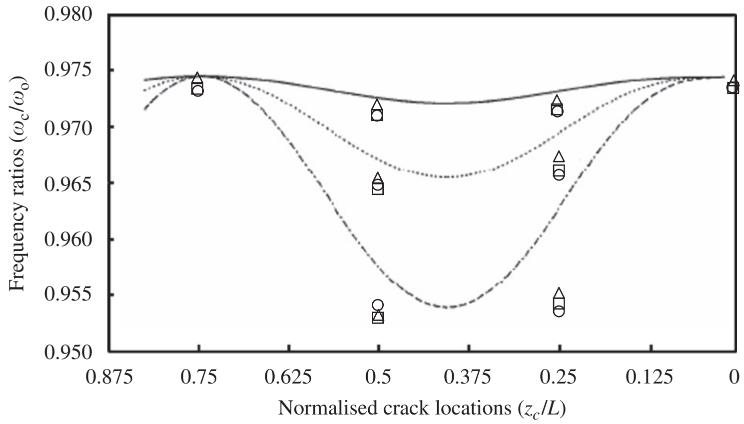

Fig. 9. Natural frequency ratios of the second mode vibration of Beam4 with variably located fourth crack having depths (——) $a_4 = 0.1h_1$, (…) $a_4 = 0.2h_1$, (- - -) $a_4 = 0.3h_1$ as given in Ref. [27], and findings of the method with 4 terms (○), 5 terms (□), and 6 terms (△).

natural frequency ratios. Negligible differences increase as the crack depth increases. It can be observed that as the number of terms in the deflection function increases, the difference between the results decreases as expected. In Fig. 9, the natural frequency ratios coincide for all considered crack depths at the normalised locations 0 and 0.75.

It should be noted that performing the finite element program with the acceptable number of elements that results in the correct solution is not possible for Beam4 having four cracks. Furthermore, processes can exceed the memory limitations of computers with the previously defined crack and meshing properties, especially when the cracks are too close to each other.

5. Conclusion

The energy-based method presented by Yang et al. [16] is modified to obtain the vibration of multiple-cracked non-uniform Euler–Bernoulli beams. Effects of the stress field caused by the angular displacement of the beam in addition to strain energy change caused by the crack are both taken into account in the energy consumed. In the cases of multiple cracks, the energy consumed caused by one crack varies with the influence of other cracks. Examples are presented on several tapered cantilever beams. The results of the method presented agree well with the results of the finite element program when the beam has single or double cracks. Additionally, the first mode frequencies obtained for the multiple-cracked Beam4 has an excellent agreement with the results of Zheng and Fan [27], although small differences are obtained in the second mode.

Instead of the analytical methods, uses of the energy distributions in numerical approaches simplify the solution of non-uniform beams. However, these approaches suffer from the interaction of crack effects in multiple-cracked beams. Proposal for the solution of this problem is presented in this paper. It is observed that a double-cracked beam behaves like a single-cracked beam when both cracks come closer to each other, as one would expect.

Coupling effects are neglected in this study. It should be remembered that bending–torsion coupling cannot be influential on the lower vibration modes of non-uniform Euler–Bernoulli beams. Furthermore, when the beams have cracks with acceptable depth ratios, bending–torsion coupling has still negligible influence on the lower vibration modes as seen in the figures representing the comparatively examined method results. However, this coupling may be more influential on the vibration of the stepped beams.

Significant advantage of the method can be performing the processes in quite short durations in the order of seconds. Thus, natural frequencies required for the frequency-based inverse methods like prediction schemes or contour graphs can be easily obtained for each different beam. In practical applications, natural frequencies may be measured in some error interval that can be kept in minimum by taking large sampling frequencies.

Effects of truncation factors are evaluated with respect to variation of the natural frequency ratios. Results show that cracks cause lower natural frequency ratios when the beam has lower truncation factors except for the cracks near the root of the beam. It is clear that the truncation factor of a beam's height is much more effective than the truncation factor of the beam's width. Another finding can be the shift of node points from the root to the tip with the decrease in truncation factor.

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