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Simplified dispersion curves for circular cylindrical shells using shallow shell theory

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Abstract

An alternative derivation of the dispersion relation for the transverse vibration of a circular cylindrical shell is presented. The use of the shallow shell theory model leads to a simpler derivation of the same result. Further, the applicability of the dispersion relation is extended to the axisymmetric mode and the high frequency beam mode.

1. Introduction

In structural acoustics, the dispersion curves of waves associated with transverse displacements are of primary interest since they are responsible for sound generation. A particularly insightful form of presenting these dispersion characteristics for circular cylindrical shells are the universal constant-frequency loci as given by Heckl [1] (in a short almost inconspicuous appendix) and later elaborately by Fahy [2] (figures 56a and 96). In particular, the transition from the flexural effect to the membrane effect is depicted simply by the bending of the idealized circular curves (which arise for the case of pure bending in flat plates).

The original derivation by Heckl starts by writing the three coupled equations of motion with the only forcing term being in the form of a convected pressure. The impedance corresponding to the ratio of the amplitude of pressure loading to the radial velocity is found. This impedance when equated to zero gives the free wavenumbers. The model used by Heckl corresponds to the Kennard theory with a group of terms neglected (the Kennard model perhaps gives the most unwieldy set of equations, see Ref. [3, equation (2.9h)]). This approximation is shown to be valid for circumferential modes $n \geqslant 2$. Thus, the simplified dispersion relation given by Fahy [2, equation 2.114] is applicable only for $n > 1$.

In the present article, we derive the same expression as Fahy [2] in an alternative simple method using the shallow shell theory (Donell–Mushtari–Vlasov model). Unlike other shell theories, for example the Donell–Mushtari theory, which gives a set of three equations (see Ref. [3, equations 2.7 and 2.9]), this model yields a single equation for the normal displacements. Additionally, we show the applicability of the

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dispersion equation for the axisymmetric ($n=0$) and the high frequency beam modes ($n=1$). The equation is unsuited for the beam mode at low frequencies.

2. The shallow shell theory

The essential features of the shallow shell theory are as follows [3,4]:

- In-plane displacements are used in calculating the membrane strain energy but not the bending strain energy. The expressions for the bending strain energy are thus considerably simplified.
- No loading, inclusive of inertia loading, is considered in the in-plane direction.
- An Airy stress function is used which satisfies the in-plane equations identically.
- The theory gives us a set of two equations; one in the normal direction and the other due to the compatibility requirement of the stress function (see Ref. [3, equation 1.136]).
- The two equations can be simplified to give just one equation (see Ref. [4, equation 6.8.9]).

It seems that traditionally this model was thought to be applicable only to slightly curved plates and hence it is known as the shallow shell theory. However, as Soedel [4] points out this is an "*unnecessarily severe restriction*".

3. Derivation of the dispersion relation

The shallow shell differential equation for the free vibration of circular cylindrical shells is given by (see Ref. [4, sections 6.7–6.9])

$$D\nabla^8 w(\theta,x) + Eh\nabla_k^4 w(\theta,x) = \rho h\omega^2\nabla^4 w(\theta,x). \tag{1}$$

In the above equation, w is the radial displacement of the cylindrical shell, h is the shell thickness, ω is the circular frequency, ρ, E and v are the density, Young's modulus of elasticity and Poisson's ratio of the shell material, respectively. The parameter D is the flexural rigidity given by $D = Eh^3/[12(1-v^2)]$.

Note, ∇^2 is the Laplacian operator and ∇_k^2 is a differential operator which arises in the derivation process. For the general form of ∇_k^2 applicable to any shell surface, the reader is referred to Refs. [4, equation (6.7.11)] or [3, equation (1.138)]. Specializing for a circular cylindrical shell of radius a, we get

$$\nabla^2 w = \frac{1}{a^2}\frac{\partial^2 w}{\partial\theta^2} + \frac{\partial^2 w}{\partial x^2}, \quad \nabla_k^4 w = \frac{1}{a^2}\frac{\partial^4 w}{\partial x^4} \quad \text{(see Ref. [4, equation 6.9.2])}.$$

We assume a solution for w of the form $w(\theta,x) = W\cos(n\theta)\mathrm{e}^{\mathrm{i}kx}$. Note, the harmonic time dependence has already been incorporated in arriving at Eq. (1). The above relations further reduce to

$$\nabla^2 w = -\left(\frac{n^2}{a^2}+k^2\right)w \Rightarrow \nabla^8 w = \left(\frac{n^2}{a^2}+k^2\right)^4 w, \quad \nabla_k^4 w = \frac{1}{a^2}k^4 w. \tag{2}$$

Thus, we have the following dispersion relation:

$$D\left(\frac{n^2}{a^2}+k^2\right)^4 + Eh\frac{k^4}{a^2} = \rho h\omega^2\left(\frac{n^2}{a^2}+k^2\right)^2.$$

Introducing the non-dimensional wavenumber $\kappa = ka$, we get

$$\frac{D}{a^8}(n^2+\kappa^2)^4 + Eh\frac{\kappa^4}{a^6} = \frac{\rho h\omega^2}{a^4}(n^2+\kappa^2)^2.$$

Note, the extensional wave speed of the shell material is $c_L = \sqrt{E/\rho}$. Thus, in terms of the non-dimensionalized frequency $\Omega = \omega a/c_L$, we have

$$\frac{D}{Eh}(n^2+\kappa^2)^4 + \kappa^4 a^2 = \frac{\omega^2 a^2}{c_L^2}a^2(n^2+\kappa^2)^2 \Rightarrow \frac{D}{Eh}(n^2+\kappa^2)^4 + \kappa^4 a^2 = \Omega^2 a^2(n^2+\kappa^2)^2.$$

Using $\beta^2 = h^2/12a^2$, we get

$$\frac{\beta^2}{(1-\nu^2)}(n^2+\kappa^2)^4 + \kappa^4 = \Omega^2(n^2+\kappa^2)^2. \tag{3}$$

This equation is the non-dimensional dispersion equation. However, that would not be *universal* in the sense that for shells with different β, these curves would have to be redrawn. A final simplification is therefore made as follows:

$$\frac{1}{(1-\nu^2)}(n^2\beta+\kappa^2\beta)^4 + \kappa^4\beta^2 = \Omega^2(n^2\beta+\kappa^2\beta)^2$$
$$\Rightarrow \frac{1}{(1-\nu^2)}(n^2\beta+\kappa^2\beta)^2 + \frac{\kappa^4\beta^2}{(n^2\beta+\kappa^2\beta)^2} = \Omega^2.$$

Finally, we get

$$\frac{1}{(1-\nu^2)}[(n\sqrt{\beta})^2+(\kappa\sqrt{\beta})^2]^2 + \frac{(\kappa\sqrt{\beta})^4}{[(n\sqrt{\beta})^2+(\kappa\sqrt{\beta})^2]^2} = \Omega^2. \tag{4}$$

But for the $1-\nu^2$ term (which can be taken close to unity for practical values of $\nu \approx 0.3$) this equation is identical to equation 2.114 in Ref. [2] (original derivation in Ref. [1]). Further, this relation holds for all n.

We obtain an easy insight into the above equation by considering the variables $\kappa\sqrt{\beta}$ and $n\sqrt{\beta}$ in polar coordinates. Accordingly, let us define $\kappa\sqrt{\beta} = r\sin\theta$ and $n\sqrt{\beta} = r\cos(\theta)$. The dispersion relation above with $\nu \approx 0$ (as used by Fahy) comes to

$$r^4 = \Omega^2 - \sin^4\theta. \tag{5}$$

As θ goes from 0 to $\pi/2$, $\sin^4\theta$ goes from 0 to 1. Thus, for $\theta = 0$, which corresponds to the line $\kappa\sqrt{\beta} = 0$, we have $n = \sqrt{\Omega/\beta}$. This square root dependence on the frequency and the thickness parameter is typical of plates.

For $\Omega < 1$, there is range of θ, with upper bound $\pi/2$, at which there is no real-valued r. This is because $\Omega^2 < \sin^4\theta$ in this range. This leads to the bending of the dispersion curves. As Ω increases beyond unity, the effect of this bending is reduced. For any Ω, there is nonetheless some bending of the curve and most appreciably in the $\theta = \pi/2$ direction or along the $\kappa\sqrt{\beta}$ axis. As noted earlier, this bending of the constant-frequency loci is the membrane effect due to the cylinder curvature. The amount of this bending characterizes the membrane effect and can now be quantified as

$$\sin^4\theta = \frac{\kappa^4\beta^2}{[\kappa^2\beta+n^2\beta]^2}.$$

4. The $n = 0$ and the $n = 1$ mode

As Fahy [2] has used the dispersion relation given in Eq. (4) for $n \geqslant 2$, the accuracy of the above relation for $n \geqslant 2$ shall be taken for granted. In this section, we shall analyze the efficacy of Eq. (4) for $n = 0$ and $n = 1$. Again following Fahy, we shall take $\nu \approx 0$ in the remainder of the discussion.

For the axisymmetric mode we have $n = 0$, which when used in Eq. (4) with $\nu = 0$, gives

$$\kappa = \sqrt[4]{\frac{\Omega^2-1}{\beta^2}}. \tag{6}$$

This shows that for $\Omega > 1$ κ is real and for $\Omega < 1$ κ is complex. The former case implies a propagating wave whereas the latter case implies an evanescent character of the wave. Note, due to the choice of non-dimensionalization, the ring frequency is given by $\Omega = 1$. Thus, the relation above correctly predicts that the axisymmetric flexural wave propagates at frequencies above the ring frequency and decays at frequencies below the ring frequency. For a discussion on the ring frequency and its significance to the axisymmetric flexural wave propagation, the reader is referred to Ref. [2, p. 27] or [5].

The expression above is the $\mathcal{O}(1)$ solution for the flexural wavenumber in the axisymmetric mode derived from the Donell–Mushtari shell equations. The exact wavenumber may be estimated to considerable accuracy by adding a correction dependent on v^2 (which anyway we are assuming to be small following Fahy's lead). Unlike the present shallow shell model, the Donell–Mushtari shell equations are fully coupled in the three directions. For a detailed derivation of these results using the asymptotics of the Donell–Mushtari shell equations, the reader is referred to our previous work [6].

With $n = 1$ and $v^2 \approx 0$, the dispersion Eq. (3) gives

$$\beta^2(1+\kappa^2)^4 + \kappa^4 - \Omega^2(1+\kappa^2)^2 = 0. \tag{7}$$

This relation is a quartic equation in κ^2 and hence solvable in principle. However, the analytical expressions obtained would be too cumbersome for any use other than plotting a graph. Here we use the asymptotic method to get approximate compact analytical expressions. The derivation proceeds in two parts for the high and the low frequencies, respectively.

4.1. High frequency ($n = 1$)

For high frequencies, we scale the variables as $\Omega = \Omega'/\varepsilon$ and $\kappa = k/\sqrt{\varepsilon}$, where $0<\varepsilon\ll 1$ is a small fictitious parameter. The new variables introduced are all supposed to be $\mathcal{O}(1)$.

Making these substitutions in Eq. (7), we get the following equation:

$$\beta^2\varepsilon^4 + 4\beta^2k^2\varepsilon^3 + 6\beta^2k^4\varepsilon^2 + 4\beta^2k^6\varepsilon + \beta^2k^8 + k^4\varepsilon^2 - \Omega'^2\varepsilon^2 - 2\,\Omega'^2k^2\varepsilon - \Omega'^2k^4 = 0.$$

Next, we substitute $k = a_0 + a_1\varepsilon$ in the above equation and balance the resulting equation at every order of ε. At the first order we get $a_0 = \sqrt{\Omega'/\beta}$. Using this result at the next order we get $a_1 = -\frac{1}{2}(\beta/\Omega')^{1/2}$. Using these values to find $k = a_0 + a_1\varepsilon$ and then changing back to the original variable we find

$$\kappa = \sqrt{\frac{\Omega}{\beta}} - \frac{1}{2}\sqrt{\frac{\beta}{\Omega}}. \tag{8}$$

These steps are typical of asymptotic methods [6,7].

Using the Donell–Mushtari shell theory we had obtained an identical expression for the two-term high frequency asymptotic expansion for the flexural wavenumbers [7]. Thus, in this case also we find that the shallow shell equation gives identical results.

4.2. Low frequency

For the low frequency, we again rescale the variables as follows, $\Omega = \varepsilon\Omega'$, $\kappa = \sqrt{\varepsilon}k$ and $\beta = \varepsilon b$, where $0<\varepsilon\ll 1$ is a small fictitious parameter. Again the new variables Ω', k and b are assumed to be $\mathcal{O}(1)$.

From this stage onwards, the same steps as described previously are repeated, namely (1) the above substitutions are plugged into Eq. (7), (2) $k = a_0 + a_1\varepsilon$ is substituted in the resulting equation, (3) the resulting equation is balanced at $\mathcal{O}(1)$ to get a_0, (4) using a_0 at the next order balance equation a_1 is obtained, (5) using a_0 and a_1, k is found, (6) finally the result is transformed back to the original variables. The final result obtained is as follows:

$$\kappa = \sqrt[4]{\Omega^2 - \beta^2} + \frac{1}{2}\frac{\Omega^2 - 2\beta^2}{\sqrt[4]{\Omega^2 - \beta^2}}. \tag{9}$$

In contrast to the high frequency case, this expression does not match (not even in the first term) with the low frequency asymptotic expansion of the Donell–Mushtari model derived in our previous work [7]. It is well-known in the literature that the low-frequency $n = 1$ mode vibration of a cylindrical shell resembles the vibration of a beam with annular cross-section [2,5]. In particular, the shell model should bear a close match with the Timoshenko beam model [8,7]. In the following, we compare the results obtained from various shell and beam theories.

The expression in Eq. (9) is compared with the following related solutions (i) numerical solution of Eq. (7) (ii) wavenumbers obtained using the Euler–Bernoulli beam model (iii) wavenumbers obtained using the Rayleigh beam model, which accounts for the inertia but not the shear deformation (iv) the wavenumbers obtained using the Timoshenko model (v) wavenumbers obtained by numerical solution of the Donell–Mushtari model. The dispersion equations for the Timoshenko beam, the Rayleigh beam, the Euler–Bernoulli beam and the Donell–Mushtari models are given in our previous work with the same notation [7]. The only difference is that in the present case we use $\nu = 0$. Plots corresponding to each of these models for $h/a = 0.05$ and $0.05 < \Omega < 0.5$ are presented in Fig. 1. Note, due to the low frequency assumption, the plot is restricted to $\Omega < 0.5$. For the sake of clarity, we plot the results for the frequency range $0 \leqslant \Omega \leqslant 0.05$ separately in Fig. 2. It has been verified that in this case, the asymptotic expression (9) is in complete agreement with the numerical solution for the shallow shell model (*viz.*, Eq. (7)). The results for the shallow shell model are presented for $\Omega \geqslant 0.015$ because below this frequency the wavenumber obtained turns complex (see the following section for details on this aspect). Also, in this case the solution arising from the Kennard shell model is plotted in Fig. 2. The relevant equations for the Kennard shell model may be found in Ref. [3].

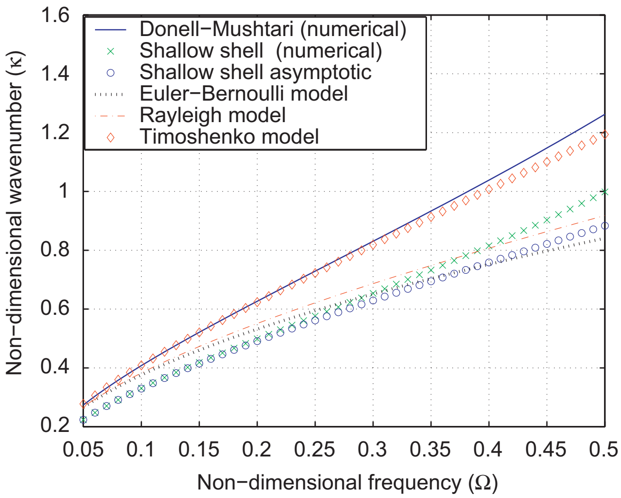

Fig. 1. Comparison of low frequency bending wavenumbers in a cylindrical shell with $h/a = 0.05$, $\nu = 0$ for $h/a \leqslant \Omega < 0.5$.

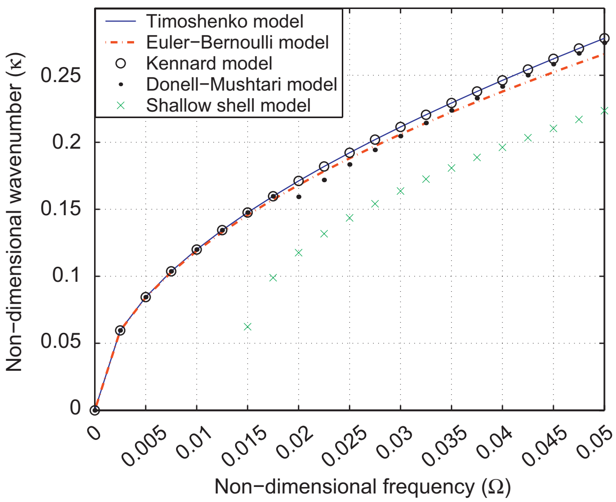

Fig. 2. Comparison of low frequency bending wavenumbers in a cylindrical shell with $h/a = 0.05$, $\nu = 0$ for $0 \leqslant \Omega \leqslant h/a$.

4.3. Discussion

In this section, we discuss some salient features of the results obtained in Section 4.2.

1. From Fig. 1, we note that the asymptotic solution in Eq. (9) is in agreement with the numerical solution of Eq. (7) for frequencies $\Omega < 0.25$. Beyond this frequency range the two solutions show a mismatch. The asymptotic solution is erroneous for $\Omega > 0.25$. This observation is in spirit with the asymptotic analysis methodology wherein the solution is supposed to be accurate only in the asymptotic limit [9].
2. As discussed earlier, it is well-known that for low frequencies, the $n = 1$ mode cylindrical shell vibration should be in close agreement with the Timoshenko model of a beam with annular cross-section [8]. As depicted in Fig. 1, the Donell–Mushtari model is in close agreement with the Timoshenko beam model for the greater portion of the frequency range. Though the results for the Kennard model have not been presented in this figure, it has been verified that it gives results identical to that obtained from the Donell–Mushtari model.
   In Fig. 2 it is seen that the Kennard model which is more accurate than the Donell–Mushtari model (see Ref. [3]), follows the Timoshenko beam characteristics perfectly for this case. The Donell–Mushtari model is in error in the frequency range $0.02 < \Omega < 0.04$. It has been reported in the literature that the Donell–Mushtari model is faulty for the very low frequency range [5].
3. Note, all beam models (Euler–Bernoulli, Rayleigh, Timoshenko) predict propagating wavenumbers for any non-zero frequency, however small. The asymptotic expression derived in Eq. (9) gives us real wavenumbers if $\Omega > \beta$. Thus, this formula cannot be used in the frequency range $\Omega < \beta$.
4. The above flaw is due to the inherent lacuna of the shallow shell model. Recall, this model yielded the dispersion relation given by Eq. (7). We now revisit this quartic equation in κ^2 and prove that if $0 < \Omega < \beta$, then there is no positive root for κ^2. By Descartes rule of sign, the maximum number of positive roots in a polynomial equation is given by the number of sign changes in the coefficients of the equation (see Ref. [10, p. 37]). We rewrite Eq. (7) as follows:
   $$\beta^2\kappa^8 + 4\beta^2\kappa^6 + (6\beta^2 + 1 - \Omega^2)\kappa^4 + (4\beta^2 - 2\Omega^2)\kappa^2 + (\beta^2 - \Omega^2) = 0.$$
   It is verified that for $0 < \Omega < \beta$, all the coefficients are positive. Thus, there are no sign changes and hence no positive roots for κ^2. Hence, the wavenumber κ turns complex for this case. This is what the asymptotic relation in Eq. (9) predicted.
5. Due to the above observation, it is expected that the shallow shell model will be erroneous at low frequencies for the $n = 1$ mode. This is further substantiated by the plots in Figs. 1 and 2. In a qualitative sense, the wavenumber characteristics do not follow any of the beam models. Thus, the shallow shell theory is inappropriate for modeling the low frequency vibration in the $n = 1$ mode. The limitations of the shallow shell theory are discussed in Ref. [3] and in greater detail in Ref. [11, pp. 84–90].
   However, as observed from Fig. 1, the numerical error between the results of the shallow shell theory and the simpler beam theories (Euler–Bernoulli or Rayleigh) lies within 10% in the frequency range $0.2 < \Omega < 0.4$. In this sense, for certain choice of parameters the shallow shell model may barely provide a crude numerical estimate.

5. Conclusions

In this article, we have derived the universal dispersion relation for a circular cylindrical shell. In comparison to the earlier method of derivation [1], the present method is much simpler as it employs the shallow shell model. There are no assumptions made in the derivation process. In contrast, the earlier derivation started from a set of three coupled equations and invoked various assumptions at intermediate steps. The perfect match of the results establishes the equivalence of the two methods. We are *exactly* solving an *approximate* problem, whereas the earlier method *approximately* solves the *exact* problem.

Further, the earlier method was applicable for $n \geqslant 2$. In this paper, we extend the range of applicability of the simplified dispersion relation to $n = 0$ and the high frequency region of $n = 1$. It is found that in the low frequency region of $n = 1$, the shallow shell model is inappropriate. Better shell models such as the

Donell–Mushtari theory or the Kennard theory yield solutions close to the Timoshenko beam model. Although the Donell–Mushtari theory has a small frequency band of discrepancy, the Kennard theory has an excellent match with the Timoshenko theory for the low frequencies. In contrast, the shallow shell model does not follow the characteristics of any of the beam models. Some discussion is presented on these aspects of the results.

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