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Stochastic averaging of energy envelope of Preisach hysteretic systems

Y. Wang, Z.G. Ying*, W.Q. Zhu

Department of Mechanics, State Key Laboratory of Fluid Power Transmission and Control, Zhejiang University, Hangzhou 310027, PR China

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Abstract

A new stochastic averaging technique for analyzing the response of a single-degree-of-freedom Preisach hysteretic system with nonlocal memory under stationary Gaussian stochastic excitation is proposed. An equivalent nonhysteretic nonlinear system with amplitude-envelope-dependent damping and stiffness is firstly obtained from the given system by using the generalized harmonic balance technique. The relationship between the amplitude envelope and the energy envelope is then established, and the equivalent damping and stiffness coefficients are expressed as functions of the energy envelope. The available range of the yielding force of the system is extended and also the strong nonlinear stiffness of the system is incorporated so as to improve the response prediction. Finally, an averaged Itô stochastic differential equation for the energy envelope of the system as one-dimensional diffusion process is derived by using the stochastic averaging method of energy envelope, and the Fokker–Planck–Kolmogorov equation associated with the averaged Itô equation is solved to obtain stationary probability densities of the energy envelope and amplitude envelope. The approximate solutions are validated by using the Monte Carlo simulation.
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1. Introduction

Hysteretic behavior exists widely in mechanical and structural systems [1–3], in which hysteretic forces depend on not only the instantaneous deformation but also the history of deformation. Furthermore, there has been an increasing interest recently in using smart materials such as piezoceramics, shape memory alloys, and electro-/magneto-rheological fluids, which exhibit significant hysteresis [4–7]. Several models such as bilinear model, Bouc–Wen model, Jenkins–Iwan model and Masing model have been proposed for representing the hysteretic constitutive relationship [8–11]. Almost all hysteresis models used in mechanical and structural disciplines, however, can only represent hysteresis with local memory. In recent years, the Preisach model extensively used for ferromagnetism has been applied to the field of engineering mechanics. The main feature of this model is its capability to describe hysteresis nonlinearity with nonlocal memory and to capture the crossing of minor loops which can arise in many real materials [2].

*Corresponding author.
E-mail address: yingzg@zju.edu.cn (Z.G. Ying).

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In mechanical and structural engineering, the dynamic loading to which hysteretic systems are subjected is usually random in nature. It is extremely difficult to analytically determine the exact random responses of hysteretic systems, and thus some approximate solution techniques have been developed. The random dynamic responses of bilinear [8,9], Bouc–Wen [12] and Duhem hysteretic systems [13] have been studied by using the equivalent linearization method and the stochastic average method. However, there are only a few papers studying the random response of Preisach hysteretic systems. The mean output of the Preisach model to stochastic input has been studied by Mayergoyz and Korman [14–16]. The second-order statistics of stochastic dynamic response of the Preisach hysteretic systems have been approximately determined by Ni et al. [17] and Ying et al. [18] based on the covariance and switching probability analysis of a nonlocal memory hysteretic constitutive model. Recently, a stochastic averaging procedure for Preisach hysteretic systems has been developed by Spanos et al. [19] and applied to the random vibration analysis of smart memory alloy systems [20]. They firstly obtained the equivalent nonlinear system by using the harmonic balance technique, and then obtained the Fokker–Planck–Kolmogorov (FPK) equation governing the evolution of probability density functions of the amplitude envelope by using the stochastic averaging method.

In the present paper, a new stochastic averaging technique is proposed based on the generalized harmonic balance technique and the stochastic averaging method of energy envelope. Firstly, an equivalent nonhysteretic nonlinear system with amplitude-envelope-dependent damping and stiffness is obtained by using the generalized harmonic balance technique. Secondly, the relationship between the amplitude envelope and the energy envelope is established, and equivalent damping and stiffness coefficients are expressed as functions of the energy envelope. Thirdly, an averaged Itô stochastic differential equation for the energy envelope of the system as one-dimensional diffusion process is derived by using the stochastic averaging method of energy envelope, and the FPK equation associated with the averaged Itô equation is solved to obtain stationary probability densities of the energy envelope and amplitude envelope. The approximate solutions are validated by using the Monte Carlo simulation, and also are compared with the analytical solutions given in Ref. [19] for a special case.

2. Preisach hysteretic model

The Preisach hysteretic model is expressed in terms of the following integral:

$$f(t) = \int\int_{\alpha \geqslant \beta} \mu(\alpha, \beta)\gamma_{\alpha\beta}(x(t))\,\mathrm{d}\alpha\,\mathrm{d}\beta \tag{1}$$

where $x(t)$ and $f(t)$ denote the displacement and hysteretic force, respectively. $\mu(\alpha, \beta)$ is a weighting function, called the Preisach function, with support on a limiting triangle D of the (α,β)-plane with line $\alpha = \beta$ being the hypotenuse and point (α_P, β_P) being the vertex. The triangle D in the half-plane $\alpha \geqslant \beta$ is called the Preisach plane (see Fig. 1). $\mu(\alpha, \beta)$ is equal to zero outside the triangle D. $\gamma_{\alpha\beta}(x)$ is the hysteretic relay operator as shown in Fig. 2. It is a two-position relay with values $+1$ and -1 corresponding to "up" and "down" positions, respectively, and is given by Refs. [1,2]

$$\gamma_{\alpha\beta}(x) = \begin{cases} +1 & x > \alpha \text{ or } x > \beta \text{ and decreasing} \\ -1 & x < \beta \text{ or } x < \alpha \text{ and increasing} \end{cases} \tag{2}$$

The Preisach model, expressed as the superposition of a continuous family of elementary rectangular loops, can be interpreted in terms of the spectral decomposition of a complicated hysteretic constitutive law with nonlocal memory into the simplest hysteretic relay operator $\gamma_{\alpha\beta}(x)$ with local memory. The Preisach hysteretic behavior is completely characterized by the weighting function $\mu(\alpha, \beta)$. For arbitrary displacement input $x(t)$, the Preisach hysteretic force output $f(t)$ can be determined by the weighting function together with a staircase line $L(t)$ (see Fig. 1). And the staircase line divides the Preisach plane into two parts: the domain $S^+(t)$ encompassing the set of the relay in the $+1$ status, and the domain $S^-(t)$ encompassing the set of the relay in the -1 status. Each vertex of the staircase line is associated with dominant maximum M_k or minimum m_k of the displacement $x(t)$. The nonlocal selective memory is then stored in this way. Therefore, the Preisach

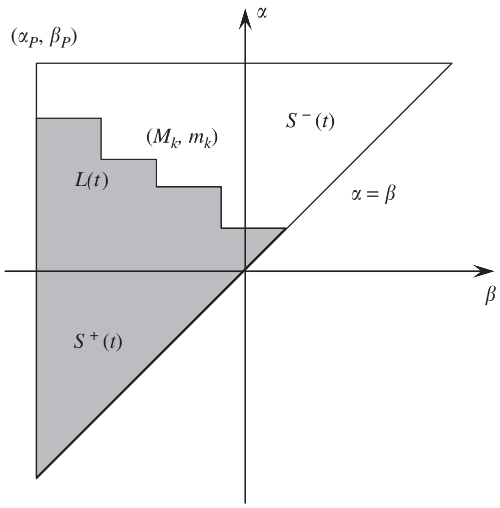

Fig. 1. Preisach plane.

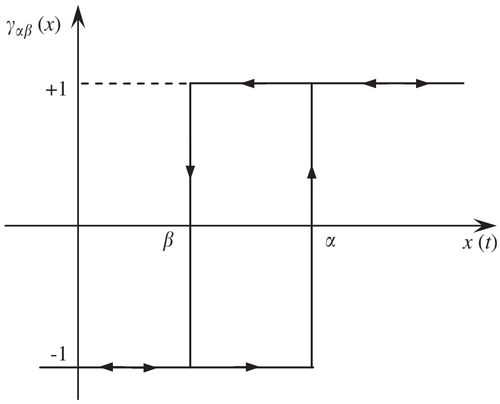

Fig. 2. Hysteretic relay operator.

hysteretic restoring force $f(t)$ can be expressed as

$$\begin{aligned} f(t) &= \int\int_{S^+(t)} \mu(\alpha,\beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta - \int\int_{S^-(t)} \mu(\alpha,\beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta \\ &= 2\int\int_{S^+(t)} \mu(\alpha,\beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta - \int\int_{D} \mu(\alpha,\beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta \end{aligned} \tag{3}$$

The Preisach function of a specific hysteretic system can be determined by the first-order transition curves [2]. Lubarda et al. [11] derived the Preisach function $\mu(\alpha,\beta)$ in a closed form for classical rheological models by using this procedure. For the Iwan–Jenkins model, it is

$$\mu(\alpha,\beta) = \frac{k_J}{2}\left\{\delta(\alpha-\beta) - \frac{k_J}{2}\frac{1}{f_{y,\max}-f_{y,\min}}\left[H\left(\alpha-\beta-2\frac{f_{y,\min}}{k_J}\right) - H\left(\alpha-\beta-2\frac{f_{y,\max}}{k_J}\right)\right]\right\} \tag{4}$$

where $\delta(\cdot)$ and $H(\cdot)$ are the Dirac delta function and the Heaviside function, respectively. The symbol k_J represents the linear stiffness of the individual Jenkins element. f_y is the yielding force, and $f_{y,\min} \leqslant f_y \leqslant f_{y,\max}$. The weighting function is defined in the domain A as shown in Fig. 3. The corresponding Preisach hysteretic

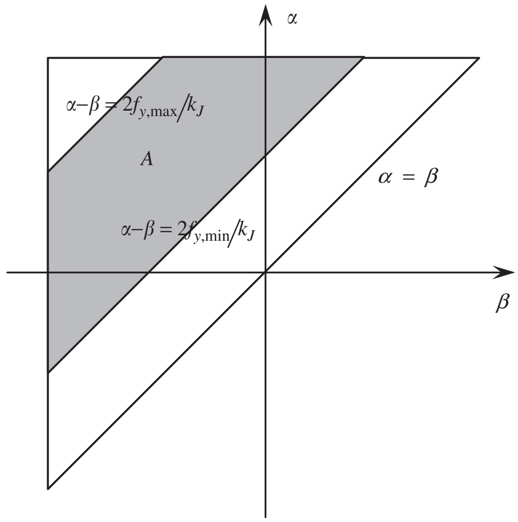

Fig. 3. Domain of the weighing function for the Iwan–Jenkins model.

force can be obtained by substituting Eq. (4) into Eq. (1) as

$$f(t) = \frac{k_J}{2}\left[\int_{-\alpha_P}^{\alpha_P} \gamma_{\alpha,\alpha}(x)\,\mathrm{d}\alpha - \frac{k_J}{2}\frac{1}{f_{y,\max} - f_{y,\min}}\int\int_A \gamma_{\alpha\beta}(x)\,\mathrm{d}\alpha\,\mathrm{d}\beta\right] \tag{5}$$

By using Eq. (3), the hysteretic force (5) can be rewritten as

$$f(t) = k_J x(t) - \frac{k_J^2}{4}\frac{1}{f_{y,\max} - f_{y,\min}}\left(2\int\int_{S_A^+(t)} \mathrm{d}\alpha\,\mathrm{d}\beta - \int\int_A \mathrm{d}\alpha\,\mathrm{d}\beta\right) \tag{6}$$

In general, the restoring force of smart materials is anti-symmetric as a function of deformation and then the restoring force of relevant stochastic hysteretic systems with zero mean will be anti-symmetric too. The stochastic hysteretic systems with nonzero mean can be converted into the corresponding one with zero mean. In the case of hysteretic systems with zero mean, let $\alpha_P = -\beta_P = f_{y,\max}/k_J$. Then the shaded area A is a triangle defined by $\alpha = \alpha_P$, $\beta = \beta_P$ and $\alpha - \beta = 2f_{y,\min}/k_J$. According to Lubarda et al. [11], the domain A is initially divided into two equal parts and thus $f(0) = 0$. Introduce function

$$F(\alpha_i, \beta_i) = \frac{[2f_{y,\min} + k_J(\beta_j - \alpha_j)]^2}{2k_J^2} \tag{7}$$

which represents the area of a triangle surrounded by $\alpha = \alpha_i$, $\beta = \beta_i$ and $\alpha - \beta = 2f_{y,\min}/k_J$ (see Fig. 4). It is assumed that the vibration amplitude varies slowly with respect to time. When the present vibration amplitude $\bar{a}$ is equal to or greater than $f_{y,\min}/k_J$, i.e., $\bar{a} \geqslant f_{y,\min}/k_J$ (see Fig. 5) [19], the hysteretic force for the ascending status is given by

$$\begin{aligned} f(t) = k_J x(t) - \frac{k_J^2}{4}\frac{1}{f_{y,\max} - f_{y,\min}}&\left[4\left(F\left(\frac{f_{y,\min}}{k_J}, \beta_P\right) - F\left(\frac{f_{y,\min}}{k_J}, \beta_1\right)\right)\right. \\ &\left. + 2H\left(x(t) - \beta_{n-1} - \frac{2f_{y,\min}}{k_J}\right)F(x(t), \beta_{n-1}) + 2\sum_{j=2}^{n-1}(F(\alpha_j, \beta_{j-1}) - F(\alpha_j, \beta_j)) - F(\alpha_P, \beta_P)\right] \end{aligned} \tag{8}$$

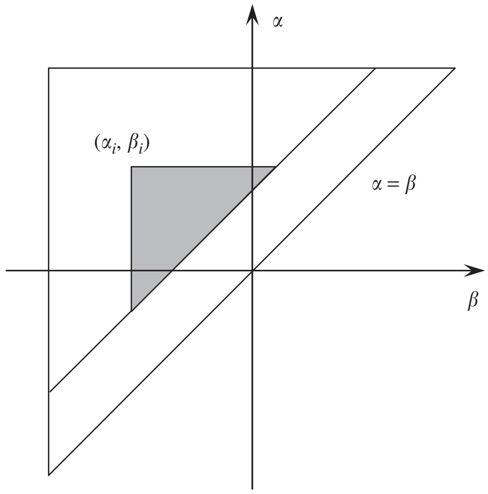

Fig. 4. Geometric interpretation of the function F.

and that for the descending status is

$$f(t) = k_J x(t) - \frac{k_J^2}{4} \frac{1}{f_{y,\max} - f_{y,\min}} \left[4\left(F\left(\frac{f_{y,\min}}{k_J}, \beta_P\right) - F\left(\frac{f_{y,\min}}{k_J}, \beta_1\right) \right) + 2F(\alpha_n, \beta_{n-1}) \right.$$
$$\left. -2H\left(\alpha_n - \frac{2f_{y,\min}}{k_J} - x(t)\right) F(\alpha_n, x(t)) + 2\sum_{j=2}^{n-1} (F(\alpha_j, \beta_{j-1}) - F(\alpha_j, \beta_j)) - F(\alpha_P, \beta_P) \right] \quad (9)$$

When the present vibration amplitude $\bar{a}$ is smaller than $f_{y,\min}/k_J$, i.e., $\bar{a} < f_{y,\min}/k_J$ (see Fig. 6), the hysteretic force for both the ascending status and descending status is as follows:

$$f(t) = k_J x(t) - k_J^2 \frac{1}{f_{y,\max} - f_{y,\min}} \left(F\left(\frac{f_{y,\min}}{k_J}, \beta_P\right) - F\left(\frac{f_{y,\min}}{k_J}, \beta_1\right) \right)$$
$$- \frac{k_J^2}{4} \frac{1}{f_{y,\max} - f_{y,\min}} \left[2\sum_{j=2}^{s} (F(\alpha_j, \beta_{j-1}) - F(\alpha_j, \beta_j)) - F(\alpha_P, \beta_P) \right] \quad (10)$$

In Eqs. (8)–(10), α_i and β_i are the dominant maximum and minimum of displacement $x(t)$, respectively. If the present vibration amplitude $\bar{a}$ is smaller than $f_{y,\min}/k_J$, the minimum of the dominant maxima α_s and the maximum of the dominant minima β_s are much close to $f_{y,\min}/k_J$ and $-f_{y,\min}/k_J$, respectively.

3. Equivalent nonhysteretic nonlinear stochastic system

Consider a single-degree-of-freedom nonlinear hysteretic system under stochastic excitation as shown in Fig. 7. The motion of the system is governed by the following equation:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) + f(t) = w(t) \quad (11)$$

where $x(t)$ is the displacement; m, c and k are mass, damping and stiffness coefficients, respectively; $w(t)$ is the zero-mean Gaussian white noise with spectral density S_w; $f(t)$ denotes the Preisach hysteretic restoring force governed by Eqs. (8)–(10). Introducing the nondimensional parameter $\phi = k/k_J$, Eq. (11) can be rewritten as

$$\ddot{x}(t) + 2\zeta\varpi\dot{x}(t) + \varpi^2 x(t) + \frac{f_H(t)}{m} = \frac{w(t)}{m} \quad (12)$$

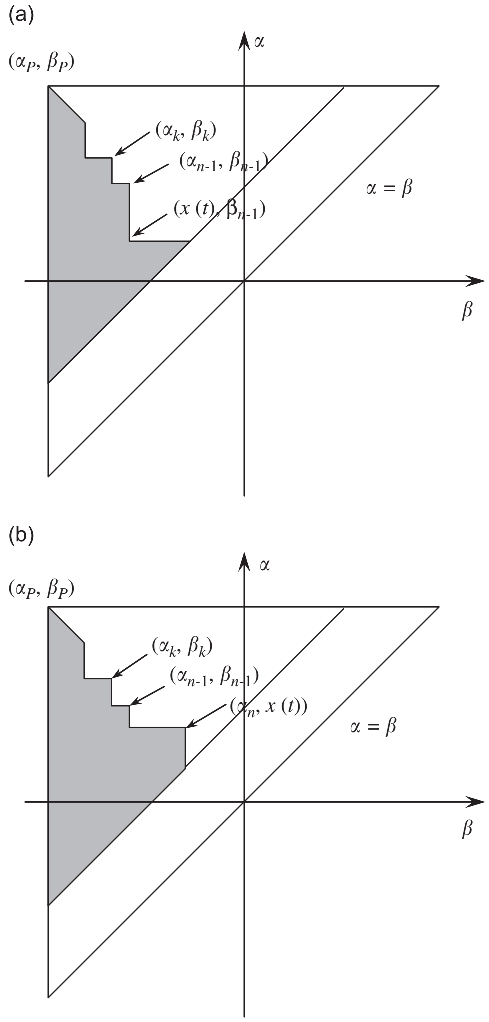

Fig. 5. Geometric interpretation of the Preisach hysteretic force when $\bar{a} \geqslant f_{y,\min}/k_J$: (a) for the ascending status; (b) for the descending status.

where $\bar{\omega} = \sqrt{(k+k_J)/m} = \omega_J\sqrt{1+\phi}$, $\omega_J = \sqrt{k_J/m}$ and $2\zeta\bar{\omega} = c/m$. $f_H(t)$ is the nonlinear part of the hysteretic force, which represents the memory of the hysteretic system. Further introducing the nondimensional displacement

$$y = \frac{x\sqrt{2\zeta\bar{\omega}^3 m^2}}{\sqrt{\pi S_w}} \tag{13}$$

Eq. (12) can be recast in the following form:

$$\ddot{y}(t) + 2\zeta\bar{\omega}\dot{y}(t) + \bar{\omega}^2(y(t) + \psi p_H(t)) = \widehat{w}(t) \tag{14}$$

where $\widehat{w}(t) = w(t)\sqrt{2\zeta\bar{\omega}^3}/\sqrt{\pi S_w}$ with $S_{\widehat{w}(t)} = 2\zeta\bar{\omega}^3/\pi$; $\psi = \bar{\omega}^2\sqrt{\pi S_w}/(f_y^*\sqrt{2\zeta\bar{\omega}^3})$ and $f_y^* = (f_{y,\max} + f_{y,\min})/2$; $p_H(t) = 2\zeta f_y^* f_H(t)/(\pi S_w \bar{\omega})$. Define nondimensional parameters

$$v = \frac{f_{y,\max} - f_{y,\min}}{2f_y^*}, \quad \frac{2f_{y,\min}}{k_J}\frac{\sqrt{2\zeta\bar{\omega}^3 m^2}}{\sqrt{\pi S_w}} = 2\frac{(1+\phi)(1-v)}{\psi} = \Delta \tag{15a,b}$$

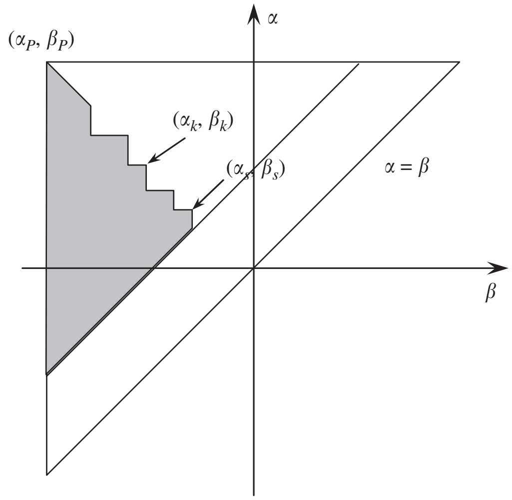

Fig. 6. Geometric interpretation of the Preisach hysteretic force when $\bar{a} < f_{y,\min}/k_J$.

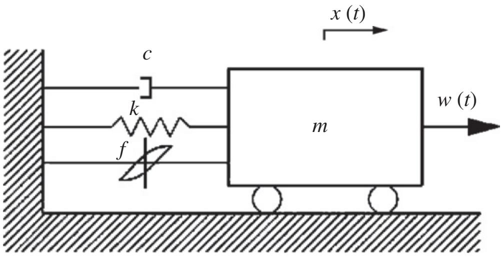

Fig. 7. Single-degree-of-freedom nonlinear hysteretic system.

The parameter v monitors the spread of the distributed element. For $v \to 0$, the domain A reduces to a straight line that represents the domain of the individual Jenkins model. For $v < 1$, the system possesses a well-defined proportionality limit. If $v = 1$, the system exhibits a strongly nonlinear behavior with a vanishing proportionality limit, and the weighting function has the support of whole domain D. When the nondimensional amplitude $a = \bar{a}\sqrt{2\zeta\bar{\omega}^3 m^2}/\sqrt{\pi S_w} > \Delta/2$, the nondimensional expression of the nonlinear part of the hysteretic force for the ascending status is

$$p_H(t) = \frac{1}{4v(1+\phi)^2}\left\{-\left[2\left(\tilde{F}\left(\frac{\Delta}{2}, \beta'_P\right) - \tilde{F}\left(\frac{\Delta}{2}, \beta'_1\right)\right) + H(y(t) - \beta'_{n-1} - \Delta)\tilde{F}(y, \beta'_{n-1})\right] - \frac{1}{2}\left[2\sum_{j=2}^{n-1}(\tilde{F}(\alpha'_j, \beta'_{j-1}) - \tilde{F}(\alpha'_j, \beta'_j)) - \tilde{F}(\alpha'_P, \beta'_P)\right]\right\} \tag{16}$$

and that for the descending status is

$$p_H(t) = \frac{1}{4v(1+\phi)^2}\left\{-\left[2\left(\tilde{F}\left(\frac{\Delta}{2}, \beta'_P\right) - \tilde{F}\left(\frac{\Delta}{2}, \beta'_1\right)\right) + \tilde{F}(\alpha'_n, \beta'_{n-1}) - H(\alpha'_n - \Delta - y(t))\tilde{F}(\alpha'_n, y(t))\right] - \frac{1}{2}\left[2\sum_{j=2}^{n-1}(\tilde{F}(\alpha'_j, \beta'_{j-1}) - \tilde{F}(\alpha'_j, \beta'_j)) - \tilde{F}(\alpha'_P, \beta'_P)\right]\right\} \tag{17}$$

When the nondimensional amplitude $a<\Delta/2$, the nondimensional expression of the nonlinear part of the hysteretic force for both the ascending status and descending status is

$$p_H(t) = \frac{1}{4v(1+\phi)^2}\left\{-2\left[\tilde{F}\left(\frac{\Delta}{2},\beta'_P\right) - \tilde{F}\left(\frac{\Delta}{2},\beta'_1\right)\right] - \frac{1}{2}\left[2\sum_{j=2}^{s}(\tilde{F}(\alpha'_j,\beta'_{j-1}) - \tilde{F}(\alpha'_j,\beta'_j)) - \tilde{F}(\alpha'_P,\beta'_P)\right]\right\} \quad (18)$$

where $\alpha'_j = \alpha_j\sqrt{2\zeta\bar{\omega}^3 m^2}/\sqrt{\pi S_w}$ and $\beta'_j = \beta_j\sqrt{2\zeta\bar{\omega}^3 m^2}/\sqrt{\pi S_w}$ are the nondimensional dominant maximum and minimum, respectively, and $\tilde{F}(\alpha'_j,\beta') = [\Delta + (\beta'_j - \alpha'_j)]^2/2$.

The Preisach hysteretic restoring force generally contains a nonlinear elastic restoring force and a nonlinear dissipative force, which are coupled. To apply the stochastic averaging method of energy envelope, the hysteretic force is firstly separated into quasi-linear damping force and elastic force. According to the generalized harmonic balance technique, let nondimensional displacement and velocity be as follows [21,22]:

$$y(t) = a(t)\cos\theta(t) \quad (19a)$$

$$\dot{y}(t) = -\omega(a)a(t)\sin\theta(t) \quad (19b)$$

where $\theta(t) = \omega(a)t + \varphi(t)$, $\sin\theta$ and $\cos\theta$ are the generalized harmonic functions. The hysteretic restoring force is replaced equivalently by the following nonhysteretic forces containing a quasi-linear elastic force and a quasi-linear damping force:

$$\bar{\omega}^2 p_H(y) = C(a)\dot{y} + D(a)y \quad (20)$$

where $C(a)$ and $D(a)$ are determined by the generalized harmonic balance as

$$C(a) = -\frac{\bar{\omega}^2}{a\pi\omega(a)}\int_0^{2\pi} p_H(a\cos\theta)\sin\theta\,\mathrm{d}\theta \quad (21a)$$

$$D(a) = \frac{\bar{\omega}^2}{a\pi}\int_0^{2\pi} p_H(a\cos\theta)\cos\theta\,\mathrm{d}\theta \quad (21b)$$

When the nondimensional amplitude $a>\Delta/2$,

$$C(a) = \frac{\bar{\omega}^2}{a\pi\omega(a)}\frac{1}{4v(1+\phi)^2}\left\{\int_0^{\pi}[\tilde{F}(\alpha'_n,\beta'_{n-1}) - H(\alpha'_n - \Delta - a\cos\theta)\tilde{F}(\alpha'_n, a\cos\theta)]\sin\theta\,\mathrm{d}\theta + \int_{\pi}^{2\pi}[H(a\cos\theta - \beta'_{n-1} - \Delta)\tilde{F}(a\cos\theta,\beta'_{n-1})]\sin\theta\,\mathrm{d}\theta\right\} \quad (22a)$$

$$D(a) = \frac{\bar{\omega}^2}{a\pi}\frac{1}{4v(1+\phi)^2}\left\{\int_0^{\pi}[H(\alpha'_n - \Delta - a\cos\theta)\tilde{F}(\alpha'_n, a\cos\theta)]\cos\theta\,\mathrm{d}\theta - \int_{\pi}^{2\pi}[H(a\cos\theta - \beta'_{n-1} - \Delta)\tilde{F}(a\cos\theta,\beta'_{n-1})]\cos\theta\,\mathrm{d}\theta\right\} \quad (22b)$$

Due to the generalized harmonic behavior of the response process, the nondimensional dominant maximum α'_n and minimum β'_{n-1} can be set approximately equal to the nondimensional amplitude a and its opposite value $-a$, respectively, that is,

$$\alpha'_n = a, \quad \beta'_{n-1} = -a \quad (23a,b)$$

Then Eq. (22) becomes correspondingly

$$C(a) = \frac{\bar{\omega}^2}{a\pi\omega(a)}\frac{1}{4v(1+\phi)^2}\left\{\int_0^{\pi}[\tilde{F}(a,-a) - H(a-\Delta-a\cos\theta)\tilde{F}(a,a\cos\theta)]\sin\theta\,\mathrm{d}\theta + \int_{\pi}^{2\pi}[H(a\cos\theta+a-\Delta)\tilde{F}(a\cos\theta,-a)]\sin\theta\,\mathrm{d}\theta\right\} \tag{24a}$$

$$D(a) = \frac{\bar{\omega}^2}{a\pi}\frac{1}{4v(1+\phi)^2}\left\{\int_0^{\pi}[H(a-\Delta-a\cos\theta)\tilde{F}(a,a\cos\theta)]\cos\theta\,\mathrm{d}\theta - \int_{\pi}^{2\pi}[H(a\cos\theta+a-\Delta)\tilde{F}(a\cos\theta,-a)]\cos\theta\,\mathrm{d}\theta\right\} \tag{24b}$$

or

$$C(a) = \frac{\bar{\omega}^2}{a\pi\omega(a)}\frac{1}{4v(1+\phi)^2}\left\{(\Delta-2a)^2 - \int_{\arccos(a-\Delta)/a}^{\pi}\frac{1}{2}[\Delta+a(\cos\theta-1)]^2\sin\theta\,\mathrm{d}\theta + \int_{2\pi-\arccos(\Delta-a)/a}^{2\pi}\frac{1}{2}[\Delta-a(\cos\theta+1)]^2\sin\theta\,\mathrm{d}\theta\right\} \tag{25a}$$

$$D(a) = \frac{\bar{\omega}^2}{a\pi}\frac{1}{4v(1+\phi)^2}\left\{\int_{\arccos(a-\Delta)/a}^{\pi}\frac{1}{2}[\Delta+a(\cos\theta-1)]^2\cos\theta\,\mathrm{d}\theta - \int_{2\pi-\arccos(\Delta-a)/a}^{2\pi}\frac{1}{2}[\Delta-a(\cos\theta+1)]^2\cos\theta\,\mathrm{d}\theta\right\} \tag{25b}$$

When the nondimensional amplitude $a<\Delta/2$,

$$C(a) = 0 \tag{26a}$$

$$D(a) = 0 \tag{26b}$$

Thus, the equivalent nonhysteretic nonlinear system can be written as

$$\ddot{y}(t) + (2\zeta\bar{\omega} + \psi C(a))\dot{y}(t) + (\bar{\omega}^2 + \psi D(a))y(t) = \widehat{w}(t) \tag{27}$$

or in the following form:

$$\ddot{y}(t) + 2\zeta\bar{\omega}\dot{y}(t) + \bar{\omega}^2 y(t) = \widehat{w}(t), \quad a<\frac{\Delta}{2} \tag{28a}$$

$$\ddot{y}(t) + (2\zeta\bar{\omega} + \psi C(a))\dot{y}(t) + (\bar{\omega}^2 + \psi D(a))y(t) = \widehat{w}(t), \quad a>\frac{\Delta}{2} \tag{28b}$$

Eq. (28) implies that the nonlinear part of the hysteretic restoring force has not any effects on the system when the nondimensional amplitude $a<\Delta/2$ and the response is governed by a linear differential equation, and when the nondimensional amplitude $a>\Delta/2$, the response is governed by an equivalent nonlinear differential equation with amplitude-envelope-dependent damping and stiffness coefficients.

The total energy of the equivalent nonhysteretic system (27) can be expressed as

$$H = \tfrac{1}{2}\dot{y}^2 + \tfrac{1}{2}(\bar{\omega}^2 + \psi D(a))y^2 \tag{29}$$

When $\dot{y}=0$, $y=a$ so that Eq. (29) yields $H=\frac{1}{2}(\bar{\omega}^2+\psi D(a))a^2$. By solving this equation, the amplitude $a=a(H)$ as a function of energy H can be obtained. Then Eq. (27) can be rewritten as

$$\ddot{y}(t) + C^*(H)\dot{y}(t) + A^*(H)y(t) = \widehat{w}(t) \tag{30}$$

where

$$C^*(H) = 2\zeta\bar{\omega} + \psi C(a(H)) \tag{31a}$$

$$A^*(H) = \bar{\omega}^2 + \psi D(a(H)) \tag{31b}$$

in which the frequency $\omega(a(H))$ is expressed as a function of the system energy. Hereafter, the superscript "*" of $C^*(H)$ and $A^*(H)$ will be omitted for simplicity.

4. Stochastic averaging of energy envelope

Now the stochastic averaging method of energy envelope [23] is applied to system (30). The state equations of the system are

$$\dot{Y}_1 = Y_2 \tag{32a}$$

$$\dot{Y}_2 = -C(H)Y_2 - A(H)Y_1 + \widehat{w}(t) \tag{32b}$$

where $Y_1 = y$ and $Y_2 = \dot{y}$ are the nondimensional displacement and velocity, respectively. The intensity of stochastic excitation $\widehat{w}(t)$ is denoted by $2D = 2\pi S_{\widehat{w}} = 4\zeta\bar{\omega}^3$. The Itô stochastic differential equations of Eq. (32) are

$$\mathrm{d}Y_1 = Y_2\,\mathrm{d}t \tag{33a}$$

$$\mathrm{d}Y_2 = [-C(H)Y_2 - A(H)Y_1]\,\mathrm{d}t + \sqrt{2D}\,\mathrm{d}W(t) \tag{33b}$$

where $W(t)$ is the unit Wiener process. The system energy is expressed as

$$H = \tfrac{1}{2}Y_2^2 + \tfrac{1}{2}A(H)Y_1^2 = H(Y_1, Y_2) \tag{34}$$

By introducing the transformation

$$Y_1 = Y_1, \quad H = H(Y_1, Y_2) \tag{35a,b}$$

and using the Itô differential rule, Eq. (33) can be transformed into the Itô equations for displacement Y_1 and energy envelope H as

$$\mathrm{d}Y_1 = \pm\sqrt{2H - A(H)Y_1^2}\,\mathrm{d}t \tag{36a}$$

$$\begin{aligned}\mathrm{d}H = &\left\{\begin{bmatrix}\frac{\partial H}{\partial Y_1} & \frac{\partial H}{\partial Y_2}\end{bmatrix}\begin{bmatrix}Y_2\\ -C(H)Y_2 - A(H)Y_1\end{bmatrix} + \frac{1}{2}tr\left(\begin{bmatrix}\frac{\partial^2 H}{\partial Y_1^2} & \frac{\partial^2 H}{\partial Y_1 \partial Y_2}\\ \frac{\partial^2 H}{\partial Y_2 \partial Y_1} & \frac{\partial^2 H}{\partial Y_2^2}\end{bmatrix}\begin{bmatrix}0\\ \sqrt{2D}\end{bmatrix}[0 \;\; \sqrt{2D}]\right)\right\}\mathrm{d}t\\ &+ \begin{bmatrix}\frac{\partial H}{\partial Y_1} & \frac{\partial H}{\partial Y_2}\end{bmatrix}\begin{bmatrix}0\\ \sqrt{2D}\end{bmatrix}\mathrm{d}W(t)\end{aligned} \tag{36b}$$

where $tr(\cdot)$ is the trace of square matrix. Note that

$$\frac{\partial H}{\partial Y_1} = \frac{A(H)Y_1}{1 - \frac{1}{2}\frac{\partial A(H)}{\partial H}Y_1^2} \tag{37a}$$

$$\frac{\partial H}{\partial Y_2} = \frac{Y_2}{1 - \frac{1}{2}\frac{\partial A(H)}{\partial H}Y_1^2} \tag{37b}$$

$$\frac{\partial^2 H}{\partial Y_2^2} = \frac{\left(1 - \frac{1}{2}\frac{\partial A(H)}{\partial H}Y_1^2\right) - Y_2\left(-\frac{1}{2}\frac{\partial A^2(H)}{\partial H^2}Y_1^2\frac{\partial H}{\partial Y_2}\right)}{\left(1 - \frac{1}{2}\frac{\partial A(H)}{\partial H}Y_1^2\right)^2} \tag{37c}$$

Then Eq. (36b) can be rearranged as follows:

$$\begin{aligned} \mathrm{d}H = {} & \left\{ -C(H)\frac{2H - A(H)Y_1^2}{1 - \frac{1}{2}\frac{\partial A(H)}{\partial H}Y_1^2} + D\frac{1}{1 - \frac{1}{2}\frac{\partial A(H)}{\partial H}Y_1^2} + D\left(1 - \frac{1}{2}\frac{\partial A(H)}{\partial H}Y_1^2\right)^{-3}\frac{1}{2}\frac{\partial A^2(H)}{\partial H^2}Y_1^2(2H - A(H)Y_1^2)\right\}\mathrm{d}t \\ & + \sqrt{2D}\frac{\pm\sqrt{2H - A(H)Y_1^2}}{1 - \frac{1}{2}\frac{\partial A(H)}{\partial H}Y_1^2}\,\mathrm{d}W(t) \end{aligned} \tag{38}$$

The energy envelope H of the system is a slowly varying process and can be replaced in the first approximation by a one-dimensional diffusion process [23]. For simplicity, the same symbol H will be used to denote this diffusion process. The Itô differential equation for this diffusion process is obtained by making the time averaging of Eq. (38). By using Eq. (36a), the time averaging can be replaced by the space averaging with respect to Y_1 under certain H. Thus

$$\mathrm{d}H = U(H)\,\mathrm{d}t + V(H)\,\mathrm{d}W(t) \tag{39}$$

where

$$T(H) = 2\int_{-a(H)}^{a(H)} \frac{\mathrm{d}Y_1}{\sqrt{2H - A(H)Y_1^2}} \tag{40a}$$

$$\begin{aligned} U(H) = {} & \frac{1}{T(H)}2\int_{-a(H)}^{a(H)} \left\{ -C(H)\frac{\sqrt{2H - A(H)Y_1^2}}{1 - \frac{1}{2}\frac{\partial A(H)}{\partial H}Y_1^2} + D\frac{1}{\left(1 - \frac{1}{2}\frac{\partial A(H)}{\partial H}Y_1^2\right)\sqrt{2H - A(H)Y_1^2}} \right. \\ & \left. + D\frac{\frac{1}{2}\frac{\partial A^2(H)}{\partial H^2}Y_1^2\sqrt{2H - A(H)Y_1^2}}{\left(1 - \frac{1}{2}\frac{\partial A(H)}{\partial H}Y_1^2\right)^3} \right\}\mathrm{d}Y_1 \end{aligned} \tag{40b}$$

$$V^2(H) = \frac{1}{T(H)}4D\int_{-a(H)}^{a(H)} \frac{\sqrt{2H - A(H)Y_1^2}}{\left(1 - \frac{1}{2}\frac{\partial A(H)}{\partial H}Y_1^2\right)^2}\,\mathrm{d}Y_1 \tag{40c}$$

in which $T(H)$ is determined by $\omega(a(H)) = 2\pi/T(H)$. The $C(H)$ can be obtained by Eqs. (25a) and (31a), and $U(H)$ and $V^2(H)$ are determined by Eqs. (40b) and (40c).

The FPK equation associated with the averaged Itô differential Eq. (39) is

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial H}[U(H)p] + \frac{1}{2}\frac{\partial^2}{\partial H^2}[V^2(H)p] \tag{41}$$

where $p = p(H, t|H_0)$ is the transition probability density of the energy envelope H. The stationary probability density of the energy envelope $p(H)$ is obtained from solving the reduced FPK equation of Eq. (41) as

$$p(H) = \tilde{C}\,\exp\left\{-\int_0^H \left[\frac{\mathrm{d}V^2(u)}{\mathrm{d}u} - 2U(u)\right]/V^2(u)\,\mathrm{d}u\right\} \tag{42}$$

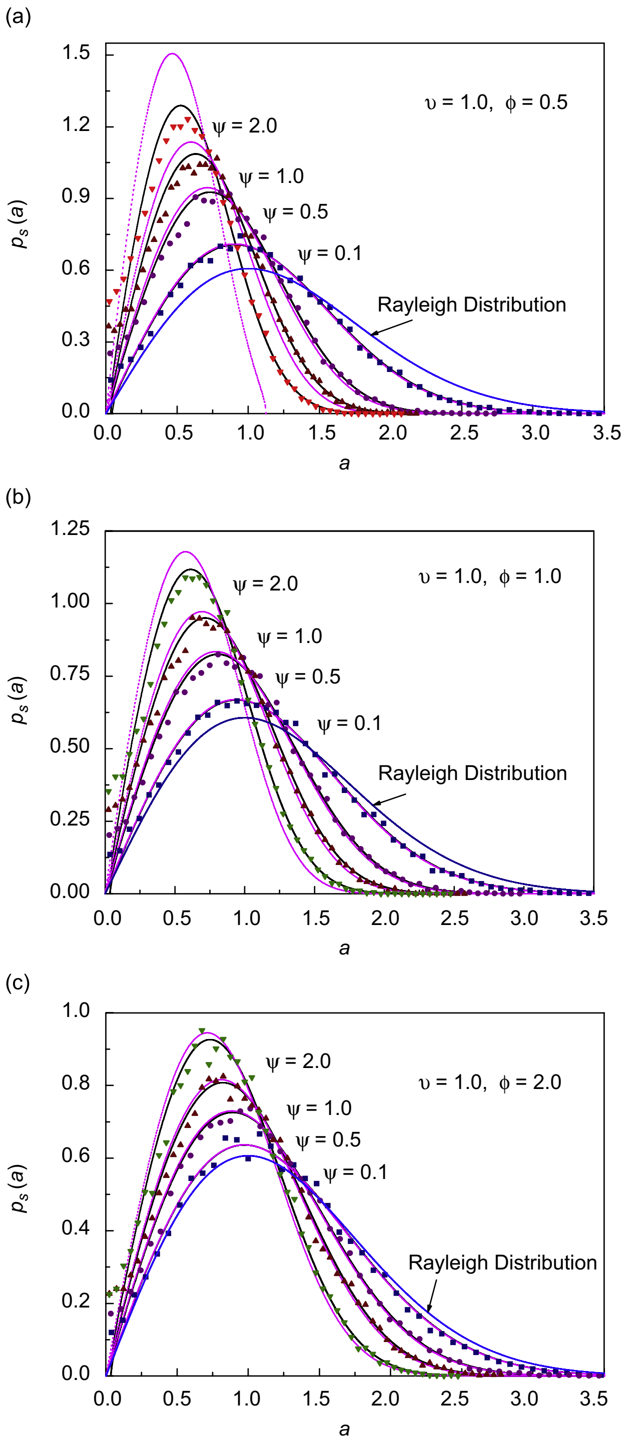

Fig. 8. Stationary probability density of the amplitude envelope for $v = 1.0$ (solid line: proposed result; dotted line: result in Ref. [19]; ▼ ▲ ● ■: Monte Carlo simulation). (a) $\phi = 0.5$; (b) $\phi = 1.0$; (c) $\phi = 2.0$.

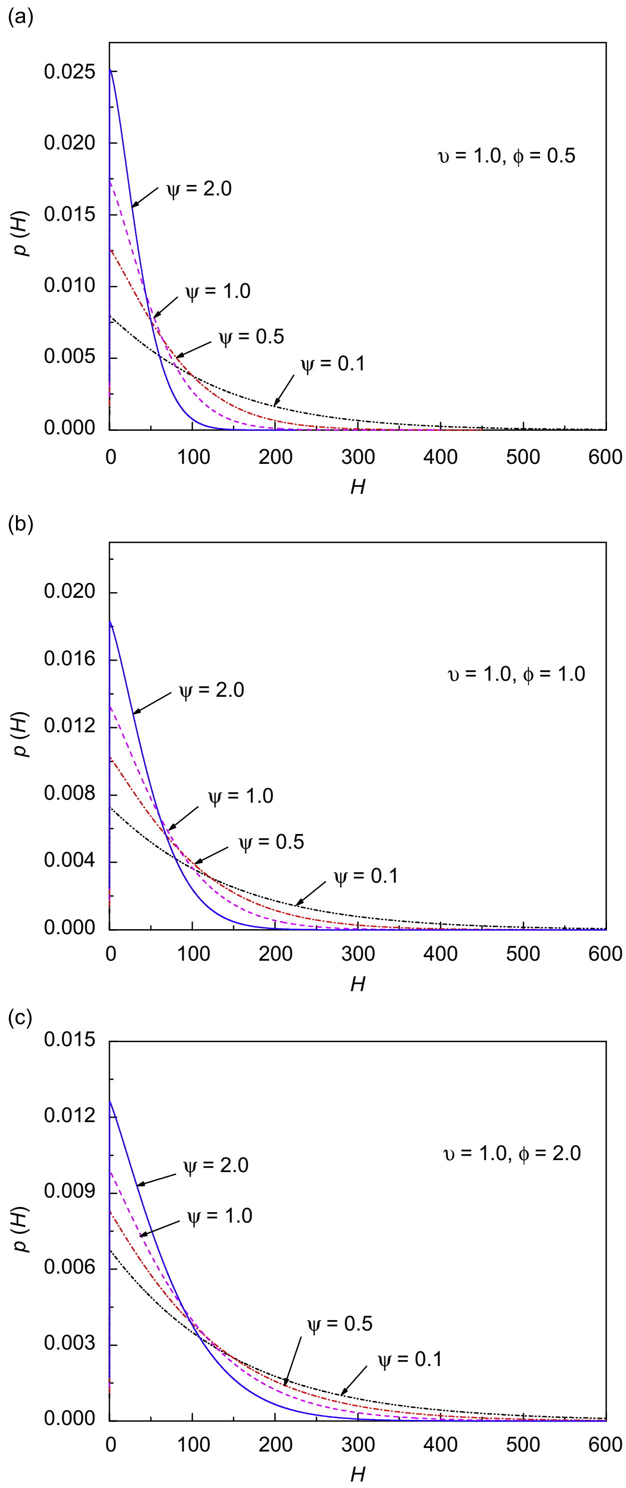

Fig. 9. Stationary probability density of the energy envelope for $v = 1.0$. (a) $\phi = 0.5$; (b) $\phi = 1.0$; (c) $\phi = 2.0$.

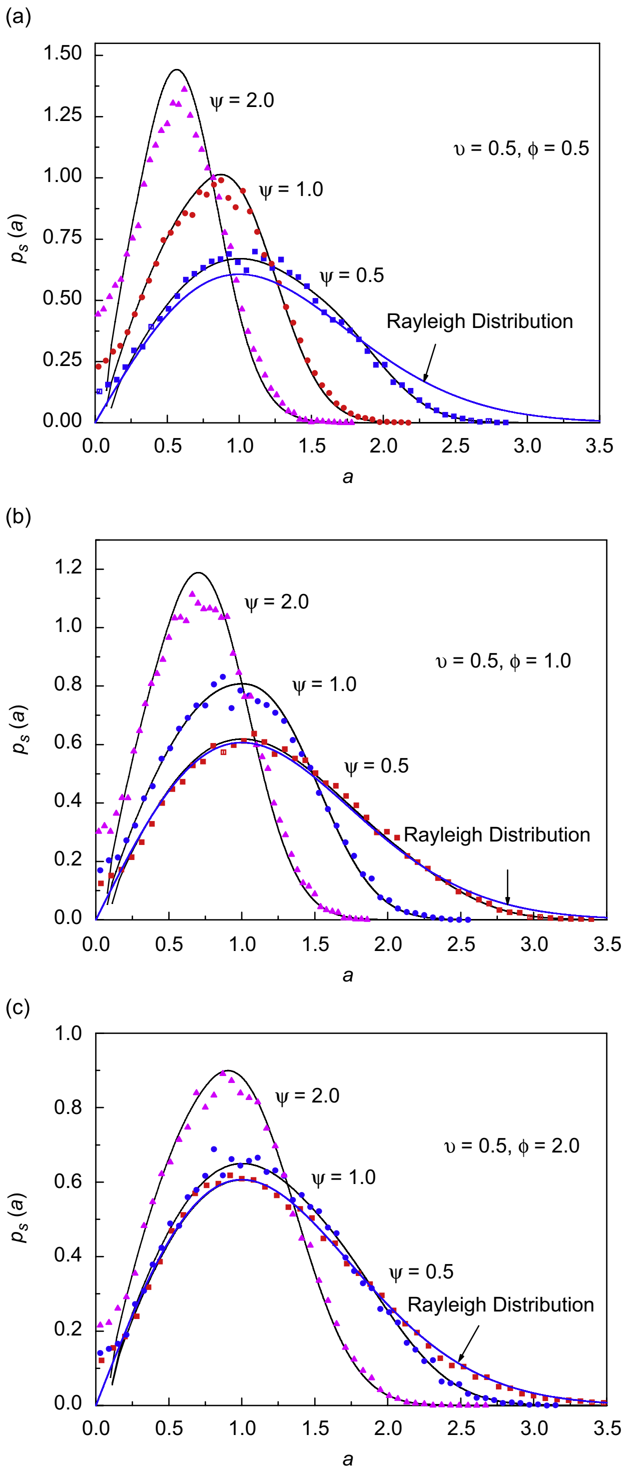

Fig. 10. Stationary probability density of the amplitude envelope for $v = 0.5$ (solid line: proposed result; ▲ ● ■: Monte Carlo simulation). (a) $\phi = 0.5$; (b) $\phi = 1.0$; (c) $\phi = 2.0$.

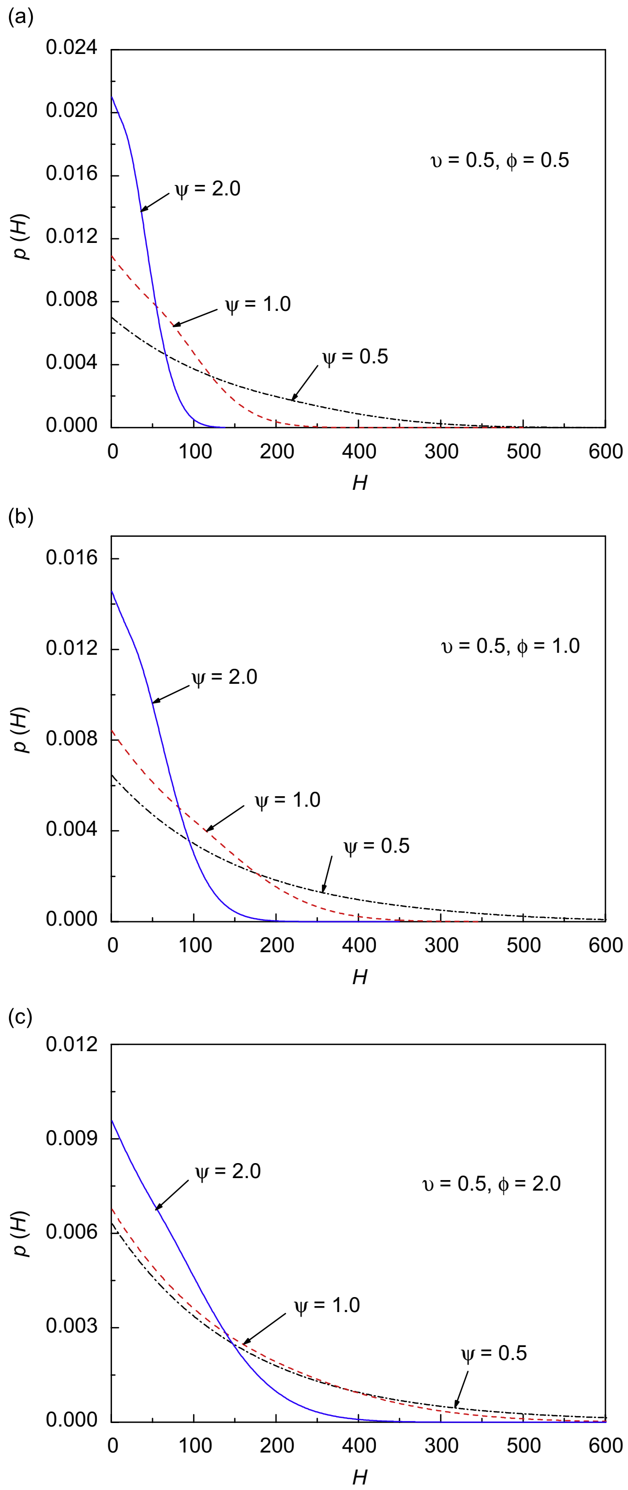

Fig. 11. Stationary probability density of the energy envelope for $v = 0.5$. (a) $\phi = 0.5$; (b) $\phi = 1.0$; (c) $\phi = 2.0$.

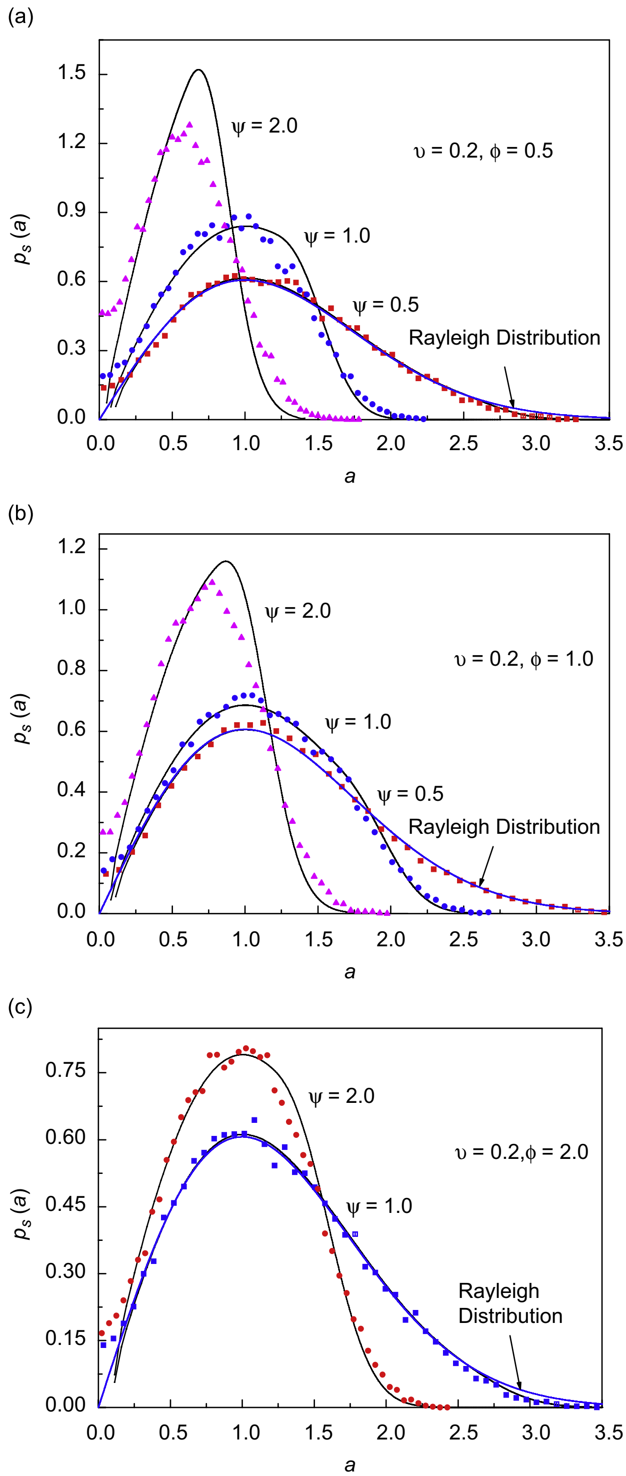

Fig. 12. Stationary probability density of the amplitude envelope for $v = 0.2$ (solid line: proposed result; ▲ ● ■: Monte Carlo simulation). (a) $\phi = 0.5$; (b) $\phi = 1.0$; (c) $\phi = 2.0$.

where $\tilde{C}$ is the normalization constant. The joint probability density of displacement Y_1 and velocity Y_2 can be expressed as

$$p(Y_1, Y_2) = \frac{p(H)}{T(H)}\bigg|_{H=H(Y_1,Y_2)} \tag{43}$$

and the probability density of the amplitude envelope a is

$$p(a) = p(H)\left|\frac{\partial H(a)}{\partial a}\right| \tag{44}$$

5. Numerical results

The proposed stochastic averaging technique is applied to system (11) with damping ratio $\zeta = 0.01$ and parameter $\bar{\omega} = 4\pi$, and numerical results are given in Figs. 8–12. Figs. 8(a)–(c) show the stationary probability densities of amplitude envelope of the system for $v = 1.0$ and different values of parameters ϕ and ψ by using the proposed technique, Monte Carlo simulation and the analytical expression given in Ref. [19]. It can be seen that the proposed results are in good agreement with those by the Monte Carlo simulation, and the results in Ref. [19] are improved, especially for small ϕ and large ψ. As ψ decreases, the system response tends to a Gaussian process and the probability density of amplitude envelope approaches the Rayleigh distribution.

Figs. 10(a)–(c) and 12(a)–(c) show the stationary probability densities of amplitude envelope of the system for, respectively, $v = 0.5$ (corresponding to $f_{y,\min}/f_{y,\max} = 1/3$) and $v = 0.2$ (corresponding to $f_{y,\min}/f_{y,\max} = 2/3$) and different values of parameters ϕ and ψ. However, the analytical expression given in Ref. [19] is usable only for $v = 1.0$. In the case of $v < 1.0$, $\Delta = 2(1+\phi)(1-v)/\psi > 0$. The system response is governed by the linear Eq. (28a) when the amplitude $a < \Delta/2$, and by the equivalent nonlinear Eq. (28b) with amplitude-envelope-dependent damping and stiffness when the amplitude $a > \Delta/2$. As ϕ decreases, the nonlinear feature of the system becomes more pronounced and as ψ increases, the effects of the nonlinear part of hysteretic force $p_H(t)$ on the response enhance. It is also noticed from Figs. 10 and 12 that the difference between the proposed results and those by the Monte Carlo simulation increases with the increase of ψ and the decrease of v and ϕ, which needs to be improved further. The stationary probability densities of energy envelope of the system for $v = 1.0$ and $v = 0.5$ are illustrated by Figs. 9 and 11, respectively, corresponding to Figs. 8 and 10.

6. Conclusions

A new stochastic averaging technique for analyzing the response of a single-degree-of-freedom Preisach hysteretic system with nonlocal memory under a stationary Gaussian stochastic excitation has been proposed. The equivalent nonhysteretic nonlinear system with amplitude-envelope-dependent damping and stiffness was obtained firstly by using the generalized harmonic balance technique. Then the relationship between the amplitude envelope and the energy envelope was established, and the equivalent damping and stiffness coefficients were expressed as functions of the energy envelope. The averaged Itô stochastic differential equation for the energy envelope of the system was derived finally by using the stochastic averaging method of energy envelope, and the FPK equation associated with the averaged Itô equation was solved to obtain the stationary probability densities of the energy envelope and amplitude envelope. The approximate solutions have been validated by comparing with the results of Monte Carlo simulation. The proposed technique improves the results given in Ref. [19] by extending the available value of v and incorporating strong nonlinear stiffness of the system. However, the accuracy of the proposed results falls for large ψ and small v and ϕ, which needs to be improved further.

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