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Multistability and chaotic beating of Duffing oscillators suspended on an elastic structure

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Abstract

We consider the dynamics of a number of externally excited chaotic oscillators suspended on an elastic structure. We show that for the given conditions of oscillations of the structure, initially uncorrelated chaotic oscillators become periodic and synchronous in clusters. In the periodic regime, we have observed multistability as two or four different attractors coexist in each cluster. A mismatch of the excitation frequency in the oscillators leads to the beating-like behaviour. We argue that the observed phenomena are generic in the parameter space and independent of the number of oscillators and their location on the elastic structure.

1. Introduction

An attractor is a fundamental concept in the theory of dynamical systems. Consider the dynamical system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) \tag{1}$$

where $f(x)$ is a function which fulfils all the conditions necessary for Eq. (1) to have a unique solution for the given initial condition $x(t=0)$ and $x \in R^n$. Let $x(t)$ be a solution of Eq. (1), for an open set of initial conditions. The n-dimensional real space R^n is a phase space of Eq. (1). The minimum subset $A \subset R^n$ with a property that $x(t) \to A$ as $t \to \infty$ is called an attractor. Typical attractors of system (1) are fixed points (equilibria), limit cycles (periodic behaviour), tori (quasiperiodic behaviour) and strange attractors (chaotic behaviour). One of the characteristic features of a nonlinear system is the simultaneous existence of different attractors, i.e., for the given parameter values depending on the initial conditions, the system trajectory can go to a different attractor. This feature is called multistability. To understand the dynamical behaviour of such systems, it is necessary to calculate the basin of attraction for each coexisting attractor. In a number of cases the structure of basins and their bifurcations *leads to the unexpected dynamical uncertainty; one cannot a piori* predict the attractor on which the system might evolve. Multistability is common in higher-dimensional

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systems: in particular, it can be observed in coupled dynamical systems [1–3]. Such systems are extensively studied in relation to the idea of synchronization.

Synchronization is a fundamental nonlinear phenomenon which is observed in science and engineering [4–6]. In the last two decades it has been demonstrated that any set of chaotic systems can synchronize by linking them with mutual coupling or with a common signal or signals [7–10]. In mechanical systems the synchronization was discovered in the XVIIth century by the Dutch researcher, Christian Huygens. He showed that a couple of mechanical clocks hanging from a common support were synchronized [11]. Currently, the high number of research activities in the field of synchronization reflects the importance of this subject. Some of the classical mechanical engineering applications are mentioned in [6]. Recently a promising direction has been to employ control theory to handle synchronization as a control problem. Particularly this approach can be applied in robotics when two or more robot-manipulators have to operate synchronously in a hazardous environment [12,13]. Pogromsky et al. [14] designed a controller for a synchronization problem comprising two pendula suspended on an elastically supported rigid beam.

In the current study we consider the dynamics of n nonlinear oscillators located on (coupled through) an elastic structure. We present a numerical study of a realistic model of identical double well-potential Duffing oscillators suspended on an elastic beam. The nonlinear oscillators are externally excited by a periodic signal with a frequency η. Each oscillator can be considered as a subsystem with its own dynamics.

A double-well potential Duffing oscillator has been taken as an example of a system that shows bistable instability in its periodic behaviour and a single chaotic attractor. Bistability of oscillators is important in our studies as it creates a number of coexisting attractors in a coupled system. It should be mentioned here that the details of the physical realisation of such double-well potential Duffing oscillators have not been the subject of our consideration.

Coupling through an elastic structure allows one to investigate how the dynamics of the particular oscillator is influenced by the dynamics of other subsystems and this is the main purpose of our research. Preliminary results of this problem have been presented in our previous works [15,16] in which we consider the dynamics of two Duffing and two van der Pol oscillators suspended on a beam and identify a simple mechanism of mutual interaction. Here, we concentrate on the possibility of making the oscillators behave periodically, the existence of different attractors, the creation of clusters (groups of oscillators with synchronous behaviour) and the influence of frequency mismatch on the behaviour of oscillators.

We would like to point out that we have concentrated on the analysis of a quite general model of the coupling through the elastic structure. This coupling is common in mechanical systems. The above-mentioned Huygens clocks are a classical example but the same coupling occurs for example when a number of machines operate in the same hall, or a crowd of people walks on a bridge.

We show that for the given conditions of the elastic structure oscillations, (i) initially uncorrelated chaotic oscillators become periodic, i.e., the behaviour of nonlinear oscillators becomes periodic as a result of an interaction with the elastic structure, (ii) symmetrical oscillators are synchronized creating clusters and, (iii) in the case of a mismatch of the excitation frequency in each oscillator it shows a chaotic beating-like behaviour. We argue that the observed phenomena are generic in the parameter space and independent of the number of oscillators and the method of discretisation of the continuous structure.

The paper is organised as follows: in Section 2 we describe the considered model. Section 3 presents the dynamics of two Duffing oscillators connected with the elastic beam and describes the phenomenon of making the oscillators behave periodically. The multistability of the system is discussed in Section 4. Section 5 deals with the influence of the mismatch of the excitation frequency η on the behaviour of oscillators. Finally, our results are summarised in Section 6.

2. The model

The system consists of an array of n oscillators which are suspended on an elastic beam as shown in Fig. 1(a). Its dynamics can be described in the following general form:

$$\frac{\partial^2 z(x,\tau)}{\partial \tau^2}+\frac{g}{\omega}\frac{\partial z(x,\tau)}{\partial \tau}+\delta^{-2}\frac{\partial^4 z(x,\tau)}{\partial x^4}=\sum_{i=1}^{n} p_i(x,\tau) \tag{2}$$

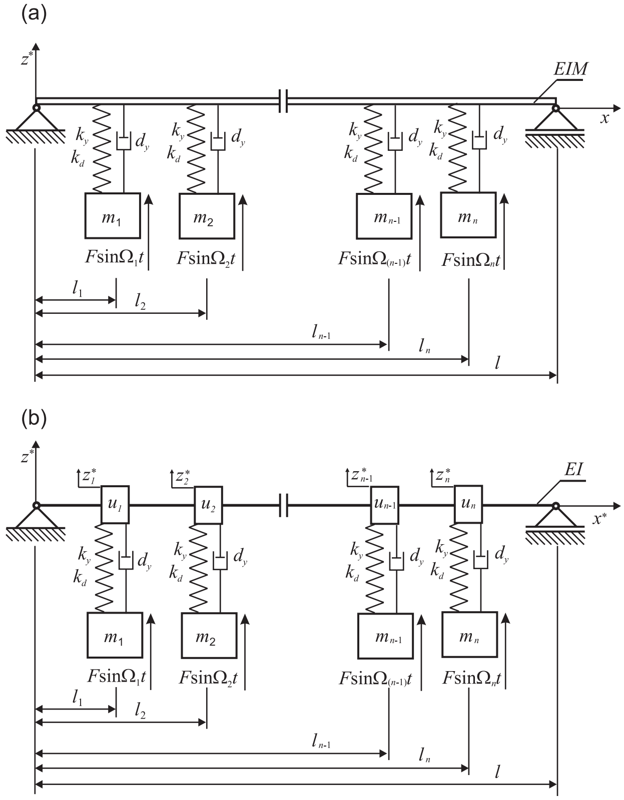

Fig. 1. Double-well potential Duffing oscillators suspended on the elastic beam: (a) continuous model and (b) discrete model.

$\tau = \omega t$, $\omega = \sqrt{E/\rho l^2}[\mathrm{s}^{-1}]$, $\delta^{-2} = EI/Ml^3\omega^2$, $z = z^*/l$ and $x = x^*/l$ (z^* and x^* are dimensional coordinates). Parameters of the beam: mass M [kg], density ρ [kg/m^3], length l [m], modulus of elasticity E [N/m^2] and the inertial momentum of cross–section I [m^4] and damping coefficient g [1/s] are constant, whereas, $p_i(x,\tau)$ describes the signal transmitted by the ith oscillator to the beam, i.e., the dynamical reaction from the ith oscillator transmitted to the beam.

We assumed an identity for the suspended oscillators, i.e., $m_i = m$ and $\Omega_i = \Omega$. As an example we consider the double-well potential Duffing oscillator described by

$$\frac{m}{M}\frac{\mathrm{d}^2 y_i}{\mathrm{d}\tau^2} + \delta^{-2}\gamma\frac{\mathrm{d}y_i}{\mathrm{d}\tau} - \delta^{-2}\beta_1 y_i + \delta^{-2}\beta_2 y_i^3 = f\cos\eta\tau \tag{3}$$

where $y = y^*/l$, $\gamma = d_y\omega l^3/EI$, $\beta_1 = k_y l^3/EI$, $\beta_2 = k_d l^5/EI$, $f = F/M\omega^2 l$ and $\eta = \Omega/\omega$. Real parameters of the oscillators are mass m [kg], damping d_y [Ns/m], linear k_y [N/m] and nonlinear k_d [N/m^3] parts of spring stiffness, amplitude F [N] and frequency Ω [1/s] of excitation. The unexcited oscillators (3) ($f = 0$) have three equilibria: unstable for $y_0 = (0,0)$ and two stable for $y_{-1} = (-\sqrt{\beta_1/\beta_2}, 0)$ and $y_{+1} = (\sqrt{\beta_1/\beta_2}, 0)$. In the periodic regime ($f \neq 0$), Eq. (3) shows bistability as two different periodic attractors coexist in the neighbourhood of y_{-1} and y_{+1}. The transition from the periodic behaviour to the chaotic one is associated with the homoclinic bifurcation. The stable limit cycle surrounding one of the non-zero equilibrium points collides with the homoclinic separatrix and a chaotic double-well attractor is created [17].

Thus, the expression $p_i(x,\tau)$ on the right-hand side of Eq. (2) is given as follows:

$$p_i(x,\tau) = \begin{cases} \delta^{-2}\gamma\left(\dfrac{\mathrm{d}y_i}{\mathrm{d}\tau} - \dfrac{\mathrm{d}z(x_i,\tau)}{\mathrm{d}\tau}\right) - \delta^{-2}\beta_1(y_i - z(x_i,\tau)) + \delta^{-2}\beta_2(y_i - z(x_i,\tau))^3 & \text{for} \quad x = x_i \\ 0 & \text{for} \quad x \neq x_i \end{cases} \tag{4}$$

The next assumption is that the beam is simply supported at both ends and so we have the following boundary conditions: $z(0,\tau)=0$, $z(l,\tau)=0$, $\mathrm{d}^2z(0,\tau)/\mathrm{d}x^2=0$ and $\mathrm{d}^2z(l,\tau)/\mathrm{d}x^2=0$.

In our study, Eq. (2) has been discretised in such a way that the continuous beam of the mass M was replaced by the massless beam on which $\hat{n}$ discrete identical bodies of mass u are located, i.e., $\hat{n}u=M$. The number of discrete masses has been selected in such a way as to obtain the first two eigenfrequencies of the continuous and discrete beam approximately equal. In our numerical simulations we assumed for the sake of simplicity that $\hat{n}$ is equal to the number of oscillators, i.e, $n=\hat{n}$ which are attached to the beam. The considered discrete model is shown in Fig. 1(b).

The discertisation is based on flexibility coefficients method [18]. The stiffness of the beam fulfils the relation $[k]=[a]^{-1}$, where $[a]$ is the $n\times n$ dimensional matrix of flexibility coefficients and it is dependent on the quantity EJl and the location of masses u. Hence, from the result of such a discretisation we obtain the following equation describing the dynamics of the ith 2 dof segment (masses u and m, $i=1,2,\ldots,n$) of the system:

$$\begin{gathered}\frac{m}{M}\ddot{y}_i+\delta^{-2}\gamma(\dot{y}_i-\dot{z}_i)-\delta^{-2}\beta_1(y_i-z_i)+\delta^{-2}\beta_2(y_i-z_i)^3=f\cos\eta\tau\\ \ddot{z}_i+\frac{g}{\omega}\dot{z}_i+n\delta^{-2}\left(\sum_j\alpha_{ij}z_j\right)=n\delta^{-2}(\gamma(\dot{y}_i-\dot{z}_i)-\beta_1(y_i-z_i)+\beta_2(y_i-z_i)^3)\end{gathered}\tag{5}$$

where $[\alpha]=[k]l^3/EJ$.

In the numerical analysis we assumed the mass of each oscillator $m=1.0\,[\mathrm{kg}]$ and the following dimensionless parameters of the Duffing oscillators (2) $\gamma=8.4$, $\beta_1=25$, $\beta_2=25$, $f=0.21$ and $\eta=1.0$. For this set of parameters each oscillator evolved on the chaotic attractor before they started to interact with the beam [19]. We also took $g/\omega=1.0$ and considered δ to be a bifurcation parameter. Physically, by changing δ we change the beam mass M without altering its stiffness EI, length l and frequency ω. We assumed that the oscillators were located symmetrically on the beam. Two particular but representative cases of two ($n=2$) and five ($n=5$) oscillators have been considered.

3. Making the Duffing oscillators periodic

3.1. Two oscillators

In Fig. 2(a–d) we show the bifurcation diagrams y_i (the values of y_i at $\tau=2\pi n/\eta$, $n=1,2,\ldots$ have been considered) versus δ, thus illustrating the behaviour of two oscillators suspended on the beam for an increasing δ (Fig. 2(a)) and for a decreasing δ (Fig. 2(b)). The corresponding two largest Lyapunov exponents are shown in Figs. 2(c and d), respectively. In the calculations of the bifurcation diagrams shown in Figs. 2(a–d) we have started with the initial conditions $z(x,0)=\mathrm{d}z(x,0)/\mathrm{d}\tau=0$, $\mathrm{d}y_1(0)/\mathrm{d}\tau=\mathrm{d}y_2(0)/\mathrm{d}\tau=\mathrm{d}y_3(0)/\mathrm{d}\tau=\mathrm{d}_4(0)/\mathrm{d}\tau=\mathrm{d}_5(0)/\mathrm{d}\tau=0$, $y_1(0)=0.1$, $y_2(0)=0.2$, $y_3(0)=0.3$, $y_4(0)=0.4$, $y_5(0)=0.5$ for the first (Figs. 2(a and c)) and the last (Figs. 2(b and d)) values of the δ parameter, respectively, $\delta=0$ and 500 and then we have used the last point of each calculation as the initial condition for the next δ parameter value. The phase shows y_1 versus y_2 for chosen values of δ and are shown in Figs. 3(a–c). In the case of an increasing δ (Fig. 2(a)) for low values of δ, the oscillators behave periodically and are synchronized, i.e., $y_1=y_2$ and $\dot{y}_1=\dot{y}_2$. The evolution of both oscillators is restricted to the synchronization manifold $y_1=y_2$, as shown in Fig. 3(a). The synchronization is lost at $\delta=250$ (Fig. 3(b)). But the oscillators are still periodic as the two largest Lyapunov exponents shown in Fig. 2(c) are negative. For larger values of the control parameter ($\delta>380$), the behaviour of the oscillators becomes uncorrelated and chaotic as can be seen in Fig. 3(c). This fact is confirmed by a sudden jump of Lyapunov exponents to positive values (Fig. 2(c)). On the other hand, when δ decreases (Figs. 2(b and c)), the oscillator behaviour becomes periodic at $\delta=360$ but the synchronization is not observed. Thus, a hysteretic effect occurs.

3.2. Five oscillators

Now, let us consider a system with 5 ($n=5$) oscillators located on an elastic beam. In Figs. 4(a–d) we show the bifurcation diagrams y_i ($i=1\ldots5$) versus δ, illustrating the behaviour of five oscillators suspended on a

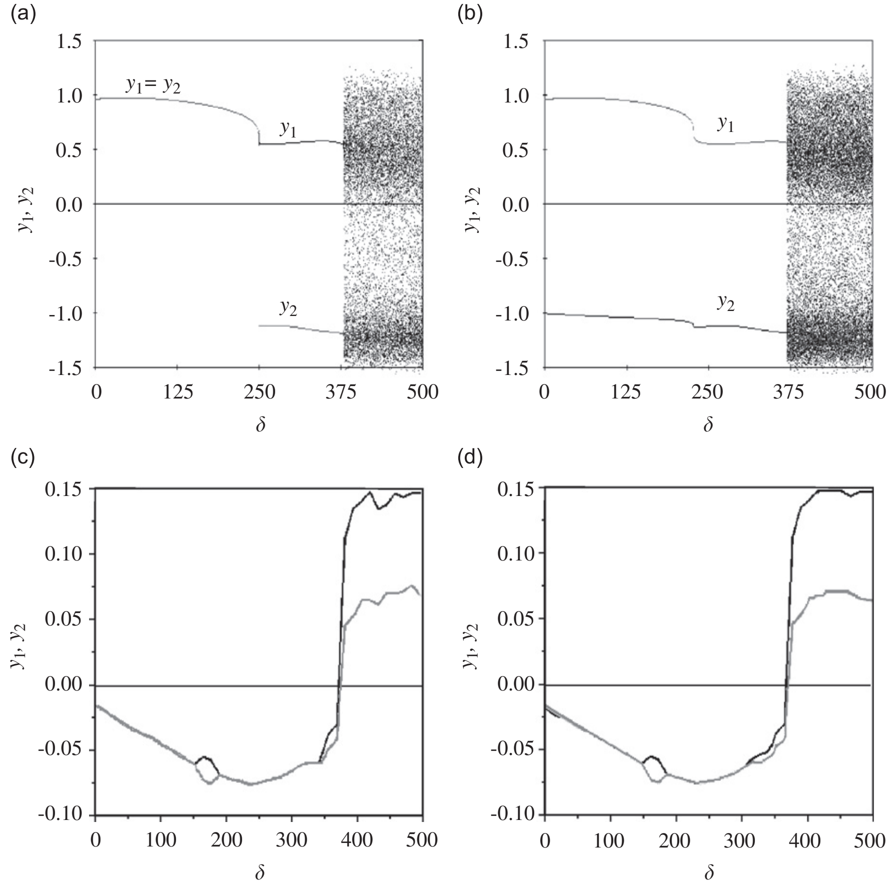

Fig. 2. Bifurcation diagrams $y_{1,2}$ versus δ for Eq. (5), $n = 2$ (two oscillators): $m = 1.0$ [kg], $f = 0.21$, $\eta = 1.0$, $g/\omega = 1.0$, $\gamma = 8.4$, $\beta_1 = 25$, $\beta_2 = 25$, $z(x,0) = dz(x,0)/d\tau = 0$, $dy_1(0)/d\tau = dy_2(0)/d\tau = dy_3(0)/d\tau = dy_4(0)/d\tau = dy_5(0)/d\tau = 0$, $y_1(0) = 0.1$, $y_2(0) = 0.2$, $y_3(0) = 0.3$, $y_4(0) = 0.4$, $y_5(0) = 0.5$; (a) δ increases and (b) δ decreases. The two largest Lyapunov exponents when δ increases (c) and δ decreases (d).

beam for an increasing δ (Fig. 4(a)), and for a decreasing δ (Fig. 4(b)). The corresponding five largest Lyapunov exponents are shown in Figs. 4(c and d), respectively. It results from the geometrical symmetry of the system that Duffing oscillators can synchronize completely in symmetrically located pairs only, i.e, $y_1 = y_5$ and $y_2 = y_4$, because only 2 dof (z_i, y_i) subsystems connected with these oscillators are identical. Thus, we can observe the clusters of synchronized subsystems. In the case of an increasing δ (Fig. 4(a)) for its low values, the oscillators behave periodically and are synchronized symmetrically in pairs. The complete synchronization is lost at $\delta = 350$. The oscillators are still periodic as the five largest Lyapunov exponents shown in Fig. 4(c) are negative. In this region of δ the oscillators are synchronized in phase. For larger values of the control parameter ($\delta > 415$), the behaviour of the oscillators becomes uncorrelated and chaotic. This fact is confirmed by a sudden jump of four Lyapunov exponents to positive values (Fig. 4(c)). On the other hand, when δ decreases (Figs. 4(b and c)), the oscillator behaviour becomes periodic at $\delta = 417$ but the synchronization is not observed. One can notice that the dynamical phenomena observed in the system with 5 oscillators are qualitatively similar to the one observed for the case of two oscillators.

In Figs. 5(a–c) the corresponding modes of the beam with the time diagrams of the mass u_3 oscillations are demonstrated. All the presented modes have been detected for the moment of maximum displacement of the mass u_3. We can observe that the harmonic mode of the beam oscillations (Fig. 5(a)) is associated with the periodic motion of the Duffing oscillator in the periodic synchronous regime and the unharmonic periodic motion of the beam (Fig. 5(b)) is observed for the periodic nonsynchronous behaviour of the Duffing oscillators. The uncorrelated chaotic mode of the beam response is associated with the chaotic behaviour of

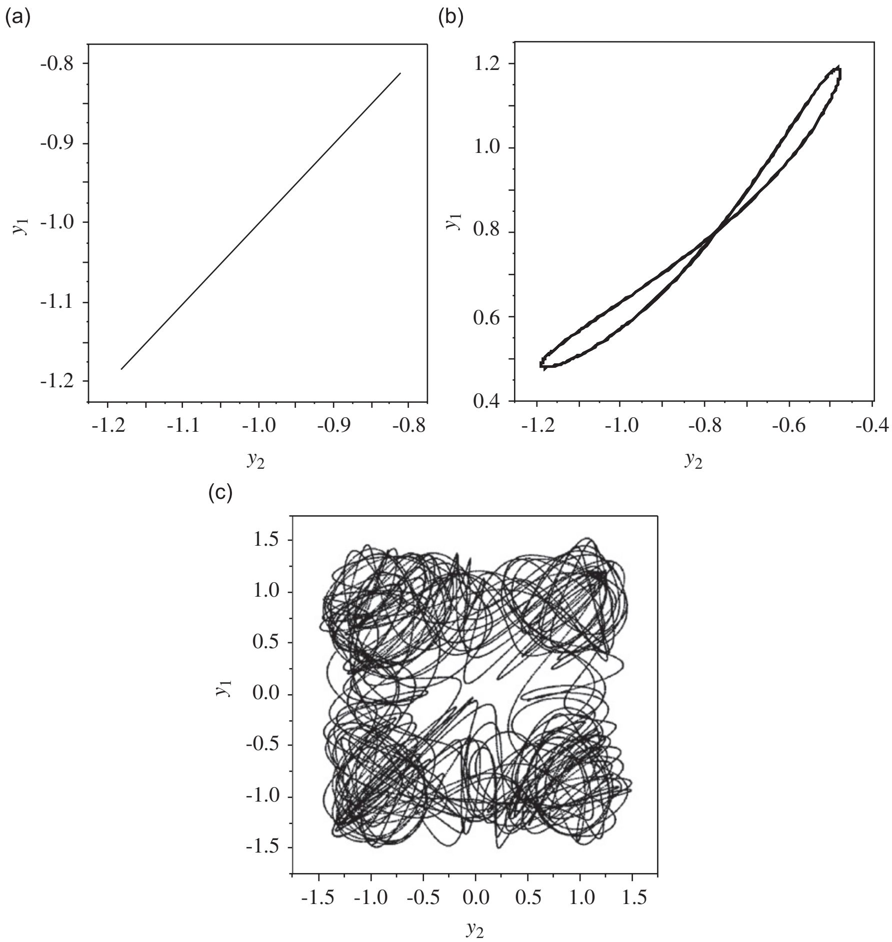

Fig. 3. Phase portraits y_1 versus y_2 for Eq. (5), $n = 2$ (two oscillators); $m = 1.0$ [kg], $f = 0.21$, $\eta = 1.0$, $g/\omega = 1.0$, $\gamma = 8.4$, $\beta_1 = 25$, $\beta_2 = 25$; (a) $\delta = 100$, (b) $\delta = 275$ and (c) $\delta = 425$.

the oscillators. Such a situation is clearly depicted in Fig. 5(c), where three forms of beam deflection randomly detected during chaotic motion are shown.

4. Multistability

A comparison of Figs. 2(a and b) shows the dynamical hysteresis in the neighbourhood of $\delta = 250$ and the multistability (coexistence of different attractors) for smaller values of δ. In the considered system in the case of the periodic behaviour of both oscillators, four different attractors (modes of oscillations) are possible: (i) both oscillators evolve around the upper stable fixed point, (ii) both oscillators evolve around the lower stable fixed point and, (iii) the left oscillator evolves around the lower stable fixed point while the right one around the upper stable fixed point and (iv) the opposite case to situation (iii) takes place. Modes (i) and (ii), in which both oscillators are synchronized, are symmetrical. On the other hand, modes (iii) and (iv) can be treated as cases of anti-synchronization.

All modes together with the initial conditions which guarantee their appearance are shown in Fig. 6(a). The range of initial conditions leading to each mode is indicated, respectively, in light grey, dark grey, white and black. In our calculations we took $z(x,0) = \mathrm{d}z(x,0)/\mathrm{d}\tau = 0$, $\mathrm{d}y_1(0)/\mathrm{d}\tau = \mathrm{d}y_2(0)/\mathrm{d}\tau = 0$ and allowed $y_1(0)$ and $y_2(0)$ to vary in the interval $(y_1(0), y_2(0)) \in [-1,1] \times [-1,1]$. At $\delta = 250$, symmetrical modes (i) and (ii) disappear. The initial conditions leading to two surviving modes ((iii) and (iv)) are shown in Fig. 6(b).

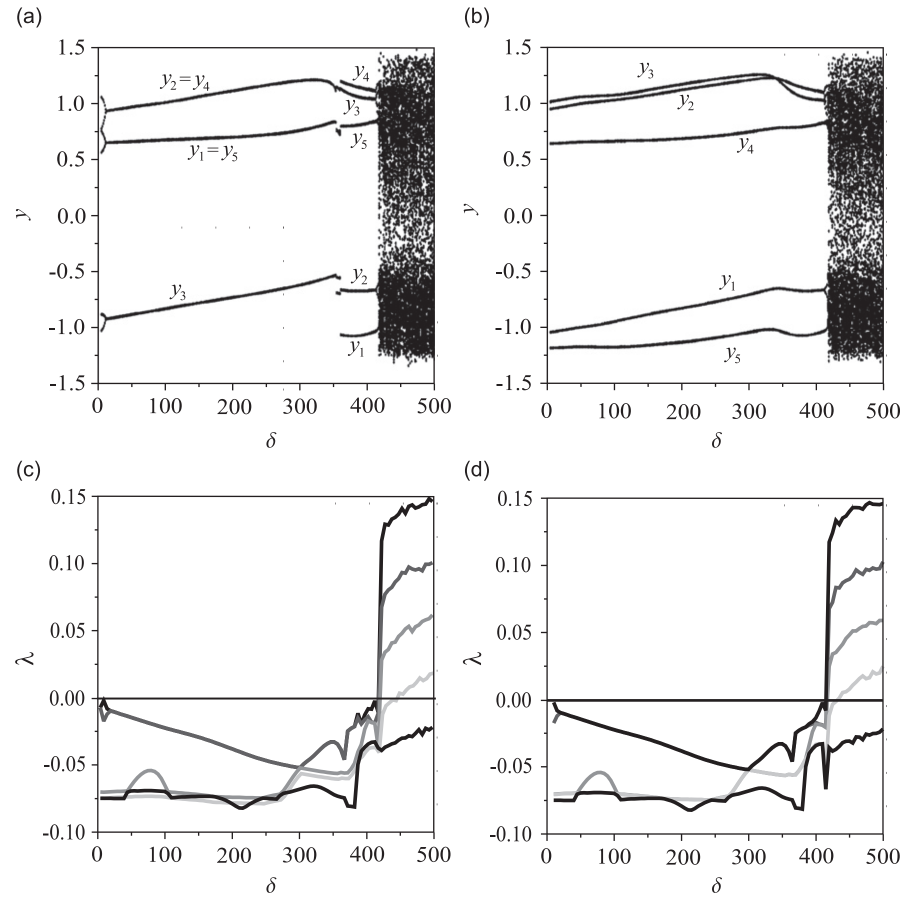

Fig. 4. Bifurcation diagrams $y_{1,2}$ versus δ for Eq. (5), $n = 5$ (five oscillators); $m = 1.0\,[\mathrm{kg}]$, $f = 0.21$, $\eta = 1.0$, $g/\omega = 1.0$, $\gamma = 8.4$, $\beta_1 = 25$, $\beta_2 = 25$; (a) δ increases and (b) δ decreases. Two largest Lyapunov exponents when δ increases (c) and δ decreases (d).

The structure of this figure shows a fractal-like nature, i.e., there exist regions in the phase space where small uncertainty of the initial conditions can lead the system behaviour to different attractors.

Multistability is more visible in the case of 5 oscillators, because the variety of different attractors is larger. The comparison of Figs. 5(a and b) shows the dynamical hysteresis in the neighbourhood of $\delta = 350$ and the coexistence of a number of different attractors for smaller values of δ. In the considered system in the case of periodic behaviour of the oscillators 32 attractors (for arbitrary n, the number of possible attractors is 2^n) are possible as each oscillator can evolve around either upper or lower equilibria.

In our systems inside each cluster, i.e., for oscillators 1 and 5 or 2 and 4 four different attractors (modes of oscillations) are possible; (i) both oscillators evolve around an upper stable fixed point, (ii) both oscillators evolve around a lower stable fixed point, (iii) the oscillator evolves (1 or 2) around a lower stable fixed point while the oscillator (5 or 4) evolves around an upper stable fixed point and, (iv) opposite to case (iii). In modes (i) and (ii) both oscillators are synchronized.

All vibration modes of the external cluster ($y_1 = y_5$) together with the basis of initial conditions which guarantee their appearance are shown in Fig. 7(a). The range of initial conditions leading to each mode is indicated, respectively, in light grey, dark grey, white and black. In our calculations we took $z(x,0) = \mathrm{d}z(x,0)/\mathrm{d}\tau = 0$, $\mathrm{d}y_1(0)/\mathrm{d}\tau = \mathrm{d}y_2(0)/\mathrm{d}\tau = \mathrm{d}y_3(0)/\mathrm{d}\tau = \mathrm{d}y_4(0)/\mathrm{d}\tau = \mathrm{d}y_5(0)/\mathrm{d}\tau = 0$, $y_2(0) = y_3(0) = y_4(0) = 0$ and allowed $y_1(0)$ and $y_5(0)$ to vary in the interval $(y_1(0), y_5(0)) \in [-1, 1] \times [-1, 1]$. This is a probe scheme: to observe all possible positions of oscillators other pairs of initial conditions have to be taken. At $\delta = 350$ symmetrical modes (i) and (ii) disappear. The initial conditions leading to the two survived modes

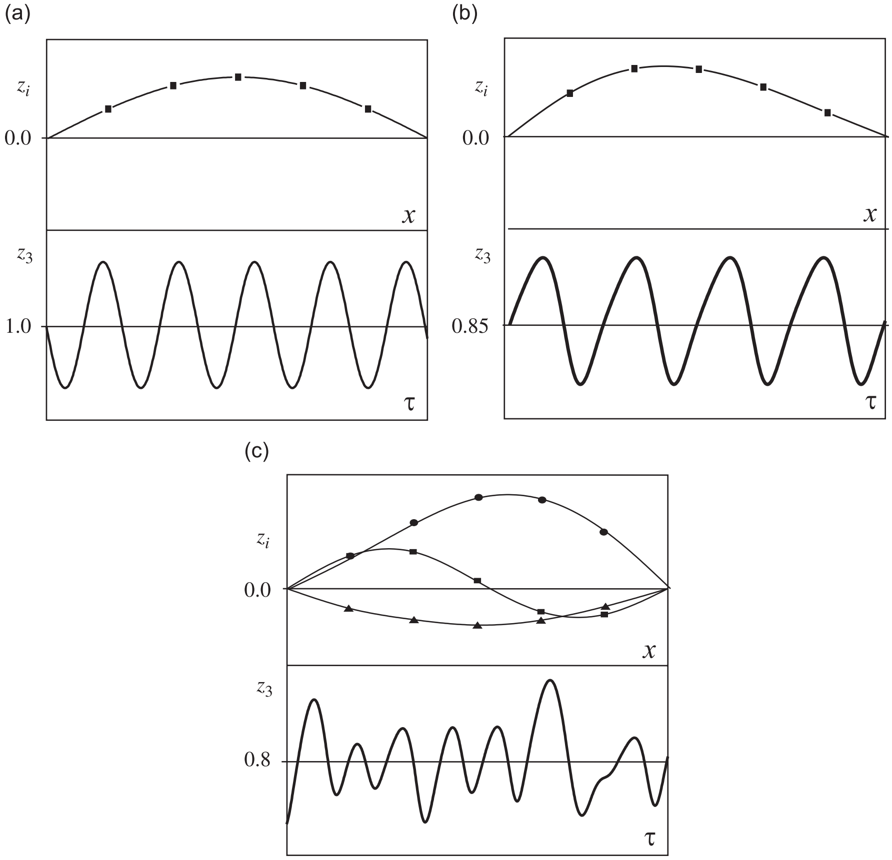

Fig. 5. Modes of the beam oscillations and the corresponding time diagrams of the mass u_3 oscillations for Eq. (5), $n = 5$ (five oscillators); $m = 1.0$ [kg], $f = 0.21$, $\eta = 1.0$, $g/\omega = 1.0$, $\gamma = 8.4$, $\beta_1 = 25$, $\beta_2 = 25$; (a) harmonic $\delta = 100$, (b) unharmonic periodic $\delta = 275$, (c) chaotic $\delta = 425$. All the presented modes have been detected for the moment of the maximal displacement of the mass u_2.

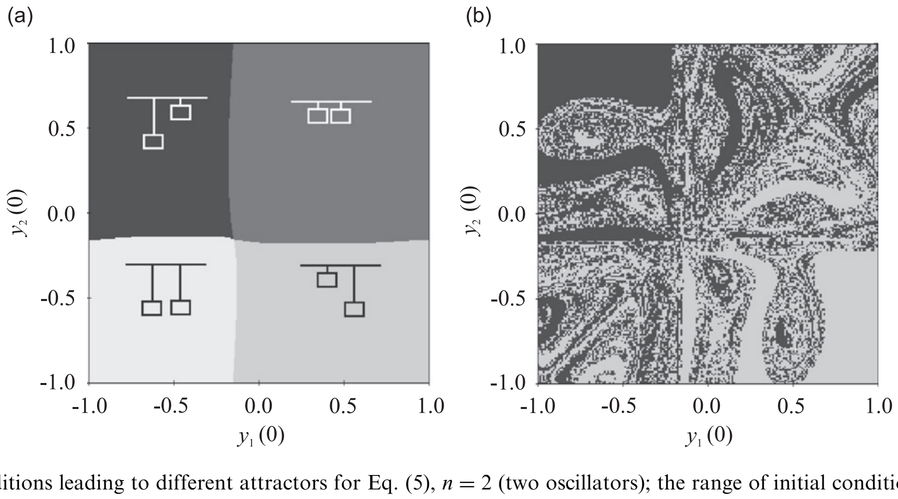

Fig. 6. Initial conditions leading to different attractors for Eq. (5), $n = 2$ (two oscillators); the range of initial conditions leading to each mode is indicated, respectively, in light grey, dark grey, white and black, $z(x,0) = \mathrm{d}z(x,0)/\mathrm{d}\tau = 0$, $\mathrm{d}y_1(0)/\mathrm{d}\tau = \mathrm{d}y_2(0)/\mathrm{d}\tau = 0$; $m = 1.0$ [kg], $f = 0.21$, $\eta = 1.0$, $g/\omega = 1.0$, $\gamma = 8.4$, $\beta_1 = 25$, $\beta_2 = 25$; (a) $\delta = 248$ four attractors exist and (b) $\delta = 252$ two attractors exist.

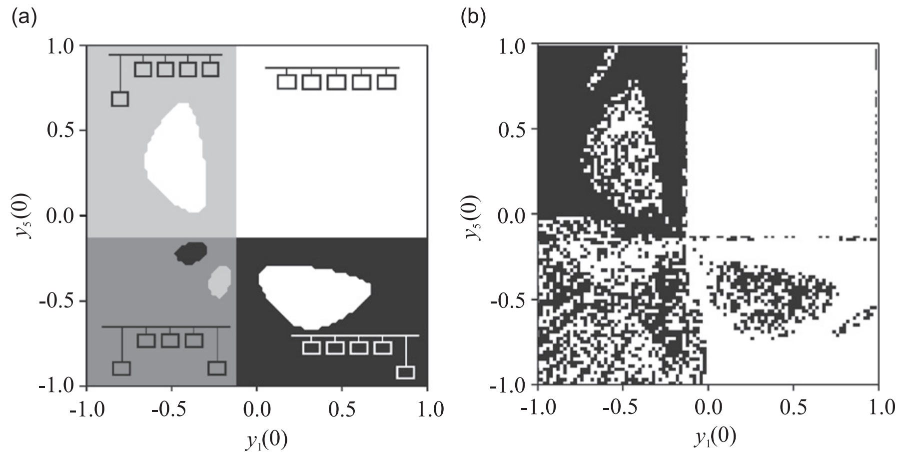

Fig. 7. Initial conditions leading to different attractors in external cluster x_1, x_5 for Eq. (5), $n = 5$ (five oscillators); the range of initial conditions leading to each mode is indicated, respectively, in light grey, dark grey, white and black, $z(x,0) = \mathrm{d}z(x,0)/\mathrm{d}\tau = 0$, $\mathrm{d}y_1(0)/\mathrm{d}\tau = \mathrm{d}y_2(0)/\mathrm{d}t = \mathrm{d}y_3(0)/\mathrm{d}\tau = \mathrm{d}y_4(0)/\mathrm{d}\tau = \mathrm{d}y_5(0)/\mathrm{d}\tau = 0$, $y_2(0) = y_3(0) = y_4(0) = 0$; $m = 1.0$ [kg], $f = 0.21$, $\eta = 1.0$, $g/\omega = 1.0$, $\gamma = 8.4$, $\beta_1 = 25$, $\beta_2 = 25$; (a) $\delta = 325$ four attractors exist and (b) $\delta = 375$ two attractors exist.

((iii) and (iv)) are shown in Fig. 7(b). The structure of this figure shows a fractal-like nature, i.e., there exist regions in the phase space where a small uncertainty of the initial conditions can lead the system behaviour to different attractors. Identical topology of basins of attraction (Figs. 4(a and b)) can be observed for the second pair of oscillators y_2 and y_4 (internal cluster).

5. Chaotic beating

In order to demonstrate the phenomenon of chaotic beating the case of $n = 2$ oscillators is considered. Let us assume that the frequencies of excitation in both oscillators ($n = 2$) are slightly different, i.e., $\eta_1 - \eta_2 = \varepsilon$, where $\varepsilon \ll 1$. In this case the evolution of system (5) is presented in Fig. 8(a–d). A mismatch in excitation frequencies η_1 and η_2 introduces a qualitative change in the system behaviour as the periodic behaviour is replaced by a quasiperiodic behaviour, i.e. coexisting limit cycle attractors are replaced by torus attractors. In this case one observes a beating behaviour shown in Figs. 8(a and b) (δ is equal to 50 and 58, respectively, $\varepsilon = 0.003$). It should be noted here that when η_1 and η_2 are commensurate, it is possible to define the period of the beating T_b as indicated in Fig. 8(a). T_b is equal to $2\pi/\varepsilon$, i.e., as known in the linear theory of oscillations [20].

With an increase of the control parameter δ, the basins of attraction of two coexisting torus attractors merge together (boundary crisis bifurcation) and are replaced by one strange chaotic attractor as shown in Figs. 8(c and d) (δ is equal to 59 and 125, respectively). Our numerical studies indicate that two previously coexisting torus attractors become repellers embedded into this strange chaotic attractor, as shortly after this bifurcation one can observe a crisis-induced (chaos–chaos) intermittency [21–24]. The system evolution is characterised by long-time intervals of evolution on one of the repellers and short–time intervals of jumps between repellers (Fig. 8(c)). With a further increase in δ, the intervals of the evolution on the repellers become shorter as can be seen in Fig. 8(d) and finally disappear at $\delta = \delta_0 = 128.75$. Let τ^* be the average length of the time-interval, in which the jumps between repellers can be observed. The scaling relation τ^* versus $\delta_0 - \delta$ fulfils the following relation:

$$\tau^* \propto (\delta - \delta_0)^{-\alpha^*},$$

where α^* has been estimated as being equal to 0.23 ± 0.01 which is characteristic of the crisis-induced (chaos–chaos) intermittency [25]. In numerical calculations of the scaling factor α^* we considered the average of 1000 simulations for randomly chosen initial conditions.

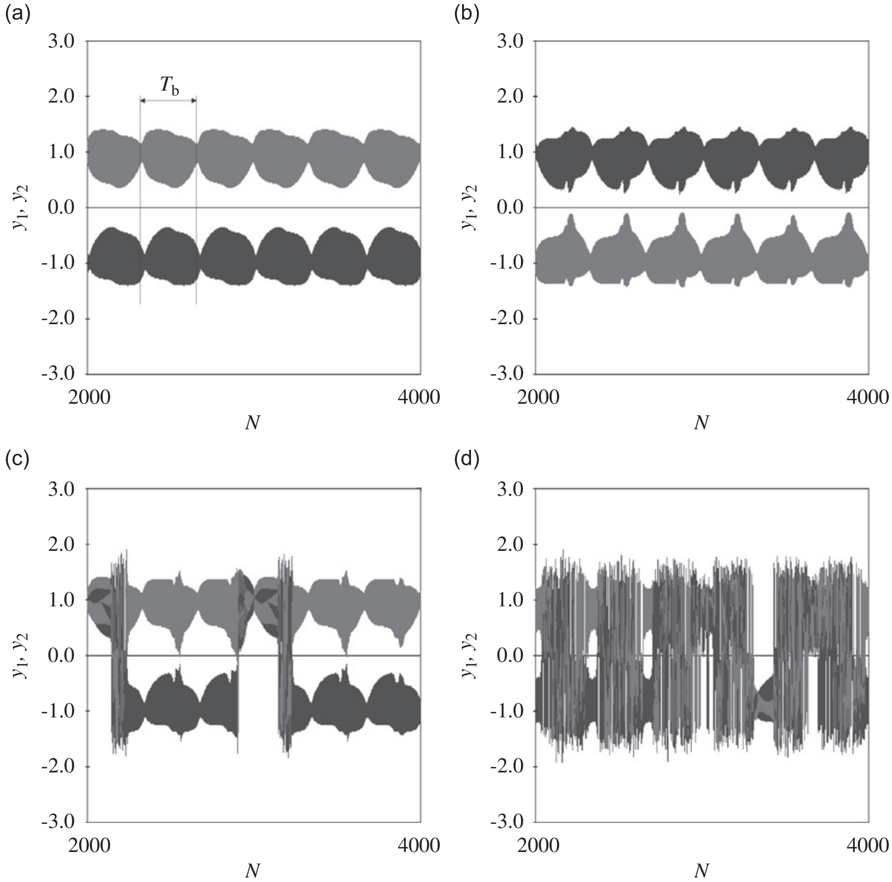

Fig. 8. Time series of Eq. (5) showing beating of oscillators; $n = 2$ (two oscillators); $m = 1.0$ [kg], $f = 0.21$, $\eta = 1.0$, $g/\omega = 1.0$, $\gamma = 8.4$, $\beta_1 = 25$, $\beta_2 = 25$; (a) $\delta = 50$, $\varepsilon = 0.003$ periodic beating, (b) $\delta = 58$, $\varepsilon = 0.003$ periodic beating, (c) $\delta = 59$, $\varepsilon = 0.003$ chaotic beating and (d) $\delta = 125$, $\varepsilon = 0.003$ chaotic beating.

6. Conclusions

To summarise, we have investigated the possibility of synchronization of nonlinear chaotic oscillators located on (coupled through) an elastic structure. In the numerical study we have considered a realistic model of two double-well potential chaotic Duffing oscillators suspended on an elastic beam. We have identified the phenomenon of making the oscillators behave periodically, in which their behaviour becomes periodic as a result of an interaction with the elastic structure. For low values of δ all oscillators synchronize in phase and the symmetrical oscillators perform complete synchronization creating clusters. The phenomenon of making the oscillators behave periodically leads to multistability as, depending on the parameter δ, two or four different attractors coexist. In the case of a mismatch of the excitation frequency $\eta_{1,2}$ in each oscillator we have observed a chaotic beating-like behaviour. We have shown an analogy of the observed behaviour to the phenomenon of beating known from the linear theory of oscillations and to the crisis-induced (chaos–chaos) intermittency.

The dynamical response of the beam has not been the main subject of interest in the presented paper. However, a strict connection between the beam response and oscillators dynamics has been confirmed by our numerical analysis. The regular modes of the beam oscillations are corresponding to the periodic behaviour of Duffing oscillators and a chaos–chaos mutual relation between the beam and the oscillators can be observed in the chaotic range.

Qualitatively similar results have been observed for different beam discretisations and a different number of oscillators. Particularly, we increased the number of discrete mass elements on the massless beam m and performed the parallel simulation carried out by means of the finite element method FEM. The phenomenon of making the oscillators behave periodically can also be observed in the case when the oscillators are not located symmetrically on the beam [15,16]. Our results allow us to argue that the observed phenomena are generic in the parameter space, independent of the number of oscillators, their location on the elastic structure and the method of discretisation of the beam.

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