Harmonic balance approach for a degenerate torus of a nonlinear jerk equation

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Abstract

The method of harmonic balance is used for the first time to find approximate expressions for the frequency and displacement amplitude of a degenerate torus arising in a third-order nonlinear oscillator, for a range of velocity amplitudes. The estimates compare favourably with numerically determined solutions. This development complements earlier works on the simpler situations of centres and limit cycles.
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1. Introduction

Jerk (third-order) differential equations of the form $\dddot{x} = J(x, \dot{x}, \ddot{x})$ have found applications in a variety of physical situations, as described for instance in Ref. [1]. Third-order nonlinear autonomous systems may exhibit regular behaviour of various types—centres, limit cycles, tori, etc.—as well as chaotic phenomena [2,3]. Explicit solutions for some simple nonlinear jerk equations were obtained in Ref. [4].

A first-order harmonic balance method (c.f. Mickens [5]) was used by Gottlieb [1] to determine approximate analytical estimates of the periods and displacement amplitudes of some regular periodic motions in nonlinear jerk equations, for a range of initial velocity amplitudes. More accurate expressions using an improved harmonic balance approach have recently been obtained by Wu et al. [6]. Approximate expressions for the periods and velocity and displacement amplitudes of stable limit cycles of some nonlinear jerk equations were obtained via the first-order harmonic balance approach by Gottlieb [7].

The present paper investigates for the first time the application of the first-order harmonic balance method to determine both the approximate frequency and displacement amplitude in a new situation, a degenerate torus, for a particular jerk equation. Degenerate tori differ from limit cycles in that, whilst they are both 1D loops in the phase space, stable limit cycles may be approached from any point off (but suitably nearby) the orbit. By contrast, degenerate tori must be started on the orbit, as found by independent numerical experiment.

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2. The nonlinear jerk equation

The system under consideration is a slight modification of the Nosé–Hoover oscillator, which is known from the work of Posch et al. [8] to exhibit degenerate tori. A slight generalization allows contact to be made with other similar systems appearing in the literature.

Consider the third-order nonlinear parametrized system

$$\dot{x} = y, \quad \dot{y} = -x + yz, \quad \dot{z} = \alpha - \beta y^2. \tag{1}$$

The system of Posch et al. [8] corresponds to $\beta = \alpha$ (with a slight change of notation; c.f. Hoover [9]). The case $\beta = 1$ and general α corresponds to Model A in a table of third-order systems given by Eichhorn et al. [10], which is itself a slight generalization of Case A in a table of algebraically simple third-order systems listed by Sprott [11] which corresponds to $\alpha = \beta = 1$.

The jerk equation corresponding to the system Eq. (1) may be shown to be

$$\dddot{x} = (\alpha - 1)\dot{x} - \beta\dot{x}^3 + \ddot{x}(x + \ddot{x})/\dot{x}. \tag{2}$$

We investigate the nonlinear equation (2) in the case $\beta = 1$ with one parameter α. The jerk equation

$$\dddot{x} = (\alpha - 1)\dot{x} - \dot{x}^3 + \ddot{x}(x + \ddot{x})/\dot{x} \tag{3}$$

has linear term involving only $\dot{x}$, and is time-reversal invariant and parity invariant. As discussed in Ref. [1], the initial conditions for the harmonic balance approach are limited to the case of zero initial acceleration $\ddot{x}(0) = 0$. Initial conditions $x(0) = 0$, $\dot{x}(0) \neq 0$, $\ddot{x}(0) = 0$ are in fact those which had been used in the numerical investigations of Posch et al. [8].

3. Harmonic balance approach

The first-order harmonic balance approximation Ansatz is [1]

$$x = (B/\Omega) \sin \Omega t \tag{4}$$

so $\dot{x}(0) = B$, the initial velocity amplitude, and $x(0) = 0 = \ddot{x}(0)$, with period $T = 2\pi/\Omega$ and displacement amplitude $A = B/\Omega$. For the application of the harmonic balance method, Eq. (3) is first multiplied through by $\dot{x}$ to obtain a form "free of fractions" (c.f. Mickens [5]). After substitutions and manipulations, terms result involving constants, $\cos(2\Omega t)$ and $\cos(4\Omega t)$. Equating constant terms yields the first order harmonic balance approximation for the angular frequency

$$\Omega_{\mathrm{HB}} = \sqrt{1 - \tfrac{1}{2}\alpha + \tfrac{3}{8}B^2}. \tag{5}$$

In this case, as mentioned above, B must be input into Eq. (5) as the unique value (for chosen value of parameter α) obtained by numerical investigation as described in Section 4.

The harmonic balance approximations to period T and orbit x-intercept A are then given by Eq. (5) and

$$T_{\mathrm{HB}} = 2\pi/\Omega_{\mathrm{HB}}, \quad A_{\mathrm{HB}} = B/\Omega_{\mathrm{HB}}. \tag{6}$$

4. Method and results

We used the software ODE Workbench [12] to investigate the jerk equation (3) computationally with initial conditions $x_0 = 0 = \ddot{x}_0$, $\dot{x}_0 = B$, for a range of values of α between 0.1 and 1. (The singularity in Eq. (3) did not seem to cause difficulties for the programme.) A periodic solution (closed orbit) was sought.

The software [12] can be used to produce the trajectory in the $x, \dot{x}$ plane, starting at the point $(0, B)$, and may record the values for $\dot{x}|_{x=0}$ as the numerical integration proceeds. The jerk equation is successively solved until the torus degenerates into a closed one-loop orbit. The value of B was successively refined until the first return value for $\dot{x}|_{x=0}$ was as close as practicable to that of B.

Thus this application of the harmonic balance method, to a degenerate torus, contrasts with the application to limit cycles, discussed in Ref. [7], where estimates for both B and A (as well as T) were obtainable in

Table 1
Values of velocity amplitude B yielding a degenerate torus, and corresponding period T and displacement amplitude A, comparing computed and harmonic balance values for several parameter values α in the jerk equation (3).

α	B_{COMP}	T_{COMP}	T_{HB} (% error)	A_{COMP}	A_{HB} (% error)
0.1	0.451	6.205	6.202 (−0.05)	0.440	0.445 (1.1)
0.2	0.644	6.129	6.116 (−0.2)	0.613	0.627 (2.3)
0.5	1.047	5.910	5.831 (−1.3)	0.925	0.972 (5.1)
1	1.550[a]	5.578[a]	5.309 (−4.8)	1.214	1.309 (7.8)

[a]c.f. Ref. [8].

advance. However, a similarity of the degenerate torus case to the even simpler case of ordinary periodic solutions (centres) for jerk equations, as discussed in Ref. [1] (and indeed in the standard application of the harmonic balance method to second-order oscillators (c.f. [5])), should be stressed. Whilst a range of values of B for a centre (not restricted to a specific value as for the degenerate torus case) may be entered into the harmonic balance estimate equation for T, some independent work in the case of complicated oscillators must actually still be carried out to determine a range of values for which the orbits are periodic. Thus even in this simpler case, some numerical integration of the jerk equation is required to determine allowable values of B which may legitimately be inserted into the estimate equations for T and A.

For example, for $\alpha = 1$, the outputs indicated a periodic behaviour arising from a degenerate torus for a value of B lying between 1.54 and 1.55. The actual value, found by successive adjustments and checking of the first return value as above, to five significant figures was $B = 1.5499$. The corresponding numerically determined period T is 5.5781. The first x-intercept (displacement amplitude) was found at $A = 1.2144$. These values for B and T, obtained by numerical integration of the single nonlinear jerk equation (3), are in agreement with the approximate values $B = 1.55$ and $T = 5.58$ for a degenerate one-loop torus found by Posch et al. in Ref. [8], where they numerically integrated the system of three first-order ODEs (1) with $\alpha = \beta = 1$ which involve polynomial (quadratic) nonlinearities in two of the equations.

Table 1 presents our results obtained for the jerk equation (3) for several values of the parameter α. The corresponding computed velocity amplitudes B_{COMP} are listed. Harmonic balance results T_{HB} and A_{HB} are compared with computed values for period T_{COMP} and displacement amplitude A_{COMP}. The relative percentage errors are also given: for T they range in magnitude from about 0.05 percent up to 5 percent; for A they are a little larger.

5. Conclusions

In summary, we conclude that the harmonic balance method, although not giving information about the velocity amplitude in this case, may give good estimates of the corresponding period and displacement amplitude for the degenerate torus of the nonlinear jerk equation (3). Since the circumstance of a degenerate torus is quite different in nature from a continuous nested set of simple periodic orbits (centre) as dealt with in Refs. [1,6], or a limit cycle as in Ref. [7], this indicates for the first time a gratifying extension of the application of the harmonic balance approach to such a wider range of oscillator phenomena.

Acknowledgements

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