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Nonlinear modes of parametric vibrations and their applications to beams dynamics

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Abstract

An iterative loop combining nonlinear modes and the Rauscher method is suggested for analyzing finite degree-of-freedom nonlinear mechanical systems with parametric excitation. This method is applied to an analysis of the parametric vibration of beams.

1. Introduction

Many methods exist for analyzing parametric vibrations of discrete nonlinear systems. The asymptotic methods (multiple-scales method, Van der Pol transformations, Melnikov method) are used for such analysis [1–4]. The parametric vibrations of essentially nonlinear systems can be analyzed by the harmonic balance method, continuation technique [5–7].

In this paper the method of parametric vibration analysis based on the combination of the Rausher method and nonlinear modes is suggested. Note this method can easily be used to analyze the parametric vibrations in the engineering systems with many degrees of freedom.

The Rauscher method is an effective tool for studying forced vibration. This method has been suggested for the analysis of single degree-of-freedom nonautonomous systems [8]. Let q be the general coordinates of such an oscillator. At first, the solution of the corresponding autonomous system is obtained. Let $q(t)$ be this solution. Then this solution is inverted into the form: $t = t(q)$. Using this function, the nonautonomous system is transformed into the autonomous system and this approximates the nonautonomous dynamical system. The generalization of the Rauscher method has been suggested in the book [9]. Chebyshev polynomials were used to obtain the approximate functions $t = t(q)$ in Ref. [10]. The existence of a function $t(q)$ for a wide class of dynamical systems was demonstrated in Ref. [11]. Rosenberg [11] used the Rauscher method for the

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qualitative analysis of a one-degree-of-freedom dynamical system. Manevitch et al. [7] suggested the combination of the Rauscher method and the NNMs to analyze discrete systems with an arbitrary number of dof. The forced vibrations close to rectilinear NNMs of a two-degree-of-freedom system were studied by means of the Rauscher method in Ref. [6]. A nonlinear two-degree-of-freedom system describing the interaction of the linear subsystem and the snap-through truss, has been investigated by the Rauscher approach [12].

Note that all the above-mentioned publications considered the Rauscher method jointly with the Kauderer–Rosenberg nonlinear modes to represent motions in the configuration space. In this paper, the Rauscher method is combined with the nonlinear modes, these being two-dimensional invariant manifolds. These normal modes have been suggested by Shaw and Pierre [13] and Shaw et al. [14]. This is the novelty of the method presented in this paper.

Nonlinear modes are effective tools for solving engineering problems. They have been used for the analysis of problems of absorption of mechanical vibrations in the papers [15,16]. Nonlinear modes are used to analyze the nonlinear vibrations of rotating pre-twisted beams [17] and to analyze the dynamics of a shallow arch [18].

This paper is organized as follows. A general method for the analysis of parametric vibration combining the Rauscher method and nonlinear modes is considered in the second section. The third and the fourth sections contain the application of this method to beams dynamics.

2. An iterative approach consisting of the Rausher method and nonlinear modes

The nonlinear system performing parametric vibrations is considered in the following form:

$$\ddot{\xi}_j + \omega_j^2 \xi_j + F_j\left(\xi_1, \ldots, \xi_n, \dot{\xi}_1, \ldots, \dot{\xi}_n\right) + \sum_{i=1}^{n} a_{ji} \xi_i \cos(2\Omega t) = 0, \quad j = \overrightarrow{1, n}, \tag{1}$$

where ω_i are eigenfrequencies of linear system; a_{ji} are constant parameters. In this paper the motions of system (1) close to the following nonlinear modes are considered:

$$\begin{aligned} &\xi_v = \xi_v(\xi_l, v_l) = a_1^{(v)} \xi_l^2 + a_2^{(v)} \xi_l v_l + \cdots, \\ &v_v = \dot{\xi}_v = v_v(\xi_l, v_l) = b_1^{(v)} \xi_l^2 + b_2^{(v)} \xi_l v_l + \cdots, \\ &v = 1, \ldots, l-1, l+1, \ldots, n. \end{aligned} \tag{2}$$

Note that such motions can be observed in system (1) if there are no internal and combination resonances.

An iterative loop is constructed for calculating the motions close to the manifold (2). A similar iterative loop for forced vibrations analysis is considered in the paper [19]. At the first iteration it is assumed that $\xi_l \neq 0$, $v_l \neq 0$ and $\xi_v = v_v = 0$. In this case one equation is derived:

$$\ddot{\xi}_l + \omega_l^2 \xi_l + \tilde{F}_l\left(\xi_l, \dot{\xi}_l, \ddot{\xi}_l\right) + a_{ll} \xi_l \cos(2\Omega t) = 0. \tag{3}$$

The motions of system (3) can be presented in the following form:

$$\xi_l = A_0 + A_1 \cos(\Omega t) + B_1 \sin(\Omega t) + A_2 \cos(2\Omega t) + B_2 \sin(2\Omega t). \tag{4}$$

The harmonic balance method is used to analyze Eq. (3). Then the system of nonlinear algebraic equations with respect to the parameters (A_0, A_1, B_1, A_2, B_2, Ω) is derived. In general, this system has the following form:

$$\Phi_\mu(A_0, A_1, B_1, A_2, B_2, \Omega) = 0, \quad \mu = \overrightarrow{1, 5}. \tag{5}$$

The aim of the present analysis is calculation of a frequency response. Therefore, the parameter A_2 is set with appropriate step size and then the system (5) is solved. For every value of A_2, system (5) is solved with respect to (A_0, A_1, B_1, B_2, Ω). Now, it is assumed that these values are calculated and function (4) is determined.

For future analysis the following notation is used:

$$z_1 = \sin \Omega t, \quad z_2 = \cos \Omega t. \tag{6}$$

Then the following equations are obtained from Eq. (4):

$$\xi_l = A_0 + A_1 z_2 + B_1 z_1 + A_2\left(z_2^2 - z_1^2\right) + 2B_2 z_1 z_2,$$
$$\dot{\xi}_l = -A_1 \Omega z_1 + B_1 \Omega z_2 - 4\Omega A_2 z_1 z_2 + 2\Omega B_2\left(z_2^2 - z_1^2\right). \quad (7)$$

The solution of Eq. (7) with respect to z_1, z_2 is presented in the form of a series:

$$z_1 = \alpha_0 + \alpha_1 \xi_l + \alpha_2 \dot{\xi}_l + \alpha_3 \xi_l^2 + \alpha_4 \dot{\xi}_l^2 + \alpha_5 \xi_l \dot{\xi}_l + \cdots,$$
$$z_2 = \beta_0 + \beta_1 \xi_l + \beta_2 \dot{\xi}_l + \beta_3 \xi_l^2 + \beta_4 \dot{\xi}_l^2 + \beta_5 \xi_l \dot{\xi}_l + \cdots, \quad (8)$$

where $\alpha_0, \alpha_1, \ldots, \beta_0, \beta_1, \ldots$ are unknown coefficients. Eq. (7) is substituted into Eq. (8) and the coefficients of the same terms $\xi_l^{j_1} \dot{\xi}_l^{j_2}$; $j_1 + j_2 = 2$; $j_1 + j_2 = 3, \ldots$; $j_1 = 0, 1, \ldots$; $j_2 = 0, 1, \ldots$ are equated. As a result two systems of linear algebraic equations are derived. These systems can be presented in the following form:

$$[\mathbf{M}]\mathbf{A} = \mathbf{R}_1,$$
$$[\mathbf{M}]\mathbf{B} = \mathbf{R}_2, \quad (9)$$

where $\mathbf{R}_1^{\mathrm{T}} = [0,0,1,0,0,0]$; $\mathbf{R}_2^{\mathrm{T}} = [0,0,0,0,1,0]$; $\mathbf{A}^{\mathrm{T}} = [\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5]$; $\mathbf{B}^{\mathrm{T}} = [\beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5]$;

$$\mathbf{M} = \begin{bmatrix} 0 & -A_2 & -2B_2\Omega & B_1^2 - 2A_0A_2 & A_1^2\Omega^2 & -B_1A_1\Omega - 2A_0B_2\Omega \\ 0 & 2B_2 & -4A_2\Omega & 4A_0B_2 + 2A_1B_1 & -2B_1\Omega^2 A_1 & B_1^2\Omega - 4A_0A_2\Omega - A_1^2\Omega \\ 0 & B_1 & -A_1\Omega & 2A_0B_1 & 0 & -A_0A_1\Omega \\ 0 & A_2 & 2B_2\Omega & 2A_0A_2 + A_1^2 & B_1^2\Omega^2 & B_1A_1\Omega + 2A_0B_2\Omega \\ 0 & A_1 & B_1\Omega & 2A_0A_1 & 0 & A_0B_1\Omega \\ 1 & A_0 & 0 & A_0^2 & 0 & 0 \end{bmatrix}.$$

Solving the two systems of linear algebraic Eq. (9), the truncated series (8) can be calculated. The following two equations are obtained from these series:

$$\sin(\Omega t) = S\left(\xi_l, \dot{\xi}_l\right),$$
$$\cos(\Omega t) = C\left(\xi_l, \dot{\xi}_l\right). \quad (10)$$

Relations (10) are substituted into Eq. (1) and as a result of this the pseudo-autonomous dynamical system is obtained. This system can be presented in the following form:

$$\ddot{\xi}_j + \omega_j^2 \xi_j + R_j\left(\xi_1, \ldots, \xi_n, \dot{\xi}_1, \ldots, \dot{\xi}_n, \ddot{\xi}_1, \ldots, \ddot{\xi}_n\right) = 0, \quad j = \overrightarrow{1, n}. \quad (11)$$

The solutions of the pseudo-autonomous dynamical system (11) are presented in the form of the nonlinear mode (2). Then, following [4], the functions $\xi_v(\xi_l, v_l)$; $v_v(\xi_l, v_l)$ are determined from the partial differential equations

$$v_v(\xi_l, v_l) = v_l \frac{\partial \xi_v}{\partial \xi_l} - \frac{\partial \xi_v}{\partial v_l}\left[\omega_l^2 \xi_l + R_l\left(\xi_1, \ldots, \xi_n, v_1(\xi_l, v_l), \ldots, v_n(\xi_l, v_l); -\omega_1^2 \xi_1; \ldots; -\omega_n^2 \xi_n\right)\right],$$
$$\omega_v^2 \xi_v(\xi_l, v_l) + R_v\left(\xi_1, \ldots, \xi_n, v_1(\xi_l, v_l), \ldots, v_n(\xi_l, v_l); -\omega_1^2 \xi_1; \ldots; -\omega_n^2 \xi_n\right) = -v_l \frac{\partial v_v}{\partial \xi_l} + \frac{\partial v_v}{\partial v_l}$$
$$\times \left[\omega_l^2 \xi_l + R_l\left(\xi_1, \ldots, \xi_n, v_1(\xi_l, v_l), \ldots, v_n(\xi_l, v_l); -\omega_1^2 \xi_1; \ldots; -\omega_n^2 \xi_n\right)\right]. \quad (12)$$

Eq. (2) is substituted into system of Eqs. (12) and the coefficients of the terms $\xi_l^{j_1} v_l^{j_2}$; $j_1 + j_2 = 2$; $j_1 + j_2 = 3$; $\ldots$; $j_1 = 0, 1, \ldots$; $j_2 = 0, 1, \ldots$ are equated. As a result the system of linear algebraic equations with respect to $a_1^{(v)}, a_2^{(v)}, \ldots, b_1^{(v)}, b_2^{(v)}, \ldots$ is derived. Solving this system, functions (2) are calculated.

Now the second iteration is considered. Function (2) is substituted into the lth equation of system (1). As a result one nonautonomous equation of the second order is obtained. In general, this equation has the following form:

$$\ddot{\xi}_l + \omega_l^2 \xi_l + F_l\left[\xi_1(\xi_l, v_l), \ldots, \xi_n(\xi_l, v_l); v_1(\xi_l, v_l), \ldots, v_n(\xi_l, v_l); -\omega_1^2 \xi_1(\xi_l, v_l); \ldots; -\omega_n^2 \xi_n(\xi_l, v_l)\right]$$
$$+ a_{ll}\xi_l \cos(2\Omega t) + \sum_{\substack{i=1 \\ i \neq l}}^{n} a_{li}\xi_i(\xi_l, v_l)\cos(2\Omega t) = 0. \quad (13)$$

The solution of Eq. (13) is presented in the form of Eq. (4). After this solution construction, Eq. (10) is determined and the normal mode of pseudo-autonomous dynamical system (11) is derived. These calculations are carried out according to the algorithm expressed by Eqs. (5)–(12). The second iteration is finished by means of nonlinear mode determination. If the coefficients $a_1^{(v)}, a_2^{(v)}, \ldots$ of two adjacent iterations are close, then the periodic motions for the particular value of A_2 is determined.

The values of A_2 are set with some incremental step value for calculating the frequency response of the periodic motions. For every value of A_2, calculations for periodic motions are carried out.

3. Parametric vibrations of beams

The vibration of a beam with a discrete end-mass is considered, as shown in Fig. 1. The vibrations of this system are described by the following partial differential equation [20]:

$$EJu^{(IV)} + 0.5EJ\left(u''u'^2\right)'' + u''\left\{P_t\cos(2\Omega t) - 0.5M\int_0^L \left(u'^2\right)''_{tt} \mathrm{d}s\right\} + \mu\ddot{u} - (Nu')' = 0,$$
$$N = 0.5\mu\int_s^l \mathrm{d}s_1 \int_0^{s_1}\left(u'^2\right)''_{tt}\mathrm{d}s_2, \quad (14)$$

where $\dot{u} = \partial u/\partial t$, $u' = \partial u/\partial s$, $u(s,t)$ is deflection of the beam; EJ is stiffness of the beam; $P_t\cos(\Omega t)$ is a longitudinal force; M is the mass attached to the beam; μ is the mass per unit length. The term $0.5EJ\left(u''u'^2\right)''$ defines the contribution of the nonlinearity to the beam curvature and the terms $0.5Mu''\int_0^L\left(u'^2\right)''_{tt}\mathrm{d}s$ and $(Nu')'$ describe the nonlinear inertia [21].

Dimensionless variables and parameters are considered for further analysis:

$$w = \frac{u}{r}, \quad \tau = t\sqrt{\frac{EJ}{\mu l^4}}, \quad \xi = \frac{s}{l}, \quad m = \frac{M}{\mu l}, \quad \chi = \frac{r^2}{l^2}, \quad f = \frac{l^2 P_t}{EJ},$$

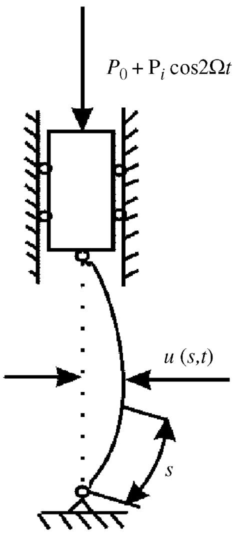

Fig. 1. Mechanical system.

where $r^2 = J/A$ is the radius of gyration of the cross section, and A is the area of the cross section. Then system (14) has the following form with respect to the dimensionless variables and parameters:

$$\begin{aligned} & w^{(IV)} + \frac{\chi}{2}\left(w'' w'^2\right)'' + w''\left\{f \cos(2\bar{\Omega}\tau) - \frac{m\chi}{2}\int_0^1 \left(w'^2\right)''_{\tau\tau} \mathrm{d}\xi\right\} + \ddot{w} - \chi(N_* w')' = 0, \\ & N_* = 0.5\int_\xi^1 \mathrm{d}\xi_1 \int_0^{\xi_1} \left(w'^2\right)''_{\tau\tau} \mathrm{d}\xi_2. \end{aligned} \tag{15}$$

The vibration of system (15) is defined by an eigenmode expansion for a simply supported beam:

$$w(x,t) = q_1(t)\sin(\pi\xi) + q_2(t)\sin(2\pi\xi).$$

Applying the Galerkin technique to system (15), a two-degree-of-freedom dynamical system is derived:

$$\begin{aligned} & \ddot{q}_1 + \pi^4 q_1 + \gamma_1 q_1^3 + \gamma_2 q_1 q_2^2 + q_1\left[2\gamma_3\left(\dot{q}_1^2 + q_1\ddot{q}_1\right) + 2\gamma_4\left(\dot{q}_2^2 + q_2\ddot{q}_2\right) + \gamma_5\left(q_1\ddot{q}_2 + q_2\ddot{q}_1 + 2\dot{q}_1\dot{q}_2\right)\right] \\ & \quad + q_2\left[2\gamma_7\left(\dot{q}_1^2 + q_1\ddot{q}_1\right) + 2\gamma_8\left(\dot{q}_2^2 + q_2\ddot{q}_2\right) + \gamma_6\left(q_1\ddot{q}_2 + q_2\ddot{q}_1 + 2\dot{q}_1\dot{q}_2\right)\right] - q_1 f_1 \cos(2\Omega\tau) = 0, \\ & \ddot{q}_2 + 16\pi^4 q_2 + \beta_1 q_2^3 + q_2\left[2\beta_3\left(\dot{q}_1^2 + q_1\ddot{q}_1\right) + 2\beta_4\left(\dot{q}_2^2 + q_2\ddot{q}_2\right) + \beta_5\left(q_1\ddot{q}_2 + q_2\ddot{q}_1 + 2\dot{q}_1\dot{q}_2\right)\right] \\ & \quad + q_1\left[2\beta_7\left(\dot{q}_1^2 + q_1\ddot{q}_1\right) + 2\beta_8\left(\dot{q}_2^2 + q_2\ddot{q}_2\right) + \beta_6\left(q_1\ddot{q}_2 + q_2\ddot{q}_1 + 2\dot{q}_1\dot{q}_2\right)\right] + \beta_2 q_2 q_1^2 - 4q_2 f_1 \cos(2\Omega\tau) = 0, \end{aligned} \tag{16}$$

where

$$\begin{aligned} & \gamma_1 = \frac{\chi\pi^6}{8}, \quad \gamma_2 = \chi\pi^6, \quad \gamma_3 = \chi\pi^2\left(\frac{\pi^2 m}{4} - \frac{3}{32} + \frac{\pi^2}{12}\right), \quad \gamma_4 = \chi\pi^2\left(\pi^2 m - \frac{5}{16} + \frac{\pi^2}{3}\right), \\ & \gamma_5 = \gamma_6 = \frac{10}{9}\chi\pi^2, \quad \gamma_7 = \frac{5}{9}\chi\pi^2, \quad \gamma_8 = \frac{20}{9}\chi\pi^2, \quad f_1 = \pi^2 f, \quad \beta_1 = 8\chi\pi^6, \quad \beta_2 = 4\chi\pi^6, \\ & \beta_3 = 2\chi\pi^2\left(\frac{\pi^2}{6} - \frac{5}{32} + \frac{\pi^2 m}{2}\right), \quad \beta_4 = 2\chi\pi^2\left(\frac{2\pi^2}{3} - \frac{3}{16} + 2\pi^2 m\right), \quad \beta_5 = \frac{40}{9}\chi\pi^2, \\ & \beta_6 = \frac{10}{9}\chi\pi^2, \quad \beta_7 = \frac{5}{9}\chi\pi^2, \quad \beta_8 = \frac{20}{9}\chi\pi^2. \end{aligned}$$

Now parametric vibrations of discrete system (16) are considered. It is assumed that the generalized coordinate q_1 shows a considerable vibration amplitude and that q_2 displays a correspondingly small amplitude. The method suggested in the previous section is used to accommodate such motions. At the first iteration it is assumed that $q_2 = 0$. System (16) takes the following form:

$$\ddot{q}_1 + \pi^4 q_1 + \gamma_1 q_1^3 + 2\gamma_3 q_1\left(\dot{q}_1^2 + q_1\ddot{q}_1\right) - f_1 q_1 \cos(2\Omega\tau) = 0. \tag{17}$$

The analysis of oscillator (17) in the asymptotic limit is considered in the Appendix A. Taking into account the results of this asymptotic analysis, periodic vibrations of system (17) is chosen in the following form:

$$q_1 = A_1 \cos \Omega t + B_1 \sin \Omega t. \tag{18}$$

The harmonic balance method is used to determine the parameters A_1, B_1. As a result, a system of two nonlinear algebraic equations with respect to A_1 and B_1 is derived:

$$\begin{aligned} & A_1\left\{-\Omega^2 + \pi^4 + (0.75\gamma_1 - \gamma_3\Omega^2)\left(A_1^2 + B_1^2\right) - 0.5 f_1\right\} = 0, \\ & B_1\left\{-\Omega^2 + \pi^4 + (0.75\gamma_1 - \gamma_3\Omega^2)\left(A_1^2 + B_1^2\right) + 0.5 f_1\right\} = 0. \end{aligned} \tag{19}$$

Two groups of solutions exist in system (19):

(I) $A_1 = 0$, $B_1 \neq 0$,
(II) $A_1 \neq 0$, $B_1 = 0$.

Group I is determined by the value B_1:

$$B_1 = \sqrt{\frac{\Omega^2 - \pi^4 - 0.5f_1}{0.75\gamma_1 - \gamma_3\Omega^2}} \tag{20}$$

and group II is characterized by the value A_1:

$$A_1 = \sqrt{\frac{0.5f_1 + \Omega^2 - \pi^4}{0.75\gamma_1 - \gamma_3\Omega^2}}. \tag{21}$$

Note that it is possible to obtain the frequency response oscillations of system (17) from Eqs. (20) and (21). In order to obtain the nonlinear mode of system (16), solution (18) is given as

$$\begin{aligned} \left(A_1^2 + B_1^2\right)\cos(\Omega t) &= A_1 q_1 + B_1\Omega^{-1}\dot{q}_1, \\ \left(A_1^2 + B_1^2\right)\sin(\Omega t) &= B_1 q_1 - A_1\Omega^{-1}\dot{q}_1. \end{aligned} \tag{22}$$

From Eq. (22) it can be shown that the following can be derived:

$$\cos(2\Omega t) = \alpha_1 q_1^2 + \alpha_2\dot{q}_1^2 + \alpha_3 q_1\dot{q}_1, \tag{23}$$

where

$$\alpha_1 = \frac{A_1^2 - B_1^2}{\left(A_1^2 + B_1^2\right)^2}, \quad \alpha_2 = \Omega^{-2}\frac{B_1^2 - A_1^2}{\left(A_1^2 + B_1^2\right)^2}, \quad \alpha_3 = \frac{4A_1B_1\Omega^{-1}}{\left(A_1^2 + B_1^2\right)^2}.$$

Eq. (23) is substituted into Eq. (16) and as a result of this the pseudo-autonomous dynamical system is derived. The nonlinear mode of this system can be presented in the following form:

$$\begin{aligned} q_2 = Q_2(q_1, v_1) &= a_1q_1^2 + a_2q_1v_1 + a_3v_1^2 + a_4q_1^3 + a_5q_1^2v_1 + a_6q_1v_1^2 + a_7v_1^3 + \cdots, \\ v_2 = V_2(q_1, v_1) &= b_1q_1^2 + b_2q_1v_1 + b_3v_1^2 + b_4q_1^3 + b_5q_1^2v_1 + b_6q_1v_1^2 + b_7v_1^3 + \cdots. \end{aligned} \tag{24}$$

Using the nonlinear mode approach of Shaw and Pierre [13] the following two partial differential equations are obtained:

$$\begin{aligned} &v_2(q_1, v_1) = v_1\frac{\partial Q_2}{\partial q_1} + \frac{\partial Q_2}{\partial v_1}\left[-\pi^4 q_1 - (\gamma_1 - f_1\alpha_1)q_1^3 - \gamma_2 q_1 Q_2^2(q_1, v_1) + f_1\alpha_2 q_1 v_1^2 + f_1\alpha_3 q_1^2 v_1 - \theta_1\right], \\ &- 16\pi^4 Q_2(q_1, v_1) + (4f_1\alpha_1 - \beta_2)Q_2(q_1, v_1)q_1^2 + 4f_1\alpha_2 Q_2(q_1, v_1)v_1^2 + 4f_1\alpha_3 Q_2(q_1, v_1)q_1v_1 \\ &\quad - \beta_1 Q_2^3 - \theta_2 = v_1\frac{\partial V_2}{\partial q_1} + \frac{\partial V_2}{\partial v_1}\left[-\pi^4 q_1 - (\gamma_1 - f_1\alpha_1)q_1^3 - \gamma_2 q_1 Q_2^2(q_1, v_1) + f_1\alpha_2 q_1 v_1^2 + f_1\alpha_3 q_1^2 v_1 - \theta_1\right], \end{aligned} \tag{25}$$

where

$$\begin{aligned} \theta_1 &= q_1\left[2\gamma_3\left(\dot{q}_1^2 + q_1\ddot{q}_1\right) + 2\gamma_4\left(\dot{q}_2^2 + q_2\ddot{q}_2\right) + \gamma_5\left(\ddot{q}_1q_2 + 2\dot{q}_1\dot{q}_2 + q_1\ddot{q}_2\right)\right] \\ &\quad + q_2\left[\gamma_6\left(\ddot{q}_1q_2 + 2\dot{q}_1\dot{q}_2 + q_1\ddot{q}_2\right) + 2\gamma_7\left(\dot{q}_1^2 + q_1\ddot{q}_1\right) + 2\gamma_8\left(\dot{q}_2^2 + q_2\ddot{q}_2\right)\right], \\ \theta_2 &= q_2\left[2\beta_3\left(\dot{q}_1^2 + q_1\ddot{q}_1\right) + 2\beta_4\left(\dot{q}_2^2 + q_2\ddot{q}_2\right) + \beta_5\left(\ddot{q}_1q_2 + 2\dot{q}_1\dot{q}_2 + q_1\ddot{q}_2\right)\right] \\ &\quad + q_1\left[\beta_6\left(\ddot{q}_1q_2 + 2\dot{q}_1\dot{q}_2 + q_1\ddot{q}_2\right) + 2\beta_7\left(\dot{q}_1^2 + q_1\ddot{q}_1\right) + 2\beta_8\left(\dot{q}_2^2 + q_2\ddot{q}_2\right)\right]. \end{aligned}$$

Matching the coefficients of the same terms $q_1^{j_1}v_1^{j_2}$; $j_1 = 0, 1, \ldots$; $j_2 = 0, 1, \ldots$, is performed. As a result of this a system of linear algebraic equations with respect to ($a_1, a_2, \ldots, b_1, b_2, \ldots$) is derived. This system of linear algebraic equations is not presented here for conciseness. The solution of this system has the following form:

$$\begin{aligned} &a_1 = a_2 = a_3 = b_1 = b_2 = b_3 = a_5 = a_7 = b_6 = b_4 = 0, \\ &a_6 = b_7 = -\frac{38\beta_7}{105\pi^4}, \quad a_4 = \frac{22\beta_7}{105}, \quad b_5 = \frac{142}{105}\beta_7. \end{aligned} \tag{26}$$

Eq. (26) is substituted into Eq. (24) and the nonlinear mode is obtained.

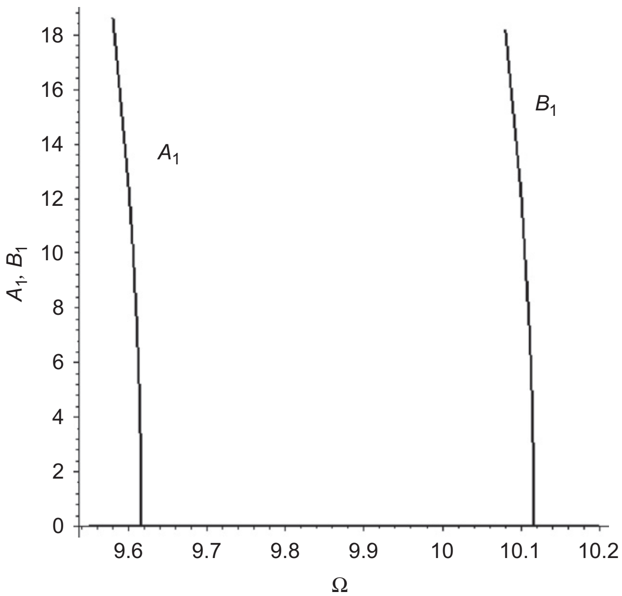

Fig. 2. The frequency response of the mechanical system (16).

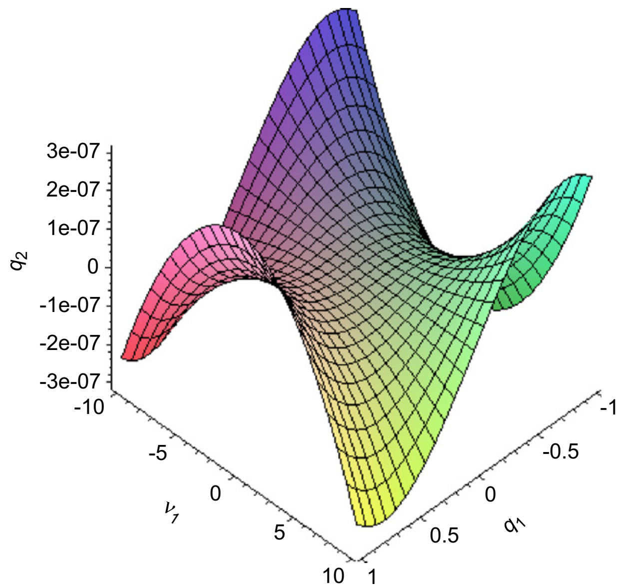

Fig. 3. The nonlinear mode of the mechanical system (16).

The numerical calculations are carried out for the beam, which is considered in the paper [20]. The following values of parameters are used:

$$\mu = 9.3 \times 10^{-2}\,\mathrm{kg/m}, \quad M = 0.162\,\mathrm{kg}, \quad E = 2.013 \times 10^{11}\,\mathrm{N/m^2}, \quad \rho = 7.8 \times 10^{3}\,\mathrm{kg/m^3},$$
$$l = 0.56\,\mathrm{m}, \quad EJ = 0.201\,\mathrm{N\,m^2}. \tag{27}$$

The quantities of (27) are used within the calculations for the frequency response in Eqs. (20) and (21). Fig. 2 shows the frequency response and the nonlinear mode is presented in Fig. 3.

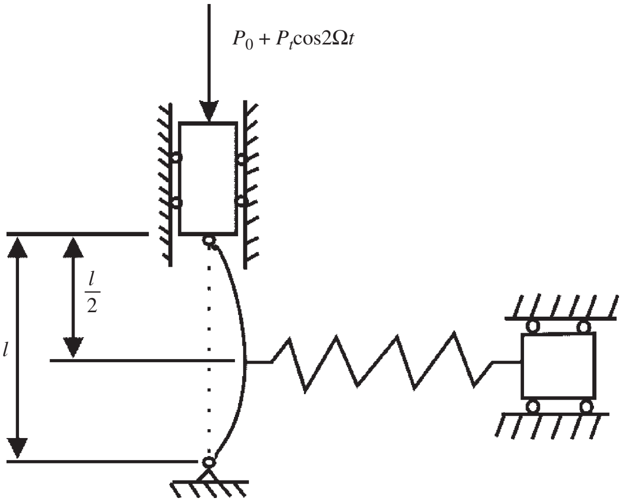

Fig. 4. The beam interacting with the oscillator.

4. Interaction of the beam with a linear oscillator

In this section, the nonlinear modes are applied for the analysis of the mechanical system consisting of the beam and an arbitrary one-degree-of-freedom system attached to the beam, as shown in Fig. 4. This oscillator can represent a vibration absorber attached in order to reduce parametric vibration in the beam. Thus the interaction of a discrete and a continuous subsystem is considered. The resulting system is defined as follows:

$$\begin{aligned} & EJw^{(IV)} + 0.5EJ\left(w''w'^2\right)'' + w''\left\{P_0 + P_t\cos(2\Omega t) - 0.5M\int_0^l \left(w'^2\right)''_{tt}\mathrm{d}s\right\} \\ & \quad + \mu\ddot{w} - (Nw')' = \delta(x-0.5L)K\{q - w(0.5l;t)\}, \\ & M_1\ddot{q} = -K[q - w(0.5l;t)], \\ & N = 0.5\mu\int_s^l \mathrm{d}s_1\int_0^{s_1}(w')''_{tt}\,\mathrm{d}s_2, \end{aligned} \tag{28}$$

where $w(s,\ t)$ is deflection of the beam; q is a generalized coordinate of the oscillator; M_1 is mass of the oscillator; EJ is bending stiffness; μ is the mass per unit length; $\delta(x-0.5l)$ is the delta function; K is a stiffness of the spring.

The following dimensionless variables and parameters are used:

$$w_* = \frac{w}{r},\quad \tau = t\sqrt{\frac{EJ}{\mu l^4}},\quad \xi = \frac{s}{l},\quad m = \frac{M}{\mu l},\quad f = \frac{P_t l^2}{EJ},\quad \chi = \frac{r^2}{l^2},\quad c = \frac{Kl^3}{EJ},\quad m_1 = \frac{M_1}{\mu l},\quad q_* = \frac{q}{r}, \tag{29}$$

where r is the radius of gyration of the cross section. The nondimensionalised dynamical system of Eq. (28) has the following form:

$$\begin{aligned} & w_*^{(IV)} + \frac{\chi}{2}\left(w_*''w_*'^2\right)'' + w_*''\left\{f\cos(2\Omega t) - 0.5m\chi\int_0^1\left(w_*'^2\right)''_{\tau\tau}\mathrm{d}\xi\right\} + \ddot{w}_* \\ & - \chi\left(\bar{N}w_*'\right)' = c\delta(\xi-0.5)\{q_* - w_*(0.5;t)\}, \\ & m_1\ddot{q}_* = -c\left[q_* - w_*(0.5;t)\right], \\ & \bar{N} = 0.5\int_\xi^1 \mathrm{d}\xi_1\int_0^{\xi_1}\left(w_*'^2\right)''_{\tau\tau}\mathrm{d}\xi_2. \end{aligned} \tag{30}$$

It is assumed that beam vibration occurs in the first mode of the simply supported beam:

$$w(\xi, t) = \theta(t) \sin(\pi\xi). \tag{31}$$

The Galerkin method is applied to the first equation of system (30). As a result the following dynamical system is derived:

$$\ddot{\theta} + \pi^4\theta + \frac{\chi}{8}\pi^6\theta^3 - \theta\pi^2\left[f \cos(2\Omega t) - 0.25m\chi\pi^2(\theta^2)^{\bullet\bullet}\right] + \chi\pi^2\left(\frac{\pi^2}{12} - \frac{3}{32}\right)\theta(\theta^2)^{\bullet\bullet} - 2c(q_* - \theta) = 0,$$
$$m_1\ddot{q}_* + c(q_* - \theta). \tag{32}$$

System (32) is re-expressed in linear modal coordinates, using the following equations:

$$\mathbf{\Xi U} = \mathbf{MUP}, \tag{33}$$

where

$$\mathbf{\Xi} = \begin{bmatrix} \pi^4 + 2c & -2c \\ -2c & 2c \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} 1 - p_1^2 v_1^{-2} & 1 - p_2^2 v_1^{-2} \\ 1 & 1 \end{bmatrix},$$
$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 2m_1 \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} p_1^2 & 0 \\ 0 & p_2^2 \end{bmatrix}, \quad v_1^2 = \frac{c}{m_1},$$
$$2p_{1,2}^2 = \pi^4 + 2c + v_1^2 \mp \sqrt{\left(\pi^4 + 2c + v_1^2\right)^2 - 4\pi^4 v_1^2}.$$

The following change of variables is applied:

$$\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \mathbf{U}^{-1} \begin{bmatrix} \theta \\ q_* \end{bmatrix}. \tag{34}$$

The dynamical system (32) takes this form in modal space:

$$\ddot{\xi}_1 + p_1^2\xi_1 + \zeta F_1\left(\xi_1, \xi_2, \dot{\xi}_1, \dot{\xi}_2, \ddot{\xi}_1, \ddot{\xi}_2, t\right) = 0,$$
$$\ddot{\xi}_2 + p_2^2\xi_2 + \zeta F_1\left(\xi_1, \xi_2, \dot{\xi}_1, \dot{\xi}_2, \ddot{\xi}_1, \ddot{\xi}_2, t\right) = 0,$$
$$F_1\left(\xi_1, \xi_2, \dot{\xi}_1, \dot{\xi}_2, \ddot{\xi}_1, \ddot{\xi}_2, t\right) = \gamma_1 P_2(\xi_1, \xi_2) + 2\gamma_3 P_1\left(\xi_1, \xi_2, \dot{\xi}_1, \dot{\xi}_2\right) + 2\gamma_3 P_1\left(\ddot{\xi}_1, \ddot{\xi}_2, \xi_1, \xi_2\right)$$
$$- (f_{11}\xi_1 + f_{12}\xi_2)\cos(2\Omega t), \tag{35}$$

where

$$f_{11} = f_1 a_1, \quad f_{12} = f_1 a_2, \quad f_1 = \pi^2 f, \quad \gamma_1 = \frac{\chi}{8}\pi^6, \quad \gamma_3 = \chi\pi^2\left(\frac{m\pi^2}{4} + \frac{\pi^2}{12} - \frac{3}{32}\right),$$
$$a_1 = 1 - p_1^2 v_1^{-2}, \quad a_2 = 1 - p_2^2 v_1^{-2}, \quad \zeta = \frac{1}{a_1 - a_2},$$
$$P_2(\xi_1, \xi_2) = a_1^3\xi_1^3 + a_2^3\xi_2^3 + 3a_1^2 a_2\xi_1^2\xi_2 + 3a_1 a_2^2\xi_1\xi_2^2,$$
$$P_1\left(\xi_1, \xi_2, \dot{\xi}_1, \dot{\xi}_2\right) = a_1^3\dot{\xi}_1^2\xi_1 + a_2^2 a_1\dot{\xi}_2^2\xi_1 + 2a_1^2 a_2\xi_1\dot{\xi}_1\dot{\xi}_2 + a_1^2 a_2\xi_2\dot{\xi}_1^2 + a_2^3\xi_2\dot{\xi}_2^2 + 2a_1 a_2^2\xi_2\dot{\xi}_1\dot{\xi}_2.$$

The method suggested in Section 2 is used to analyze the system expressed in Eq. (35). The motions with considerable values of ξ_1, $\dot{\xi}_1$ and small values of ξ_2, $\dot{\xi}_2$ are analyzed. Let us consider the iterative loop for the analysis of periodic motion. For the first iteration it is assumed that $\xi_2 = 0$. From then on Eq. (35) are transformed into the following form:

$$\ddot{\xi}_1 + p_1^2\xi_1 + \zeta\left[\gamma_1 a_1^3\xi_1^3 + 2\gamma_3 a_1^3\left(\dot{\xi}_1^2\xi_1 + \xi_1^2\ddot{\xi}_1\right) - f_{11}\xi_1 \cos(2\Omega t)\right] = 0. \tag{36}$$

The solutions of Eq. (36) are given in the form of Eq. (18), and the methodology relating to Eqs. (18)–(21) is used to solve Eq. (36), for which the following solutions are admissible:

- the equilibrium given by $A_1 = B_1 = 0$;
- periodic motions with amplitudes:

$$A_1 = 0, \quad B_1 = \sqrt{\frac{\Omega^2 - p_1^2 - 0.5f_{11}}{0.75\gamma_1 - 0.5\gamma_3\Omega^2}}. \tag{37}$$

- another periodic motion condition with the following amplitudes:

$$A_1 = 0, \quad B_1 = \sqrt{\frac{\Omega^2 - p_1^2 + 0.5f_{11}}{0.75\gamma_1 - 0.5\gamma_3\Omega^2}}.$$

From these motions function $\cos(2\Omega t)$ can be obtained using Eq. (23). Thus, the first iteration is finished. Eq. (23) is substituted into Eq. (35) and from this the following pseudo-autonomous dynamical system is derived:

$$\ddot{\xi}_i + p_i^2\xi_i + r_i\zeta\left[P_2(\xi_1,\xi_2) + P_1(\xi_1,\xi_2,\dot{\xi}_1,\dot{\xi}_2) + P_1\left(\ddot{\xi}_1,\ddot{\xi}_2,\xi_1,\xi_2\right) - \beta_1\xi_1^2\dot{\xi}_1 - \beta_2\xi_1\dot{\xi}_1\xi_2\right] = 0, \tag{38}$$

where

$$r_1 = 1, \quad r_2 = -1, \quad \beta_1 = f_{11}\alpha_3, \quad \beta_2 = f_{12}\alpha_3,$$
$$P_2(\xi_1,\xi_2) = \alpha_1^{(1)}\xi_1^3 + \alpha_2^{(1)}\xi_1^2\xi_2 + \alpha_3^{(1)}\xi_2^3 + \alpha_4^{(1)}\xi_1\xi_2^2,$$
$$P_1(\xi_1,\xi_2,\dot{\xi}_1,\dot{\xi}_2) = \alpha_1^{(2)}\dot{\xi}_1^2\xi_1 + \alpha_2^{(2)}\dot{\xi}_2^2\xi_1 + \alpha_3^{(2)}\dot{\xi}_2\dot{\xi}_1\xi_1 + \alpha_4^{(2)}\dot{\xi}_1^2\xi_2 + \alpha_5^{(2)}\dot{\xi}_2^2\xi_2 + \alpha_6^{(2)}\dot{\xi}_2\dot{\xi}_1\xi_2.$$

The nonlinear modes of Shaw and Pierre [13] are used to study the dynamical system (38):

$$\xi_2 = \xi_2(\xi_1, v_1) = \pi_4\xi_1^3 + \pi_5\xi_1^2 v_1 + \pi_6\xi_1 v_1^2 + \pi_7 v_1^3 + \cdots,$$
$$v_2 = \dot{\xi}_2 = v_2(\xi_1, v_1) = \rho_4\xi_1^3 + \rho_5\xi_1^2 v_1 + \rho_6\xi_1 v_1^2 + \rho_7 v_1^3 + \cdots \tag{39}$$

A nonlinear mode of Eq. (39) is represented as a power series, and following on from Refs. [13,17] it is known that if linear parts of system (38) are defined in modal space then the linear terms in the series of Eq. (39) are also equal to zero. System (38) only exhibits cubic nonlinearities. Therefore the quadratic terms in Eq. (39) are also equal to zero.

At this point the cubic terms from expansion (39) are calculated and by applying the nonlinear modes [13] the following equations are derived:

$$v_2(\xi_1, v_1) = \frac{\partial\xi_2(\xi_1, v_1)}{\partial\xi_1}v_1 + \frac{\partial\xi_2(\xi_1, v_1)}{\partial v_1}\dot{v}_1,$$
$$\dot{v}_2(\xi_1, v_1) = \frac{\partial v_2(\xi_1, v_1)}{\partial\xi_1}v_1 + \frac{\partial v_2(\xi_1, v_1)}{\partial v_1}\dot{v}_1. \tag{40}$$

Eq. (38) are substituted into Eq. (40) and the coefficients of ξ_1^3; $\xi_1^2 v_1$; $\xi_1 v_1^2$; v_1^3 are equated. As a result a system of linear algebraic equations in $\pi_4, \pi_5, \ldots$ is obtained. The solution of this system is the following:

$$\pi_5 = \zeta\beta_1\left(1 - 3p_1^2 p_2^{-2}\right)D^{-1},$$
$$\rho_6 = 2\zeta\beta_1 D^{-1},$$
$$\rho_4 = p_1^2\zeta\beta_1\left(3p_1^2 p_2^{-2} - 1\right)D^{-1},$$
$$\pi_7 = -2p_2^{-2}\zeta\beta_1 D^{-1},$$

$$\rho_5 = D^{-1}p_2^{-2}\zeta\left[3\left(p_1^2\alpha_1^{(3)} - \alpha_1^{(1)}\right)(p_2^2 - 3p_1^2) + 2p_1^2p_2^2\alpha_1^{(2)}\right],$$
$$\rho_7 = \pi_6 = p_2^{-2}D^{-1}\zeta\left[\alpha_1^{(2)}(3p_1^2 - p_2^2) - 6\left(p_1^2\alpha_1^{(3)} - \alpha_1^{(1)}\right)\right],$$
$$\pi_4 = p_2^{-2}D^{-1}\zeta\left[\left(p_1^2\alpha_1^{(3)} - \alpha_1^{(1)}\right)(p_2^2 - 7p_1^2) + 2p_1^4\alpha_1^{(2)}\right], \tag{41}$$

where

$$D = 10p_1^2 - 9p_1^4p_2^{-2} - p_2^2. \tag{42}$$

The next step requires the nonlinear mode of Eq. (39) to be substituted into Eq. (35) in order to generate

$$\ddot{\xi}_1 + p_1^2\xi_1 + \zeta\left\{\gamma_1 a_1^3\xi_1^3 + 2\gamma_3 a_1^3\dot{\xi}_1^2\xi_1 + 2\gamma_3 a_1^3\xi_1^2\ddot{\xi}_1 - f_{11}\xi_1\cos(2\Omega t)\right.$$
$$\left. - f_{12}\left(\pi_4\xi_1^3 + \pi_5\xi_1^2 v_1 + \pi_6\xi_1 v_1^2 + \pi_7 v_1^3\right)\cos(2\Omega t)\right\}. \tag{43}$$

Eq. (42) more accurately describes the motions in the nonlinear mode than does Eq. (36) and these two equations are seen to differ essentially from one another.

The solution of Eq. (42) is presented as Eq. (18). Then the harmonic balance method is applied and consequently a system of two nonlinear algebraic equations is obtained:

$$(p_1^2 - \Omega^2)A_1 + \zeta\left[\delta_5 A_1^3 + \delta_6 A_1 B_1^2 - \delta_3 A_1^2 B_1 - \delta_4 B_1^3 - 0.5f_{11}A_1\right] = 0,$$
$$(p_1^2 - \Omega^2)B_1 + \zeta\left[\delta_1 A_1^2 B_1 + \delta_2 B_1^3 - \delta_3 A_1 B_1^2 - \delta_4 A_1^3 + 0.5f_{11}B_1\right] = 0, \tag{44}$$

where

$$\delta_1 = 0.75\gamma_1 a_1^3 - \gamma_3\Omega^2 a_1^3 + 0.5f_{12}\pi_6\Omega^2,$$
$$\delta_2 = 0.75\gamma_1 a_1^3 - \gamma_3\Omega^2 a_1^3 + 0.5f_{12}\pi_4,$$
$$\delta_3 = 0.5f_{12}\pi_5\Omega; \quad \delta_4 = 0.5f_{12}\pi_7\Omega^3,$$
$$\delta_5 = 0.75\gamma_1 a_1^3 - \gamma_3\Omega^2 a_1^3 - 0.5f_{12}\pi_4,$$
$$\delta_6 = 0.75\gamma_1 a_1^3 - \gamma_3\Omega^2 a_1^3 - 0.5f_{12}\pi_6\Omega^2. \tag{45}$$

The second iteration concludes with the solution of Eq. (43).

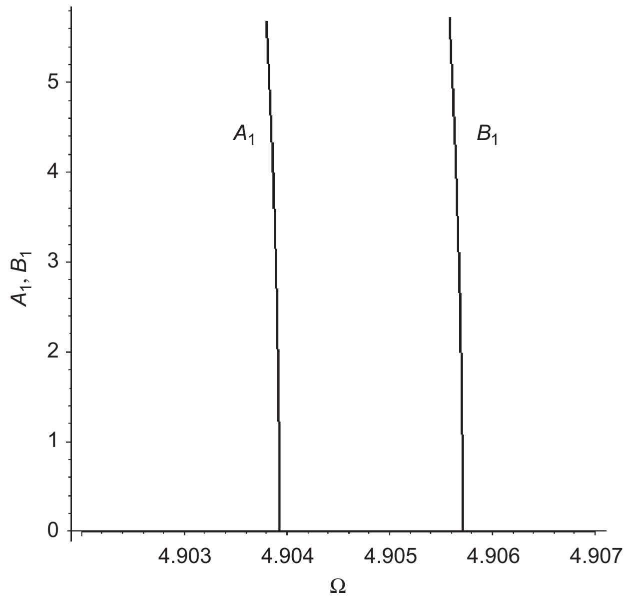

Fig. 5. The frequency response.

Table 1
The results of the calculations of the frequency response

ω	A_1 for the first iteration	A_1 for the second iteration
4.905707	0.8076918841	0.7184797517
4.90570	1.557934983	1.505888506
4.90568	2.737854823	2.707908819
4.90564	4.199224210	4.180446464
4.90561	5.023894997	5.008755142

Finally a third iteration is considered, and the solution of Eq. (43) is taken in the form of Eq. (23). Then the nonlinear mode (39) is calculated and system (42) is analyzed by solving nonlinear algebraic Eq. (43). At that point the third iteration is concluded and the iteration is complete when solutions of Eq. (43) for two subsequent iterations are regarded as being close enough.

The calculation of the frequency response is the aim of this analysis. The values of A_1 are preset by means of some chosen iterative step. For every value of A_1 an iterative loop is constructed. The values of B_1 and Ω are determined from this iterative loop.

The results of an example consisting of a beam interacting with an arbitrary linear oscillator are considered. The parameters of Eq. (27) and further data comprising

$$f = 1, \quad r = 0.289 \times 10^{-3}\,\text{m}, \quad M_1 = 0.0521\,\text{kg}, \quad c = 70,$$

are used within the calculations. The results are presented in Fig. 5 as frequency response. The dependence of the vibrations amplitudes A_1, B_1 on Ω is shown. The results of the first and the second iterations are presented in this figure. These results are so close that they cannot be identified in the scale of this figure. The data of the frequency response calculations are shown additionally in the table, in which for some values of the excitation frequency Ω, the results of the first and the second iterations are presented. As shown in Table 1 the results of the first and the second iterations are very close.

5. Concluding remarks

An iteration in which nonlinear modes and the Rauscher method are combined has been suggested in this paper. This approach can be used to determine all the nonlinear modes, and the number of these is equal to the number of degree of freedom of the nonlinear system. A dynamical system without an internal resonance has been considered here.

The main advantage of the method suggested here is that it does not need a small parameter in the equations of motion. Moreover, this method has a simple numerical implementation and reduces to the harmonic balance method for a one degree of freedom, and it also simplifies to a system of linear algebraic equations.

This method has been applied to an investigation of the parametric vibration of a beam and to the analysis of the interaction of the beam with an arbitrary one-degree-of-freedom oscillator for which calculations demonstrate that the iteration converges in two cycles. The second iteration contributes significantly to the solution of the problem of the beam interacting with the linear oscillator, with the first iteration giving the nonlinear mode emanating from Eq. (36) whereas the second iteration relates to the form of Eq. (42), with clear differences between the equations. The second iteration does not contribute to this problem of parametric vibration in a beam. Therefore, this iteration is not needed.

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Appendix A. Asymptotic analysis of system (17)

It is shown that in the asymptotic limit the solutions of Eq. (17) have the form (18). Using the small parameter ε, system (17) can be presented as

$$\ddot{q}_1 + \pi^4 q_1 + \varepsilon\left[\gamma_1 q_1^3 + 2\gamma_3 q_1\left(\dot{q}_1^2 + q_1\ddot{q}_1\right) - f_1 q_1 \cos(2\Omega\tau)\right] = 0. \quad (A.1)$$

The vibrations of system (A.1) are considered in the case of the main parametric resonance

$$\Omega = \pi^2 + \varepsilon\sigma. \quad (A.2)$$

The vibration of the system given by Eq. (A.1) is defined by

$$q_1(T_0, T_1, \ldots) = q_{10}(T_0, T_1, \ldots) + \varepsilon q_{11}(T_0, T_1, \ldots) + \cdots, \quad (A.3)$$

where $T_0 = \tau$; $T_1 = \varepsilon\tau$. Using the multiple-scales method the following equations are obtained:

$$\frac{\partial^2 q_{10}}{\partial T_0^2} + \pi^4 q_{10} = 0,$$
$$\frac{\partial^2 q_{11}}{\partial T_0^2} + 2\frac{\partial^2 q_{10}}{\partial T_0 \partial T_1} + \pi^4 q_{11} + \gamma_1 q_{10}^3 + 2\gamma_3 q_{10}\left(\frac{\partial q_{10}}{\partial T_0}\right)^2 + 2\gamma_3 q_{10}^2 \frac{\partial^2 q_{10}}{\partial T_0^2} - q_{10} f_1 \cos(2\Omega\tau) = 0. \quad (A.4)$$

The motions of the first equation of Eq. (A.4) can be shown to be given by

$$q_{10} = A(T_1)\exp(i\pi^2 T_0) + \bar{A}(T_1)\exp(-i\pi^2 T_0).$$

Using the change of variables $A = 0.5a\exp(i\psi)$ and equating to zero the secular terms of the second equation of system (A.4), the following system of modulation equations is derived:

$$a' - \frac{f_1 a}{4\pi^2}\sin(2\theta) = 0,$$
$$\theta' - \sigma + \frac{a^2}{\pi^2}\left(\frac{3}{8}\gamma_1 - \frac{\pi^4\gamma_3}{2}\right) - \frac{f_1}{4\pi^2}\cos(2\theta) = 0. \quad (A.5)$$

As follows from Eq. (A.5) three kinds of steady-state solutions exist. The first group satisfies $a_1 = 0$. The second group of solutions is described by

$$\theta_2 = 0, \quad a_2^2 = \frac{1}{\chi_1}(0.25 f_1 + \pi^2\sigma), \quad \chi_1 = \frac{3}{8}\gamma_1 - \frac{\pi^4\gamma_3}{2}.$$

The third group of solutions is determined by

$$\theta_3 = \frac{\pi}{2}, \quad a_3^2 = \frac{1}{\chi_1}(\pi^2\sigma - 0.25 f_1).$$

The vibrations of system (A.1) have the following form:

$$q_1 = a\cos(\psi + \pi^2 T_0).$$

The second group of the solutions has the form

$$q_1 = a\cos(\Omega t)$$

and the third group of motions has the following form:

$$q_1 = a\sin(\Omega t).$$

The analysis given in this appendix shows that the vibration defined by Eq. (17) has the form of Eq. (18).

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