Available online at www.sciencedirect.com
ScienceDirect
Journal of Sound and Vibration 322 (2009) 381–392
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

A hyperbolic perturbation method for determining homoclinic solution of certain strongly nonlinear autonomous oscillators

S.H. Chen[a,*], Y.Y. Chen[a], K.Y. Sze[b]

[a]Department of Applied Mechanics and Engineering, Sun Yat-sen University, Guangzhou 510275, PR China
[b]Department of Mechanical Engineering, The University of Hong Kong, Pokfulam, Hong Kong SAR, PR China

Received 19 March 2008; received in revised form 7 July 2008; accepted 9 November 2008
Handling Editor: L.G. Tham
Available online 23 December 2008

Abstract

A hyperbolic perturbation method is presented for determining the homoclinic solution of certain strongly nonlinear autonomous oscillators of the form $\ddot{x} + c_1 x + c_2 x^2 = \varepsilon f(\mu, x, \dot{x})$ in which hyperbolic functions can be employed instead of the usual periodic functions in the perturbation procedure. The generalized van der Pol oscillator in which $f(\mu, x, \dot{x}) = (\mu + \mu_1 x - \mu_2 x^2)\dot{x}$ is studied. To illustrate the accuracy of the present method, its predictions are compared with those of Runge-Kutta method.

1. Introduction

With the classical perturbation methods such as Lindstedt–Poincaré (L–P) method, the average method, Krylov–Bogoliubov–Mitropolsky (KBM) method and the multiple scales methods [1,2] as the bases, new techniques have been presented in the last few decades for solving the periodic solution of the strongly nonlinear oscillator equation in the form of

$$\ddot{x} + g(x) = \varepsilon f(x, \dot{x}), \tag{1}$$

where $g(x)$ is a nonlinear function of x, $f(x, \dot{x})$ is a polynomial function of x and $\dot{x}$, ε is a small positive parameter. These techniques can be grouped into three categories with respect to the nature of periodic functions employed in the solutions.

The first category can be termed the circular (trigonometric) function perturbation procedures. They include the modified L–P method [3,4], the modified multiple scales method [5], the generalized average method [6], the generalized KBM method [7], etc. They are applicable when the generating equation:

$$\ddot{x} + g(x) = 0, \tag{2}$$

*Corresponding author. Tel.: +86 20 84110176.
E-mail address: stscsh@mail.sysu.edu.cn (S.H. Chen).

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2008.11.015

which is obtained by setting $\varepsilon = 0$ in Eq. (1), is linear, i.e.

$$\ddot{x} + c_1 x = 0. \tag{3}$$

As the solutions of Eq. (3) are trigonometric functions, the solution of Eq. (1) is expressed by trigonometric functions.

The second category can be termed the elliptic function perturbation procedures. They include the elliptic KB method [8,9], the elliptic average method [10,11], the elliptic L–P method [12,13] and the modified elliptic L–P method [14], the elliptic perturbation method [15,16], the elliptic KBM method [17], etc. They are applicable when the generating equation is a nonlinear differential equation with quadratic or cubic nonlinearity, i.e.,

$$\ddot{x} + c_1 x + c_m x^m = 0 \quad \text{where } m = 2 \text{ or } 3. \tag{4}$$

As the solutions of Eq. (4) are elliptic functions, the solution of Eq. (1) is expressed by elliptic functions.

The third category can be termed the generalized harmonic function perturbation procedures. They include the generalized harmonic average method [18], the generalized harmonic L–P method [19], the generalized harmonic KBM method [20], the generalized harmonic multiple scales method [21], etc. These methods are applicable when the generating equation contains an arbitrary nonlinear function $g(x)$. Hence, the solutions of the generating equation are the generalized harmonic functions and the solution of Eq. (1) is expressed by the generalized harmonic functions.

Besides the periodic solution of the strongly nonlinear oscillator described by Eq. (1), much effort has been paid to investigate its stability and bifurcation of the limit cycles and to predict its homoclinic bifurcation value. For example, Xu et al. [22] presented a perturbation-incremental method to study the separatrices and the limit cycles of strongly nonlinear oscillators. Chan et al. [23] used the perturbation-incremental method to study the stability and the bifurcations of limit cycles. Chen et al. [24] used the perturbation-incremental method to study the semi-stable limit cycles. Recently, Belhaq and his coworkers presented the elliptic averaging method [25] and the elliptic Linstedts–Poincaré method [26] to predict the homoclinic bifurcations of oscillators. Their techniques lead to the same results as the standard Melnikov technique.

In this paper, a new perturbation method is presented to determine the homoclinic orbits of the following quadratic nonlinear system:

$$\ddot{x} + c_1 x + c_2 x^2 = \varepsilon f(\mu, x, \dot{x}). \tag{5}$$

Noting that the homoclinic orbit is not periodic when the time $t \to +\infty$ or $t \to -\infty$, the phase point on the phase path in the phase portrait approaches the same saddle point. After proofing that the infinite periodic solution of the generating equation can be expressed by hyperbolic functions, hyperbolic functions instead of the usual elliptic functions are selected as the basis functions throughout the perturbation procedure. To show the essence and the effectiveness of the present method, typical examples of the generalized van der Pol equations are presented. Predictions from Runge–Kutta (R–K) integration method and the present methods are compared. It can be seen that the present method possesses excellent accuracy.

2. Homoclinic solution of the generating equation

We start from solving the homoclinic solution of the generating equation of Eq. (5), i.e.,

$$\ddot{x} + c_1 x + c_2 x^2 = 0. \tag{6}$$

Integral of the equation is

$$\tfrac{1}{2}\dot{x}^2 + V(x) = E, \tag{7}$$

where $\dot{x}^2/2$ is kinetic energy, $V(x)$ is potential energy and E is total energy of the system. In particular,

$$V(x) = \int_0^x (c_1 x + c_2 x^2)\,\mathrm{d}x = \frac{1}{2}c_1 x^2 + \frac{1}{3}c_2 x^3. \tag{8}$$

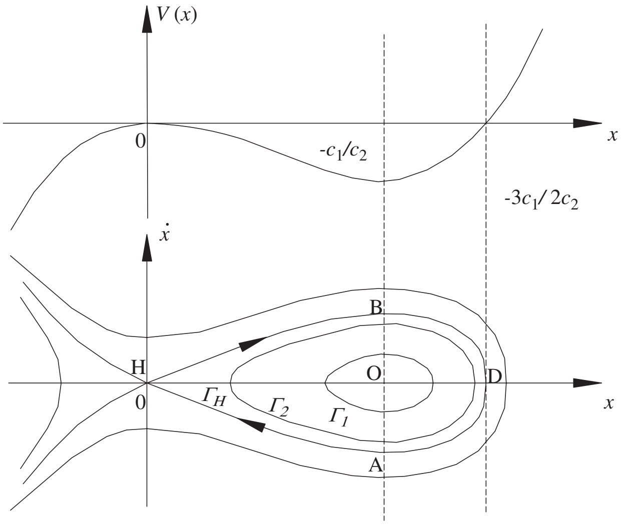

Fig. 1. The potential energy curves and phase portraits of Eq. (6) for $c_1<0$, $c_2>0$.

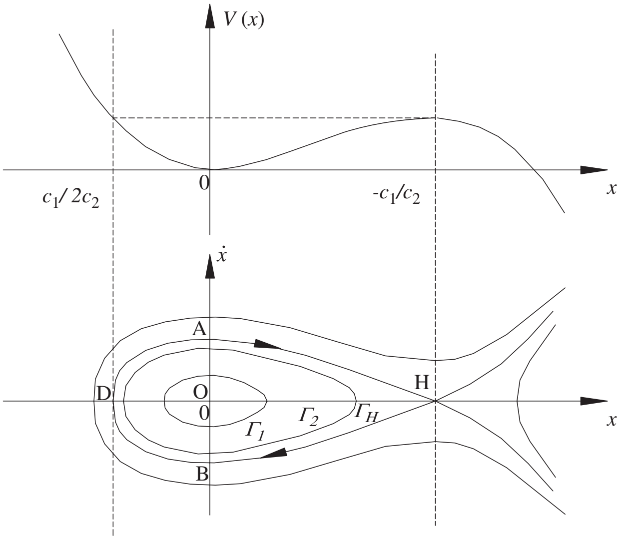

Fig. 2. The potential energy curves and phase portraits of Eq. (6) for $c_1>0$, $c_2<0$.

The potential energy curves and phase portraits of the systems with ($c_1<0$, $c_2>0$) and ($c_1>0$, $c_2<0$) are shown in Figs. 1 and 2, respectively. For $c_1<0$, $c_2>0$, the saddle point H is at (0,0) and the center O is at $(-c_1/c_2,0)$. For $c_1>0$, $c_2<0$, the saddle point H is at $(-c_1/c_2,0)$ and the center O is at point (0,0). In the figures, Γ_{H} marked with arrow heads are the homoclinic orbits. When time $t\to+\infty$, the phase point approaches saddle point H along DAH. When time $t\to-\infty$, the phase point approaches the same saddle point H but along the reverse direction. Γ_1 and Γ_2 are periodic orbits around the center O. The phase portraits in the other two cases ($c_1<0$, $c_2<0$) and ($c_1>0$, $c_2>0$) are similar to Figs. 1 and 2, respectively, only the position of the saddle point and center are different.

Chen et al. [13] obtained the exact periodic solution for Eq. (6). The solution can be expressed as

$$x_0 = a_0\,\mathrm{cn}^2(\tau, k) + b_0, \tag{9}$$

$$\tau = \omega_0 t, \tag{10}$$

$$a_0 = 6\omega_0^2 k^2/2, \tag{11}$$

$$b_0 = -[4\omega_0^2(2k^2 - 1) + c_1]/2c_2, \tag{12}$$

$$\omega_0^4 = c_1^2/16(k^4 - k^2 + 1), \tag{13}$$

where $\mathrm{cn}(\tau,k)$ is the cosine Jacobian elliptic function, a_0 is the amplitude, ω_0 is the angular frequency, k is the modulus of the elliptic function and b_0 is the bias. The time derivative of x_0 is

$$\dot{x}_0 = -2a_0\omega_0\,\mathrm{cn}(\tau, k)\,\mathrm{sn}(\tau, k)\,\mathrm{dn}(\tau, k), \tag{14}$$

where $\mathrm{sn}(\tau,k)$ is the sine Jacobian elliptic function and $\mathrm{dn}(\tau,k)$ is the delta Jacobian elliptic function. Obviously, x_0 is periodic when $0<k<1$. The solution becomes the trivial solution when $k=0$. When $k=1$, the elliptic functions will reduce to the hyperbolic functions.

The solution (9)–(13) represent the closed orbits around the orbital center O as shown in Figs. 1 and 2. Each closed orbit corresponds to a periodic solution. The constants a_0, b_0, ω_0 and k can be determined by the initial conditions $x(0) = a_0 + b_0 = d$ and Eqs. (11)–(13).

The objective of this paper is to find the solution of the homoclinic orbit. The coordinates of the saddle point depend on the value of c_1 and c_2. Two cases should be considered.

Case 1: $c_1<0$, $c_2>0$. In this case, the saddle point is H(0,0). Thus, it is required that

$$\lim_{\tau\to\infty} x_0 = 0, \tag{15}$$

$$\lim_{\tau\to-\infty} x_0 = 0 \tag{16}$$

which are equivalent to the following conditions:

$$x_0 = a_0\,\mathrm{cn}^2(\tau, k) + b_0 = 0, \tag{17}$$

$$\dot{x}_0 = -2a_0\omega_0\,\mathrm{cn}(\tau, k)\,\mathrm{sn}(\tau, k)\,\mathrm{dn}(\tau, k) = 0. \tag{18}$$

To satisfy Eq. (18), one of $\mathrm{cn}(\tau,k)$, $\mathrm{sn}(\tau,k)$ and $\mathrm{dn}(\tau,k)$ must vanish.

The condition $\mathrm{cn}(\tau,k) = 0$ and Eq. (17) lead to

$$b_0 = 0. \tag{19}$$

From Eqs. (12) and (13), it can be obtained that $k=0$ or $k=1$.

The condition $\mathrm{sn}(\tau,k)=0$ leads to $\mathrm{cn}^2(\tau,k)=1$. Thus

$$a_0 + b_0 = 0. \tag{20}$$

From Eqs. (11)–(13), it can be obtained that $k=0$.

The condition $\mathrm{dn}(\tau,k)=0$ leads to $\mathrm{cn}^2(\tau,k) = (k^2-1)/k^2$. Thus

$$a_0(k^2 - 1)/k^2 + b_0 = 0. \tag{21}$$

From Eqs. (11)–(13), $k=1$ can be obtained.

Case 2: $c_1>0$, $c_2<0$. In this case, the saddle point is H($-c_1/c_2$,0). Hence, it is requited that

$$\lim_{\tau\to\infty} x_0 = -c_1/c_2, \tag{22}$$

$$\lim_{\tau\to-\infty} x_0 = -c_1/c_2, \tag{23}$$

which are equivalent to the following conditions:

$$x_0 = a_0 \,\mathrm{cn}^2(\tau, k) + b_0 = -c_1/c_2, \tag{24}$$

$$\dot{x}_0 = -2a_0\omega_0 \,\mathrm{cn}(\tau, k)\,\mathrm{sn}(\tau, k)\,\mathrm{dn}(\tau, k) = 0. \tag{25}$$

Same as Case 1, the conditions require that k must be equal to either 0 or 1.

Since $k = 0$ corresponds to the trivial solution, the homoclinic solution of Eq. (6) requires $k = 1$.

It is known [27] that

$$\mathrm{cn}(\tau, 1) = \mathrm{dn}(\tau, 1) = \mathrm{sech}\,\tau, \tag{26}$$

$$\mathrm{sn}(\tau, 1) = \tanh\tau. \tag{27}$$

Hence, the homoclinic solution of Eq. (6) can be obtained by letting $k = 1$ in Eqs. (9)–(13) i.e.

$$x_0 = a_0 \,\mathrm{sech}^2\,\tau + b_0, \tag{28}$$

$$\dot{x}_0 = -2a_0\omega_0 \,\mathrm{sech}^2\,\tau \tanh\tau, \tag{29}$$

$$\tau = \omega_0 t, \tag{30}$$

$$\omega_0^2 = |c_1|/4, \tag{31}$$

$$a_0 = 3|c_1|/2c_2, \tag{32}$$

$$b_0 = -(|c_1| + c_1)/2c_2. \tag{33}$$

As $\lim_{\tau\to\infty} \mathrm{sech}\,\tau = \lim_{\tau\to-\infty} \mathrm{sech}\,\tau = 0$, it is obviously that $\lim_{\tau\to\infty} x_0 = \lim_{\tau\to-\infty} x_0 = b_0$. It can easily be proved that Eq. (28) satisfies Eq. (6) and the homoclinic conditions in (Eqs. (15) and (16)) or (Eqs. (22) and (23)).

3. The hyperbolic perturbation method

To demonstrate the hyperbolic perturbation procedure, we consider the nonlinear autonomous system of the form

$$\ddot{x} + c_1 x + c_2 x^2 = \varepsilon f(\mu, x, \dot{x}), \tag{34}$$

where ε is a small positive parameter, μ is a control parameter.

The periodic solution of this equation has been studied by Chen et al. [13] using the elliptic perturbation method. It is assumed that the approximate solution x_n ($n = 0,1,2,\ldots$) at each order has the form of the solution to its generating equation, i.e., Eq. (6). With this assumption, the cumbersome integral involving the elliptic function in determining x_n can be avoided. All the numerical results in Ref. [13] are in excellent agreement with those obtained by the R–K method, even if the value of parameter ε is moderately large. Hence, the elliptic perturbation method has been demonstrated to be an efficient method.

The purpose of this paper is focused on obtaining the homoclinic solution of Eq. (34). Similar perturbation procedure and assumption of the elliptic perturbation method will be employed in the present perturbation method.

When $\varepsilon = 0$, the homoclinic solution of Eq. (34) is the same as Eq. (28). When $\varepsilon \neq 0$, one can assume [13] that the homoclinic solution of Eq. (34) can still be written in the form of Eq. (28), i.e.

$$x = a \,\mathrm{sech}^2\,\tau + b_0. \tag{35}$$

However, a and τ will depend on the parameter ε. By expanding a in the powers of ε, i.e.

$$a = a_0 + \varepsilon a_1 + \cdots \tag{36}$$

and letting

$$\frac{d\tau}{dt} = \omega(\tau) = \omega_0 + \varepsilon\omega_1(\tau) + \cdots, \tag{37}$$

Eq. (35) can be rewritten as

$$x = x_0 + \varepsilon x_1 + \cdots, \tag{38}$$

where

$$x_n = a_n \operatorname{sech}^2 \tau + b_n \quad (n = 0, 1, \ldots; \; b_n = 0 \quad \text{for} \; n \geqslant 1). \tag{39}$$

After substituting Eqs. (37) and (38) into Eq. (34), equating coefficients will yield the following equations:

$$\varepsilon^0: \;\; \omega_0^2 x_0'' + c_1 x_0 + c_2 x_0^2 = 0, \tag{40}$$

$$\varepsilon^1: \;\; \omega_0 \frac{d}{d\tau}(\omega_1 x_0') + \omega_1 \frac{d}{d\tau}(\omega_0 x_0') + \omega_0 \frac{d}{d\tau}(\omega_0 x_1') + (c_1 + 2c_2 x_0)x_1 = f(\mu, x_0, \omega_0 x_0'), \tag{41}$$

$$\begin{aligned}\varepsilon^2: \omega_0 \frac{d}{d\tau}(\omega_0 x_0') + \omega_2 \frac{d}{d\tau}(\omega_0 x_0') + \omega_1 \frac{d}{d\tau}(\omega_0 x_2') + \omega_0 \frac{d}{d\tau}(\omega_1 x_1') + \omega_1 \frac{d}{d\tau}(\omega_0 x_1') + \omega_1 \frac{d}{d\tau}(\omega_1 x_0') \\ + (c_1 + 2c_2 x_0)x_2 + c_2 x_1^2 = f_x'(\mu, x_0, \omega_0 x_0')x_1 + f_{\dot{x}}'(\mu, x_0, \omega_0 x_0')(\omega_0 x_1' + \omega_1 x_0'), \;\; \text{etc.}\end{aligned} \tag{42}$$

in which $x' = dx/d\tau$, $f_x' = \partial f/\partial x$ and $f_{\dot{x}}' = \partial f/\partial \dot{x}$.

It can be seen that Eq. (40) is obtained from the Eq. (6) via the transformation in Eq. (37). Hence, the homoclinic solution of Eq. (40) can be expressed in the form of Eq. (28).

Multiplying both sides of Eq. (41) by x_0' and integrating it from τ_0 to τ, one can obtain

$$\omega_0\omega_1 x_0'^2\Big|_{\tau_0}^{\tau} = \int_{\tau_0}^{\tau} f(\mu, x_0, \omega_0 x_0')x_0' \, d\tau - \frac{a_1}{a_0}(\omega_0 x_0')^2\Big|_{\tau_0}^{\tau} - x_1(c_1 x_0 + c_2 x_0^2)\Big|_{\tau_0}^{\tau} \tag{43}$$

From Eq. (39),

$$x_n' = -2a_n \operatorname{sech}^2 \tau \tanh \tau. \tag{44}$$

Recalling that $x_n(\pm\infty) = b_n$, $x_n(0) = a_n + b_n$ ($b_n = 0$ for $n \geqslant 1$), $x_n'(0) = 0$ and letting $\tau_0 = -\infty$, $\tau = +\infty$ in Eq. (43), one obtains

$$\int_{+\infty}^{+\infty} f(\mu, x_0, \omega_0 x_0')x_0' \, d\tau = 0. \tag{45}$$

Letting $\tau_0 = 0$, $\tau = +\infty$ in Eq. (43), one obtains

$$\int_0^{+\infty} f(\mu, x_0, \omega_0 x')x_0' \, d\tau + a_1[c_1(a_0 + b_0) + c_2(a_0 + b_0)^2] = 0. \tag{46}$$

Furthermore, substitution of Eq. (28) into Eq. (43) leads to

$$\int_0^{\tau} f(\mu, x_0, \omega_0 x_0')x_0' \, d\tau - \omega_0\omega_1 x_0'^2 - \frac{a_1}{a_0}(\omega_0 x_0')^2 - x_1(c_1 x_0 + c_2 x_0^2) + a_1[c_1(a_0 + b_0) + c_2(a_0 + b_0)^2] = 0. \tag{47}$$

The values of μ, a_1 and ω_1 can be determined from Eqs. (45), (46) and (47), respectively. Therefore, the necessary condition for the homoclinic solution being the solution of the nonlinear autonomous system in Eq. (34) is that μ in Eq. (45) can be solved.

One can follow the perturbation procedure to determine the next order solution x_2 and ω_2. However, the procedure would be increasingly cumbersome as the order goes up. More importantly, the computation results show that the solution up to the order εx_1 is fairly accurate even for the moderately large parameter ε. A few remarks on the hyperbolic perturbation method are made.

Remark 1. Eq. (45), with which the value of parameter μ can be determined, is derived step-by-step in the present perturbation procedure. This result agrees with that obtained by Belhaq et al. [26] using both the elliptic Lindstedt–Poincaré method and the Melnikov method. The same result obtained by different methods demonstrates that the derivation of formulae in the present perturbation procedure is correct. Thus, the hyperbolic perturbation method is reliable.

Remark 2. Comparing with the elliptic Lindstedt–Poincaré method [13], the main distinguishing feature of the present method is that the hyperbolic functions are employed instead of the elliptic functions. The differential and integral operations on the hyperbolic functions are easier than those of the elliptic functions.

Remark 3. With the assumption in (35) and the nonlinear time transformation (37), the operations to determine $x_1(\tau)$ and $\dot{x}_1(\tau)$ in the present method by calculating the constant a_1 and $\omega_1(\tau)$ are more straight forward than those in other perturbation procedures.

4. A study of the generalized van der Pol oscillator

As an application of the present method, the following generalized van der Pol equation is studied:

$$\ddot{x} + c_1 x + c_2 x^2 = \varepsilon(\mu + \mu_1 x - \mu_2 x^2)\dot{x}. \tag{48}$$

In other words,

$$f(\mu, x, \dot{x}) = (\mu + \mu_1 x - \mu_2 x^2)\dot{x} \tag{49}$$

in which μ_1 and μ_2 are constants whilst μ is considered as a control parameter. Let

$$I(\tau) = \int f(\mu, x_0, \omega_0 x_0')x_0' \, \mathrm{d}\tau = \int \omega_0(\mu + \mu_1 x_0 - \mu_2 x_0^2)x_0'^2 \, \mathrm{d}\tau \tag{50}$$

in which x_0 is the zero order solution. By expressing x_0, $\dot{x}_0$ and ω_0 in the form of Eqs. (28), (29) and (31), respectively, and substituting them into Eq. (50), the latter becomes

$$I(\tau) = 4\alpha_0 a_0^2(A_1 + A_2 \operatorname{sech}^2 \tau + A_3 \operatorname{sech}^4 \tau + A_4 \operatorname{sech}^6 \tau)\tanh^3 \tau, \tag{51}$$

where

$$A_1 = \frac{2}{15}\mu + \left(\frac{8}{105}a_0 + \frac{2}{15}b_0\right)\mu_1 - \left(\frac{16}{315}a_0^2 + \frac{2}{15}b_0^2 + \frac{16}{105}a_0 b_0\right)\mu_2, \tag{52}$$

$$A_2 = \frac{1}{5}\mu + \left(\frac{4}{35}a_0 + \frac{1}{5}b_0\right)\mu_1 - \left(\frac{8}{105}a_0^2 + \frac{1}{5}b_0^2 + \frac{8}{35}a_0 b_0\right)\mu_2 = \frac{3}{2}A_1, \tag{53}$$

$$A_3 = \frac{1}{7}a_0\mu_1 - \left(\frac{2}{7}a_0 b_0 + \frac{2}{21}a_0^2\right)\mu_2, \tag{54}$$

$$A_4 = -\frac{1}{9}a_0^2\mu_2. \tag{55}$$

Thus, Eq. (45) can be rewritten as:

$$I(\tau)|_{-\infty}^{+\infty} = 0. \tag{56}$$

To have the equation satisfied, A_1 should be zero, i.e.

$$A_1 = 0. \tag{57}$$

Thus, from Eq. (53), one obtains

$$A_2 = 0. \tag{58}$$

From Eq. (57), one can determine the parameter with which the generalized van der Pol equation has a homoclinic orbit.

Substituting Eq. (50) into Eq. (46) gives

$$a_1 = -\frac{I(\tau)|_0^{+\infty}}{c_1(a_0+b_0)+c_2(a_0+b_0)^2}. \tag{59}$$

With the condition in Eq. (53),

$$a_1 = 0. \tag{60}$$

By incorporating Eqs. (50), (51), (57) and (60) into Eq. (47),

$$\omega_1(\tau) = \frac{I(\tau)}{\alpha_0 x_0'^2} = (A_3 + A_4 \operatorname{sech}^2\tau)\tanh\tau \tag{61}$$

Finally, the homoclinic solution of Eq. (48) can be expressed as

$$x = a_0 \operatorname{sech}^2\tau + b_0 + O(\varepsilon^2), \tag{62}$$

$$\dot{x} = -2a_0[\omega_0 + \varepsilon\omega_1(\tau)]\operatorname{sech}^2\tau\tanh\tau + O(\varepsilon^2). \tag{63}$$

5. Examples

In this section, three examples will be considered. To show the efficacy and accuracy of the present method, its results will be compared with those obtained by R–K method.

Example 1. In this example, the following equation is considered:

$$\ddot{x} + 2x - x^2 = \varepsilon(\mu + x - x^2)\dot{x}, \tag{64}$$

which is a case of the oscillator in Eq. (48) with $c_1 = 2$, $c_2 = -1$ and $\mu_1 = \mu_2 = 1$. From Eqs. (31), (32), (33) and (57), $a_0 = -3$, $b_0 = 2$, $\omega_0 = 1/\sqrt{2}$ and $\mu = 0.2857142857$ can be obtained. Through Eqs. (54) and (55), $A_3 = 3/7$ and $A_4 = -1$. The homoclinic solution of Eq. (64) is solved to be

$$x = -3\operatorname{sech}^2\tau + 2 + O(\varepsilon^2), \quad \dot{x} = 6\left[\frac{1}{\sqrt{2}} + \varepsilon\omega_1(\tau)\right]\operatorname{sech}^2\tau\tanh\tau + O(\varepsilon^2),$$

$$\omega_1(\tau) = \left(\frac{3}{7} - \operatorname{sech}^2\tau\right)\tanh\tau.$$

The homoclinic orbits for the cases of $\varepsilon = 0.5$ and 1.5 are shown in Figs. 3(a) and (b) respectively. It can be seen that the saddle point is (2,0) and the center is (0,0). The limit cycles obtained by the elliptic perturbation method [13] at $k = 0.6$ and 0.8 are also shown in the figures for illustrating the features of the phase portraits. Comparisons are made with R–K method.

In this paper, the procedure of using R–K integration method to determine the value of parameter μ of the homoclinic orbit follows that of Merkin and Needham [28]. Numerical integration is conducted for a given value of ε starting from a value of μ with which there is a limit cycle. It is repeated for increasing or reducing μ until a value of μ is reached such that there is no limit cycle. Then, by successfully reducing the interval of μ within which a limit cycle is destroyed, a critical value μ_c can be identified such that a limit cycle can be found at $\mu = \mu_c$ but not at $\mu = \mu_c \pm \Delta$ where Δ is a small preset tolerance. Here, Δ is taken to be 10^{-10}. Using this trial and error approach, $\mu_c = 0.2870602991$ when $\varepsilon = 0.5$ in Eq. (63). The value is very closed to but slightly smaller than the value $\mu_c = 0.2857142857$ obtained by the present hyperbolic perturbation method.

Example 2. In this example, the following equation is considered:

$$\ddot{x} + x + x^2 = \varepsilon(\mu - x - x^2)\dot{x} \tag{65}$$

which is a case of the oscillator in Eq. (48) with $c_1 = c_2 = 1$, $\mu_1 = -1$ and $\mu_2 = 1$. From Eqs. (31), (32), (33) and (57), $a_0 = 3/2$, $b_0 = -1$, $\omega_0 = 1/2$ and $\mu = 0$. Through Eqs. (54) and (55), $A_3 = 0$ and $A_4 = -1/4$.

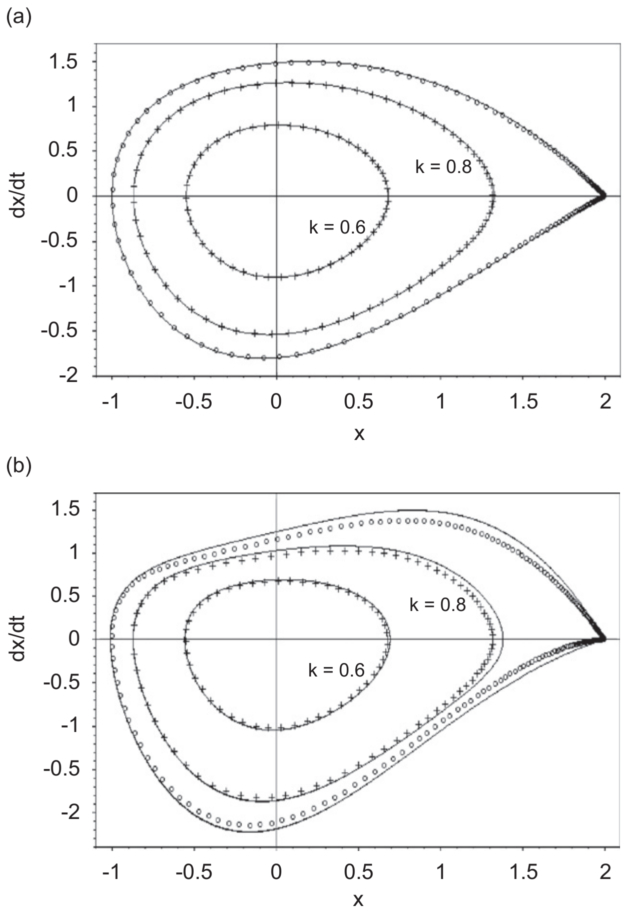

Fig. 3. Homoclinic orbits and limit cycles of Eq. (64) for (a) $\varepsilon = 0.5$ and (b) $\varepsilon = 1.5$. ○ ○ ○ denotes the homocyclic orbit predicted by the present method; + + + denotes the limit cycle predicted by the elliptical perturbation method for $k = 0.6$ and 0.8; — denotes the limit cycle predicted by Runge–Kutta method.

The homoclinic solution of Eq. (65) is solved to be

$$x = \tfrac{3}{2}\operatorname{sech}^2\tau - 1 + O(\varepsilon^2), \quad \dot{x} = -3\left[\tfrac{1}{2} + \varepsilon\omega_1(\tau)\right]\operatorname{sech}^2\tau\tanh\tau + O(\varepsilon^2),$$

$$\omega_1(\tau) = -\tfrac{1}{4}\operatorname{sech}^2\tau\tanh\tau.$$

The homoclinic orbits for the cases of $\varepsilon = 0.5$ and 2 are shown in Figs. 4(a) and (b), respectively. The saddle point is $(-1{,}0)$, while the center is (0,0). The limit cycles obtained by the elliptic perturbation method [13] for the cases $k = 0.75$ and 0.85 are also shown in the figures to illustrate the features of phase portraits. Comparisons are made with R–K method.

Example 3. In this example, the following equation is considered:

$$\ddot{x} - x - 2x^2 = \varepsilon(\mu + 0.5x - x^2)\dot{x} \tag{66}$$

which is a case of the oscillator in Eq. (48) with $c_1 = -1$, $c_2 = 2$, $\mu_1 = 0.5$ and $\mu_2 = 1$. From Eqs. (31), (32), (33) and (57), $a_0 = -3/4$, $b_0 = 0$, $\omega_0 = 1/2$ and $\mu = 0.4285714290$. Through Eqs. (54) and (55), $A_3 = -3/28$

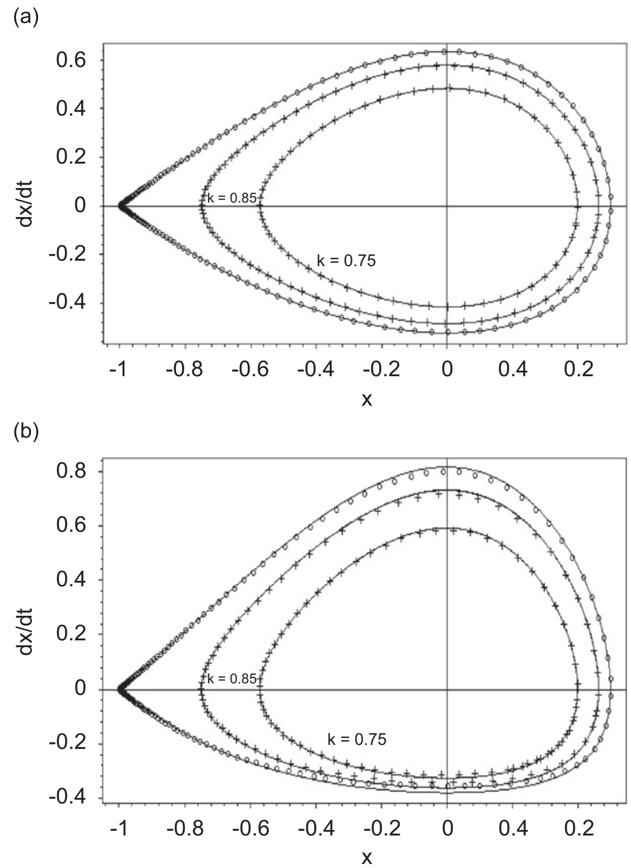

Fig. 4. Homoclinic orbits and limit cycles of Eq. (65) for (a) $\varepsilon = 0.5$; (b) $\varepsilon = 2$. ○ ○ ○ denotes the homocyclic orbit predicted by the present method; + + + denotes the limit cycle predicted by the elliptical perturbation method for $k = 0.75$ and 0.85; — denotes the limit cycle predicted by Runge–Kutta method.

and $A_4 = -1/16$. Then, the homoclinic solution of Eq. (66) is solved to be

$$x = -\tfrac{3}{4}\operatorname{sech}^2\tau + O(\varepsilon^2), \quad \dot{x} = \tfrac{3}{2}\left[\tfrac{1}{2} + \varepsilon\omega_1(\tau)\right]\operatorname{sech}^2\tau\tanh\tau + O(\varepsilon^2),$$

$$\omega_1(\tau) = -\left(\tfrac{3}{28} + \tfrac{1}{16}\operatorname{sech}^2\tau\right)\tanh\tau.$$

The homoclinic orbits for the cases of $\varepsilon = 0.8$ and 1.5 are shown in Figs. 5(a) and (b), respectively. The saddle point is (0, 0) while the center is (−0.5, 0). The limit cycle phase portraits obtained by the elliptic perturbation method [13] for the cases $k = 0.75$ and 0.85 are also shown in the figures. Comparisons are made with R–K method.

It can be seen from the Figs. 3–5 plotted for Examples 1–3 that all the results obtained by the elliptical perturbation method are in good agreement with those obtained by the R–K method even for the moderately large value of ε. The homoclinic orbit is closed to the result of R–K method at the critical value of $\mu = \mu_c$. Nevertheless, it is worth pointing out that the result of R–K method at $\mu = \mu_c$ is still a limit cycle which is a periodic solution. After that, a homoclinic orbit, which is a solution with infinite period, is formed by destruction of the limit cycle.

6. Conclusions

(1) The hyperbolic perturbation method presented in this paper is an effective method for determining homoclinic solutions of certain strongly nonlinear autonomous oscillators in which hyperbolic functions are the exact homoclinic solution of the generating equation. Based on the functions, the hyperbolic

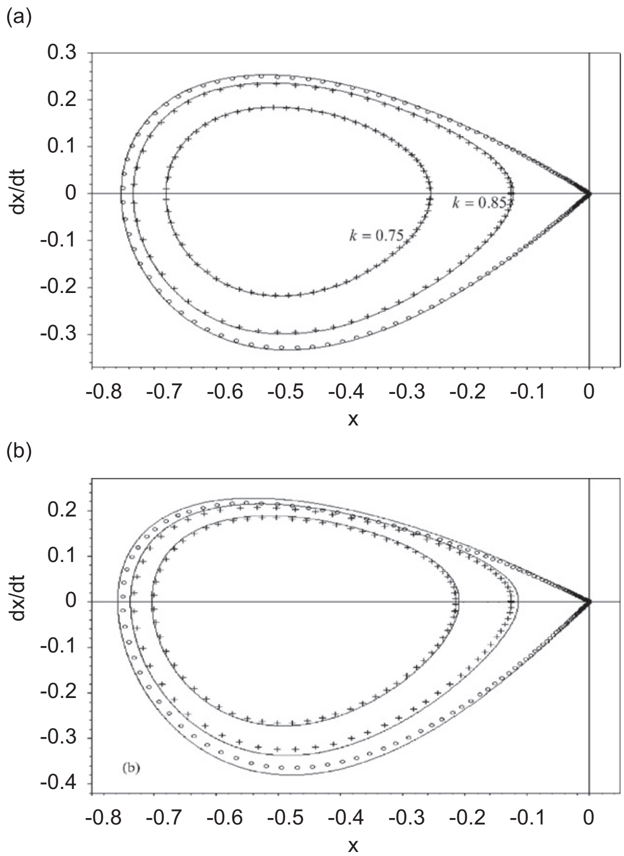

Fig. 5. Homoclinic orbits and limit cycles of Eq. (66) for (a) $\varepsilon = 0.8$; (b) $\varepsilon = 1.5$. ○ ○ ○ denotes the homocyclic orbit predicted by the present method; + + + denotes the limit cycle predicted by the elliptical perturbation method for $k = 0.75$ and 0.85; — denotes the limit cycle predicted by Runge–Kutta method.

perturbation method can lead to the analytical expression for the homoclinic solutions of the nonlinear autonomous oscillators.

(2) With the assumption that the approximate solution x_n ($n = 0,1,2,\ldots$) at each order has the same form as the solution of its generating equation and the nonlinear time transformation in (37), the operations for determining $x_1(\tau)$ and $\dot{x}_1(\tau)$ in the present method is more straight forward than those in other perturbation procedures.

(3) All the homoclinic orbits obtained by present method in the examples are closed to those obtained by R–K method at the critical parameter $\mu = \mu_c$ even for moderately large value of ε.

(4) The present hyperbolic perturbation method can be generalized for determining the heteroclinic solutions of certain strongly nonlinear autonomous oscillators.

Acknowledgement

The financial supports from the National Natural Science Foundation of China (10672193 and 10272117) and the University of Hong Kong (CRGC grant) are gratefully acknowledged.

References

[1] A.H. Nayfeh, *Introduction to Perturbation Techniques*, Wiley, New York, 1981.
[2] R.E. Mickens, *Nonlinear Oscillations*, Cambridge University Press, New York, 1981.

[3] Y.K. Cheung, S.H. Chen, S.L. Lau, A modified Lindstedt–Poincaré method for certain strongly nonlinear oscillators, *International Journal of Non-linear Mechanics* 26 (3/4) (1991) 367–378.
[4] S.H. Chen, Y.K. Cheung, A modified Lindstedt–Poincare method for a strongly non-linear two degree-of-freedom system, *Journal of Sound and Vibration* 193 (4) (1996) 751–762.
[5] T.D. Burton, Z. Rahman, On the multi-scale analysis of strongly non-linear forced oscillators, *International Journal of Non-linear Mechanics* 21 (2) (1986) 135–146.
[6] J.P. Ottoy, A perturbation method for a set of purely non-linear differential equations, *International Journal of Control* 30 (4) (1979) 587–595.
[7] S.Q. Dai, Asymptotic analysis of strongly non-linear oscillator, *Applied Mathematics and Mechanics (English Edition)* 6 (1985) 409–415.
[8] P.G.D. Barkham, A.C. Souback, An extension to the method of Kryloff and Bogoliuboff, *International Journal of Control* 10 (1969) 337–392.
[9] S.B. Yuste, J.D. Bejarano, Extension and improvement to the Krylov–Bogoliubov methods using elliptic functions, *International Journal of Control* 49 (1989) 1127–1141.
[10] V.T. Coppola, R.H. Rand, Averaging using elliptic function: approximation of limit cycles, *Acta Mechanica* 81 (1990) 125–142.
[11] R.V. Roy, Averaging method for strongly non-linear oscillators with periodic excitations, *International Journal of Non-Linear Mechanics* 29 (1994) 737–753.
[12] S.H. Chen, Y.K. Cheung, An elliptic Lindstedt–Poincaré method for certain strongly non-linear oscillators, *Nonlinear Dynamics* 12 (1997) 199–213.
[13] S.H. Chen, X.M. Yang, Y.K. Cheung, Periodic solutions of strongly quadratic non-linear oscillators by the elliptic Lindstedt–Poincaré method, *Journal of Sound and Vibration* 227 (1999) 1109–1118.
[14] C.H. Yang, S.M. Zhu, S.H. Chen, A modified elliptic Lindstedt–Poincaré method for certain strongly non-linear oscillators, *Journal of Sound and Vibration* 273 (2004) 921–932.
[15] S.H. Chen, Y.K. Cheung, An elliptic perturbation method for certain strongly non-linear oscillators, *Journal of Sound and Vibration* 192 (1996) 453–464.
[16] S.H. Chen, X.M. Yang, Y.K. Cheung, Periodic solutions of strongly quadratic non-linear oscillators by the elliptic perturbation method, *Journal of Sound and Vibration* 212 (1998) 771–780.
[17] J.P. Otty, Study of a non-linear perturbed oscillator, *International Journal of Control* 32 (3) (1980) 475–487.
[18] Z. Xu, Y.K. Cheung, Averaging method using generalized harmonic functions for strongly non-linear oscillators, *Journal of Sound and Vibration* 174 (1994) 563–576.
[19] Z. Xu, Non-linear time transformation method for strongly nonlinear oscillation systems, *Acta Mechanica Sinica (English Edition)* 8 (1992) 279–288.
[20] Z. Xu, L. Zhang, Asymptotic method for analysis of nonlinear systems with two parameters, *Acta Mathematica Scientia (English Edition)* 6 (1986) 453–462.
[21] Z. Xu, Y.K. Cheung, Non-linear scales method for strongly non-linear oscillators, *Nonlinear Dynamics* 7 (1995) 285–289.
[22] Z. Xu, H.S.Y. Chan, K.W. Chung, Separatrices and limit cycles of strongly nonlinear oscillators by the perturbation-incremental method, *Nonlinear Dynamics* 11 (1996) 213–233.
[23] H.S.Y. Chan, K.W. Chung, Z. Xu, Stability and bifurcations of limit cycles by the perturbation-incremental method, *Journal of Sound and Vibration* 206 (1997) 589–604.
[24] S.H. Chen, J.K.W. Chan, A.Y.T. Leung, A perturbation method for the calculation of semi-stable limit cycles of strongly nonlinear oscillators, *Communication in Numerical Methods in Engineering* 16 (2000) 301–313.
[25] M. Belhaq, F. Lakrad, Prediction of homoclinic bifurcation: the elliptic averaging method, *Chaos Solitons & Fractals* 11 (2000) 2251–2258.
[26] M. Belhaq, B. Fiedler, F. Lakrad, Homoclinic connections in strongly self-excited nonlinear oscillators: the Melnikov function and the elliptic Lindstedt–Poincaré method, *Nonlinear Dynamics* 23 (2000) 67–86.
[27] P.F. Byrd, M.D. Friedman, *Handbook of Elliptic Integrals for Engineers and Physicists*, Springer, Berlin, 1954.
[28] J.H. Merkin, D.J. Needham, On infinite period bifurcations with an application to roll waves, *Acta Mechanica* 60 (1986) 1–16.