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Fundamental problems with the model of uniform flow over acoustic linings

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Abstract

The walls of ducts containing flow are often acoustically lined in order to reduce sound. Many simple models of acoustic linings assume the lining to be linear and locally reacting; examples considered here include the three-parameter, mass–spring–damper, Helmholtz resonator and enhanced Helmholtz resonator models. All of these models have been found to have stability issues with uniform mean grazing flow, and there has been some confusion over the existence of hydrodynamic instability surface waves over such linings. Mathematically, the standard proven Briggs–Bers stability analysis is not applicable. Computationally, the hydrodynamic modes are routinely ignored (in the frequency domain) and instabilities filtered out (in the time domain). This confusion also causes significant problems for mode-matching, Green functions, and scattering analyses. In this paper, it is shown that any situation not capable of being analysed using the Briggs–Bers criterion is illposed, and that this is the root cause of the confusion over hydrodynamic instabilities. A large class of lining models, including all those mentioned above, are shown to be illposed with uniform mean flow. An explanation is given of the effects of this illposedness in practice, and it is argued that illposed models should not be used.

An alternative stability criterion to Briggs–Bers has been occasionally used in the literature. This alternative criterion, involving analytic continuation to purely imaginary frequencies, was recently christened the "Crighton–Leppington" stability criterion. However, this stability criterion is incorrect, and several counter-examples are given here.

1. Introduction

Acoustic linings are used in many practical applications to line the surfaces of ducts carrying mean flow, in order to reduce sound emissions. Examples include the exhaust pipes of automobiles and the intakes and bypass ducts of turbofan aircraft engines. As a model of such a situation, consider flow along a duct with velocity $\mathbf{U} + \nabla\phi \exp\{i\omega t\}$, where $\mathbf{U}(\mathbf{x})$ is the mean flow and $\phi(\mathbf{x})$ is a small acoustic perturbation. The duct wall is impermeable to the mean flow, so that $\mathbf{U} \cdot \mathbf{n} = 0$ there, where $\mathbf{n}$ is the normal to the duct wall, oriented out of the fluid. If the duct wall were perfectly rigid and impermeable, the acoustic perturbation would also satisfy $\mathbf{n} \cdot \nabla\phi = 0$ on the duct wall. For a lined duct wall, rather than modelling the physics of the lining, a simplified mathematical model is usually assumed which modifies this boundary condition [1–4]. Typically, the lining is

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assumed to be linear and locally reacting. The response of the lining is then characterized by its impedance, $Z(\omega) = p/v$, where a time-harmonic pressure forcing $p\exp\{i\omega t\}$ produces a time-harmonic fluid velocity $v\exp\{i\omega t\}$ normal to the lining. With no slipping mean flow ($\mathbf{U} = \mathbf{0}$ at the boundary), the boundary condition at the duct wall is therefore $\mathbf{n}\cdot\nabla\phi = p/Z(\omega)$. With slipping mean flow, the limit of an infinitely thin boundary layer is taken. This was shown to be equivalent to the model of an infinitely thin vortex sheet separating the moving fluid on one side and the fluid on the lining surface on the other [5–7]. The jump conditions across the vortex sheet lead to the boundary condition (quoted in the form due to Myers [8])

$$i\omega\mathbf{n}\cdot\nabla\phi = (i\omega + \mathbf{U}\cdot\nabla - (\mathbf{n}\cdot\nabla\mathbf{U})\cdot\mathbf{n})p/Z. \tag{1}$$

Some common models for the dependence of Z on ω will be considered here. The simplest is the mass–spring–damper model [2,9], or equivalently the three-parameter model of Tam and Auriault [4], all of which will be referred to here as the MSD model. If the vortex-sheet displacement in the normal direction is given by $w(\mathbf{x}, t)$, then this model is

$$d\frac{\partial^2 w}{\partial t^2} + R\frac{\partial w}{\partial t} + bw = p, \quad \Rightarrow \quad Z(\omega) = R + id\omega - ib/\omega. \tag{2}$$

The justification of this model is that it captures three physical quantities, the inertia or mass d, the springiness b, and the damping R. These parameters may be set to give the correct behaviour for $Z(\omega)$ *locally* about some target frequency ω_0 and target impedance Z_0.

Another model is the enhanced Helmholtz resonator (EHR) model proposed by Rienstra [10] (similar in form to some of the models listed in [3]), which is an extension of a model of a typical acoustic lining consisting of an array of Helmholtz resonators (HR) behind a perforated facing sheet. For this model

$$Z(\omega) = R + id\omega - iv\cot(\omega L - i\varepsilon/2), \tag{3}$$

where L is the depth of the HR, v is a parameter scaling the cavity reactance, ε is a damping within the fluid in the cavity, and the speed of sound has been normalized to unity. Setting $v = 1$ and $\varepsilon = 0$ yields the original HR model.

Experiments with acoustic linings and grazing flow show that these models, and particularly the EHR model, are a reasonably good approximation to reality [11–13]. When the above lining models are used in numerical time-domain simulations of the acoustics of lined ducts with uniform flow, the simulations are found to be unstable. Some experiments [11,13] also show an instability being excited in certain, but not all, situations. However, the numerical instability is of a different nature and occurs on the lengthscale of the computational mesh; using a finer mesh results in instability on a finer lengthscale which grows more rapidly. To enable convergence, such numerical simulations always include an artificial damping term to filter out the instability [4,14–19]. For frequency domain numerical simulations, the problem of instability is replaced by the problem of choosing the direction of propagation of modes. For example, Özyörük et al. [20] used a pseudo-time method which converges to a solution, but it is unclear which of the many potential solutions is converged to, and whether this is the causal one.

There has been much theoretical discussion about the stability of such lining models when used with uniform (slipping) mean flow. Tester [2] suggested that "strange" modes existed in a rectangular lined duct with uniform flow which might be convective instabilities, based on a Briggs–Bers [21,22] stability analysis. However, Tester gave the warning that this conclusions "must be regarded as provisional because the analysis is not fully rigorous"; in particular, it was assumed that the maximum temporal growth rate was finite, which is shown in Section 3 not to be the case. The "strange" modes found by Tester for a rectangular duct were analysed and classified by Rienstra [9], this time for a cylindrical duct, and were termed *surface modes* since they are localized close to the duct boundary (this analysis was subsequently extended in [23]).

Nilsson and Brander [24] identified a mode in a cylindrical duct with uniform flow as a potential instability, and selectively included or excluded it depending on the solution desired. They concluded that the Briggs–Bers method could not be applied for this problem owing to the unbounded maximum temporal growth rate, and instead used an interpretation of the stability criterion of Jones and Morgan [25]. However, their discussion of the Briggs–Bers criterion is confused, and their alternative procedure of analytically continuing the solution for $\arg(\omega)$ from 0 to $-\pi/2$ is unjustified. This unjustified procedure was later used by Koch and Möhring [26] and by Rienstra [27,28], who, in the latter paper, named this procedure the "Crighton–Leppington" criterion, apparently after [29]. Unfortunately, we demonstrate in Section 2 that this stability criterion is flawed.

The stability criterion proposed by Jones and Morgan [25] is also not universally valid. They state that "a necessary and sufficient condition for $\psi(\omega)$ to be causal is that it is a regular function of ω in the lower half-plane and that there are finite real numbers b and $d>0$ such that as $|\omega| \to \infty$, $\psi(\omega)\exp\{(b+\mathrm{i}d)\omega\} = O(|\omega|^s)$ for some finite s and $\mathrm{Im}\,\omega \leqslant -\varepsilon < 0$" (where the notation has been changed to that used here). However, taking $\tilde{\psi}(t) = H(t)\mathrm{e}^t$ gives $\tilde{\psi}(t) \equiv 0$ for $t<0$, so is causal, but

$$\psi(\omega) = \int_{-\infty}^{\infty} \tilde{\psi}(t)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t = \frac{-\mathrm{i}}{\omega+\mathrm{i}},$$

so that $\psi(\omega)$ has a pole at $\omega = -\mathrm{i}$ and so is not regular in the lower-half ω-plane. This is because, in Briggs–Bers terms, $\tilde{\psi}(t)$ is an absolute instability, which Jones and Morgan do not consider.

The unbounded maximum temporal growth rate that prevents the Briggs–Bers criterion being applied is connected with illposedness (as defined in the functional analysis sense, e.g., [30]). In Section 3, it is shown that the Briggs–Bers criterion is inapplicable if and only if the problem is illposed, and it is demonstrated what this means from a practical (computational and analytic) perspective. Moreover, we show that locally reacting linear impedance models of a certain type, which include all models considered here, are illposed when used with uniform (slipping) flow.

2. Stability criteria

Consider the linear partial differential equation

$$\Delta\left(\mathrm{i}\frac{\partial}{\partial x}, -\mathrm{i}\frac{\partial}{\partial t}\right)G(x,t) = 0, \tag{4}$$

where $\Delta(\cdot,\cdot)$ is the differential operator. Assuming a wave solution of the form $G(x,t) = G_0\exp\{\mathrm{i}\omega t - \mathrm{i}kx\}$ gives the dispersion relation $\Delta(k,\omega) = 0$. If ω is real and k is complex, we would like to know whether k corresponds to a left- or right-propagating mode, and therefore whether it is exponentially growing or decaying. This is very closely linked with causality.

2.1. The Briggs–Bers criterion

In order to analyse the stability of a mode at a frequency ω_f, the Briggs–Bers method [21,22] (henceforth referred to as BB) considers the response of the system to a forcing $\delta(x)H(t)\exp\{\mathrm{i}\omega_f t\}$; that is, a harmonic point forcing at frequency ω_f at position $x = 0$ turned on at time $t = 0$. The response of the system is analysed using a Fourier–Laplace transform, and causality is invoked to imply that the solution be identically zero for $t<0$. This gives the full exact solution in integral form. The long-time (large-t) limit of this solution is then investigated. If the long-time limit is time-harmonic with frequency ω_f, then we conclude that modes occurring to the left of the forcing in $x<0$ are left-propagating and modes occurring in $x>0$ are right-propagating. However, it is not necessarily the case that the long-time limit is time harmonic with frequency ω_f; for example, it is not the case if absolute instabilities are present. Note that the description of BB given by Rienstra [28] is overly simplified, and does not consider absolute instability. The full rigorous details of BB are given in Appendix A.

For BB to be valid, we required that the temporal inversion contour $\mathscr{C}_\omega$ be chosen below all values of ω for which $\Delta(k,\omega) = 0$ for any real k. However, in some problems, such as for the Kelvin–Helmholtz instability of a vortex sheet, there exist solutions of $\Delta(k,\omega) = 0$ for real k and arbitrarily negative $\mathrm{Im}(\omega)$, and so it is not possible to choose such an inversion contour. Such problems are illposed, as we will investigate in Section 3. It is for these problems that the "Crighton–Leppington" criterion was proposed.

2.2. The "Crighton–Leppington" criterion

We use here the description of the "Crighton–Leppington" criterion (henceforth referred to as CL) given by Rienstra [28], which is as it was used in [24,26,27]. For this criterion, the motion of poles is traced in the k-plane as ω varies with $|\omega|$ fixed and $\arg(\omega)$ running from $-\pi/2$ to 0. If the frequency is negative, the deformation is for $\arg(\omega)$ running from $-\pi/2$ to $-\pi$. Modes are predicted to be right-running if they originate in the lower-half k-plane for ω purely imaginary, and left-running if they originate in the upper-half k-plane. In contrast to BB, the full exact solution is never considered.

Rienstra [28] included the proviso that $|\omega|$ must be large enough for this procedure to be applicable, although how large $|\omega|$ must be was not discussed. This proviso was omitted in [24,26,27]. It is easy to show that taking $|\omega|$ larger than $-\mathrm{Im}(\omega)$ for any zero of $\Delta(\omega,k)$ for $k\in\mathbb{R}$ (in other words, so that $-\mathrm{i}|\omega|$ is below the initial BB $\mathscr{C}_\omega$ contour) means that this criterion reduces to BB as long as absolute instabilities are not present. However, CL was introduced for use when there is no bound to $-\mathrm{Im}(\omega)$ for real k, so how large $|\omega|$ must be in such cases remains unspecified.

2.3. Model stability example

We now look at the stability of an example system, chosen to allow a simple mathematical analysis while still including some relevant physics. The example is of a convected wave-like equation with diffusive and self-exciting terms,

$$\frac{\partial^2 G}{\partial x^2}-\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)^2 G+\lambda^2 G+\mu\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\frac{\partial^2 G}{\partial x^2}=0, \tag{5}$$

giving

$$\Delta(k,\omega)=(\omega-Uk)^2-k^2+\lambda^2-\mathrm{i}\mu(\omega-Uk)k^2.$$

The constants U, λ, and μ represent the convection, self-excitation and damping, respectively, with the wave speed normalized to unity. Since this is quadratic in ω and cubic in k, there are three modes (zeros of Δ) for a given frequency ω, whereas for a given wavenumber k there are two. Solving $\Delta(k,\omega)=0$ for ω in terms of k gives the two modes

$$\omega_\pm(k)=Uk+\frac{\mathrm{i}}{2}\mu k^2\pm\sqrt{k^2-\lambda^2-\frac{1}{4}\mu^2k^4}.$$

Note that $\omega_\pm(k)=Uk+\mathrm{i}\mu k^2(1\pm1)/2\mp\mathrm{i}/\mu+O(k^{-2})$ as $|k|\to\infty$. Using this, and by inspection, $\mathrm{Im}(\omega_\pm(k))>-\lambda$ for all $k\in\mathbb{R}$, so that BB is valid for this example. A graph of $\omega_\pm(k)$ for $k\in\mathbb{R}$ is given by the close-dashed line in the ω-plane in Figs. 1 and 2.

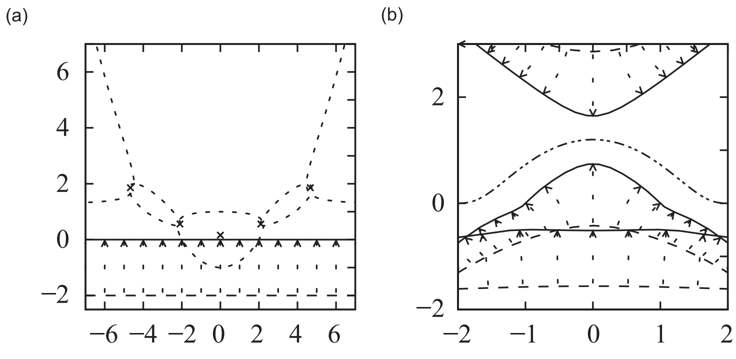

Fig. 1. Plots of Eq. (5) in (a) the ω-plane and (b) the k-plane for $U=2$, $\lambda=1$, $\mu=0.8$. The close dashed lines are solutions of $\Delta(\omega,k)=0$ for real k. Crosses denote values of ω from Eq. (6) giving a double root of $k(\omega)$. The long dashed line shows the initial $\mathscr{C}_\omega$ contour and $k(\omega)$ on this contour. The solid line shows the final $\mathscr{C}_\omega$ contour and $k(\omega)$ on this contour. The dashed arrowed lines show the motion between the two. The dash-dot line is the final $\mathscr{C}_k$ contour.

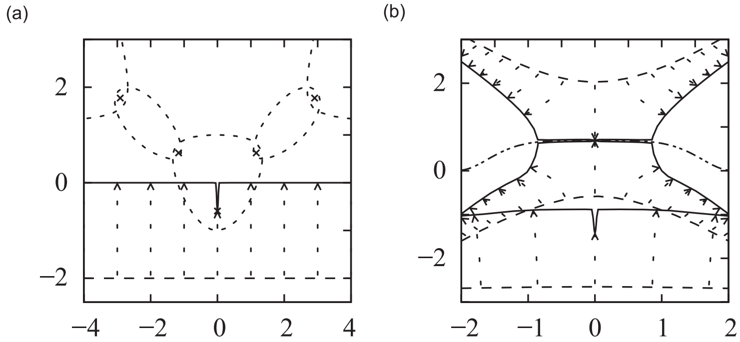

Fig. 2. As for Fig. 1 but with $U = 1.2$.

In order to analyse the stability of Eq. (5) using BB (as described in detail in Appendix A), we must first locate the double-roots in the k-plane that may lead to pinches of the $\mathscr{C}_k$ contour hence to absolute instability. These double roots are found by solving $\Delta = \partial\Delta/\partial k = 0$ simultaneously. Eliminating k gives the quintic equation for ω

$$2U\omega A^2 + 2(\beta^2 + \mathrm{i}\omega\mu)AB - 3\mathrm{i}U\mu B^2 = 0, \tag{6}$$

$$A = 2\beta^2 + 4\mathrm{i}\omega\mu(1 + 2U^2) - 2\omega^2\mu^2, \quad B = 7\mathrm{i}U\mu\omega^2 + 9\mathrm{i}U\mu\lambda^2 - 2U\omega\beta^2,$$

where $\beta^2 = 1 - U^2$. The numerical solutions of Eq. (6) are shown as crosses in Figs. 1 and 2.

We now apply BB to this example for $U = 2$, $\lambda = 1$, and $\mu = 0.8$, as shown in Fig. 1. Initially, we take the $\mathscr{C}_k$ contour along the real-k-axis and the $\mathscr{C}_\omega$ contour to be $\mathrm{Im}(\omega) = -2$. We then deform the $\mathscr{C}_\omega$ contour upwards onto the real axis, as shown by the dashed arrowed lines. In so doing, the poles in the k-plane move, and we must deform the $\mathscr{C}_k$ contour upwards in order to avoid poles crossing the contour. No double roots hinder us in this process, so that no absolute instabilities exist for these parameters. For $|\mathrm{Re}(\omega)| < 2$ poles in the lower-half k-plane have moved into the upper-half, and since these are below the $\mathscr{C}_k$ contour, they represent right-propagating waves which are therefore exponentially growing downstream. The system is therefore convectively unstable for real frequencies $|\omega| < 2$, and stable for $|\omega| \geqslant 2$. For $|\omega| = 2$ the system supports neutrally stable (i.e. propagating) modes.

The situation is significantly different for $U = 1.2$, as shown in Fig. 2. There is now a double-root in the lower-half ω-plane that involves two poles in the k-plane from opposite sides of the $\mathscr{C}_k$ contour which converge and pinch the contour. The $\mathscr{C}_\omega$ contour must therefore be deformed around this pinch frequency as shown. The dominant large-t contribution comes from this frequency, rather than any imposed forcing frequency ω_f, showing that in this case the system is absolutely unstable with dominant frequency $\omega_p \approx -0.592\mathrm{i}$.

What if we had instead applied CL? Considering as an example CL applied for $\omega = 0.5$, as shown by the solid lines in Fig. 3, our conclusions would have been different. In both cases considered, Fig. 3 shows no modes crossing from one half of the k-plane to the other, predicting no instabilities to be present for either $U = 2$ or 1.2 at a frequency $\omega = 0.5$, and so in both cases we would have predicted the system to have two stable upstream-propagating modes and one stable downstream-propagating mode. However, as shown by the trajectories in Fig. 3a with $\mathrm{Re}(\omega)$ fixed, in the first case there is an unstable mode, and in fact there are two downstream- and one upstream-propagating mode. In the second case, the system is absolutely unstable, and it does not make sense to consider a time-harmonic solution with frequency $\omega = 0.5$. Since BB is provably correct and valid in for these cases (see Appendix A), we are therefore forced to conclude that CL is erroneous for this example.

The above analysis has assumed $\mu \neq 0$. If we eliminate the damping term by setting $\mu = 0$ in Eq. (5), we lose one of the solutions for k in terms of ω, and BB may be applied entirely analytically. In the subsonic case $0 < U < 1$ this system is absolutely unstable, with a double root when $\omega = -\mathrm{i}\lambda\beta$ and $k_\pm = \mathrm{i}U\lambda/\beta$. In the supersonic case $U > 1$ the initial $\mathscr{C}_k$ contour lies above all poles in the k-plane, and therefore there is no

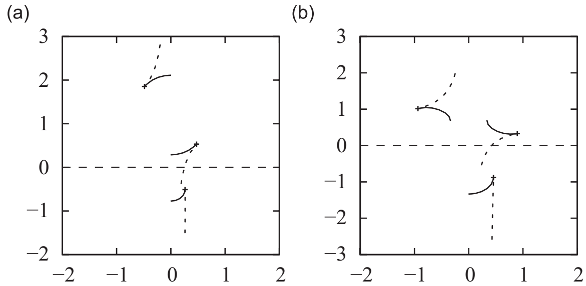

Fig. 3. In the k-plane, values of k solving $\Delta(\omega,k)=0$ for $\lambda=1$, $\mu=0.8$, and (a) $U=2.0$ and (b) $U=1.2$. Solid lines (CL trajectories) are $|\omega|=0.5$ with $-\pi/2<\arg(\omega)<0$, dashed lines are $\mathrm{Re}(\omega)=0.5$ and $-2<\mathrm{Im}(\omega)<0$, and crosses are for $\omega=0.5$.

absolute instability, since there are no poles above the contour to pinch it: all modes are downstream-propagating. One of the two modes is a convective instability for $|\omega|<\lambda\sqrt{U^2-1}$. For $\lambda\sqrt{U^2-1}\leqslant|\omega|\leqslant\lambda U$ both modes are neutrally stable (i.e. propagating), although one mode is anomalous in that it has an upstream-pointing group velocity. For $|\omega|>\lambda U$ the system supports two neutrally stable (i.e. propagating) modes, with the group velocities of both pointing downstream. CL incorrectly predicts the direction of propagation of one of the modes (and hence its stability) for $|\omega|<\lambda$, and of course also incorrectly predicts the subsonic case, as CL does not consider absolute instability.

3. Illposedness

BB is only applicable when the system being analysed has a maximum exponential growth rate, η say, meaning that it is guaranteed that $\Delta(k,\omega)\neq 0$ whenever $k\in\mathbb{R}$ and $\mathrm{Im}(\omega)<-\eta$. However, for sound in a cylindrical duct with uniform mean flow and a MSD lining, we show in the next section that no such η exists. This was the argument for using CL, which we have just seen is inadequate. In fact, any system for which BB is inapplicable is illposed. We now investigate this further.

What do we mean by illposedness? The definition used here, chosen since it leads to a helpful characterization of growth rates, is that the linear partial differential equation (4) is *wellposed* if

$$\sup_{x\in\mathbb{R}}|G(x,t+s)-G(x,t)|\to 0\quad\text{as } s\to 0$$

for all t and $s>0$. As shown by Yosida (Ref. [30, pp. 232–233]), any wellposed differential equation necessarily has a finite maximum exponential growth rate η. Another result of Yosida (Ref. [30, pp. 240–241]) can be used to show that the Fourier transform $G(x,\omega)$ exists and is well defined for $\mathrm{Im}(\omega)<-\eta$, implying that $\Delta(k,\omega)\neq 0$ for all $\mathrm{Im}(\omega)<-\eta$ and $k\in\mathbb{R}$. Therefore, if we have a system for which $\Delta(k,\omega)=0$ for arbitrarily negative $\mathrm{Im}(\omega)$ and $k\in\mathbb{R}$, the problem must be illposed, and so the condition above must not always be true. This is, in effect, because arbitrarily quickly growing instabilities exist that can grow arbitrarily large between time $t=0$ and time 0^+, and the solution can "blow up instantly".

One practical implication of illposedness is that numerical simulations do not converge, however, small a timestep is used. For example, for sound in a cylindrical duct with a MSD lining and uniform mean flow, we show in the next section that there are surface modes for which $\mathrm{Im}(\omega)\to\infty$ as $k\to\pm\infty$, so that the arbitrarily quick exponential growth occurs at arbitrarily short wavelengths. When numerically simulating such a system on a fine mesh, it is found that the numerics are unstable at the grid scale, with even finer meshes becoming unstable even more rapidly. These instabilities are therefore routinely filtered out [4,14–19]. However, this instability can now be seen as the numerics attempting to accurately simulate the underlying mathematical differential equation, which has no regular mathematical solution. Having filtered out the unwanted part of the numerical solution, there is no justification that what is left is of any relevance to the physical problem being modelled. We have seen above that the illposedness causes problems with stability analysis, and in addition it causes problems for mode-matching [31–33], Green's function derivations [34,35] and scattering analyses [26,28,36].

3.1. The illposedness of a large class of impedance lining models

Consider a uniform cylindrical duct with centreline in the x-direction and cross-section described by polar coordinates r, θ. We nondimensionalize lengths by the duct radius, speeds by the mean-flow speed of sound, and densities by the mean-flow density. The velocity of the fluid is given by $\mathbf{u} = U\mathbf{e}_x + \nabla\phi$, where ϕ is the acoustic perturbation and U is the steady mean flow Mach number. This gives the wave equation for ϕ,

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 - \nabla^2\phi = 0 \quad \text{with } p = \rho = -\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\phi,$$

with solution

$$\phi = J_m(\alpha r)\exp\{\mathrm{i}\omega t - \mathrm{i}kx - \mathrm{i}m\theta\}, \quad \alpha^2 = (\omega - Uk)^2 - k^2,$$

where J_m is the mth Bessel function of the first kind (for details, see, e.g., [23]). The boundary condition (1) becomes

$$\frac{\alpha J_m'(\alpha)}{J_m(\alpha)} - \frac{(\omega - Uk)^2}{\mathrm{i}\omega Z(\omega)} = 0,$$

where $Z(\omega)$ is the lining impedance. We restrict attention here to subsonic mean flows, so that $0<U<1$.

When considering the boundedness of $\mathrm{Im}(\omega)$ for zeros of $\Delta(k, \omega) = 0$ with $k \in \mathbb{R}$, it is the surface modes that cause the potentially unbounded behaviour, with the other acoustic modes being well behaved. The surface modes are predicted using the asymptotic dispersion relation of [23] (Eq. (7), a modification of the original approximation of [9]), which, when rearranged, gives

$$(k^2 + m^2 - (\omega - Uk)^2)(\mathrm{i}\omega Z)^2 - (\omega - Uk)^4 = 0 \quad \text{with } \mathrm{Re}\left(\frac{(\omega - Uk)^2}{\mathrm{i}\omega Z}\right) > 0. \tag{7}$$

We now consider the behaviour of ω as $k \to \pm\infty$ with k real. Assuming $\mathrm{i}\omega Z = A\omega^\mu$ to leading order, we have three cases depending on whether $\mu>1$, $\mu = 1$, or $\mu<1$,

$$\mu>1, \quad \omega = Nk^{1/\mu}, \;\; N^\mu = \pm U^2/(A\beta),$$

$$\mu = 1, \quad \omega = Nk, \;\; (\beta^2 + 2UN - N^2)A^2N^2 - (N - U)^4 = 0,$$

$$\mu<1, \quad \omega = Uk + Nk^{(\mu+1)/2}, \;\; N^2 = \pm AU^\mu, \tag{8}$$

where, as before, $\beta^2 = 1 - U^2$. The $\mu = 1$ case involves solving a quartic equation for N, which simplifies significantly if either $|A| \ll 1$ or $|A| \gg 1$. If $|A| \ll 1$, then the solutions are

$$N = U + \sqrt{AU}\mathrm{e}^{\mathrm{i}n\pi/2} + O(A),$$

where the condition in Eq. (7) implies that $n = 0, 2$ if $k \to \infty$ and $n = 1, 3$ if $k \to -\infty$. If $|A| \gg 1$ then the solutions are

$$N = \pm\frac{U^2}{\beta A} + O(A^{-2}) \quad \text{and} \quad N = U \pm 1 + O(A^{-2}),$$

where the condition in Eq. (7) implies that $N = U^2/\beta A$ as $k \to \infty$, with the opposite sign as $k \to -\infty$. The solutions $N = U \pm 1$ represent acoustic modes, rather than surface modes, and have $\mathrm{Re}(\sqrt{\cdots}) = 0$.

For the MSD lining, Eq. (2) implies that

$$\mathrm{i}\omega Z = -d\omega^2 + \mathrm{i}\omega R + O(1).$$

If $d>0$ then we are in the case $\mu = 2$ and $A = -d$, so that there exists a surface mode with $\mathrm{Im}(\omega) \sim -(|k|U^2/d\beta)^{1/2}$, and hence $\mathrm{Im}(\omega) \to -\infty$ as $|k| \to \infty$. This is exactly the scaling mentioned by Rienstra and Peake [37]. The problem of sound in a duct with uniform flow and a MSD lining is therefore illposed if $d \neq 0$. If $d = 0$ then we are in the different regime $\mu = 1$ and $A = \mathrm{i}R$, so that as $k \to \pm\infty$, there exists

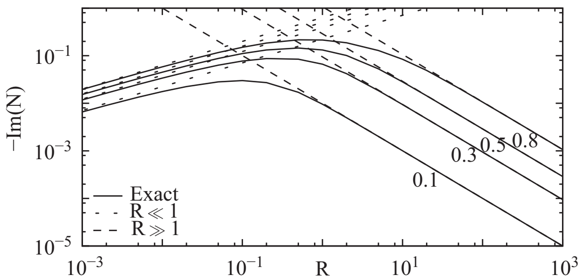

Fig. 4. Plot of $-\mathrm{Im}(N)$ against R, defined by Eq. (8) with $A = \mathrm{i}R$, for four mean flow velocities $U = 0.1, 0.3, 0.5, 0.8$. Also plotted are the asymptotics given in Eq. (9).

a surface mode for which

$$\mathrm{Im}(\omega) \sim -\sqrt{RU/2}|k| \quad \text{if } R \ll 1,$$

$$\mathrm{Im}(\omega) \sim -\frac{U^2|k|}{\beta R} \quad \text{if } R \gg 1, \tag{9}$$

so that the problem is still illposed in these regimes. If $R = O(1)$ we must proceed by solving the full quartic equation. Fig. 4 plots $-\mathrm{Im}(N)$ against R for a variety of subsonic Mach numbers U, and demonstrates that in all cases there exists a solution N with $\mathrm{Im}(N) < 0$, while also showing the accuracy of the asymptotics from Eq. (9) in their respective regimes. This demonstrates that the MSD model is illposed for all values of R even when $d = 0$.

We next consider the EHR model in Eq. (3). We are interested in $\mathrm{Im}(\omega)$ being large and negative, and since $\cot(x + \mathrm{i}y) \to \mathrm{i}$ as $y \to -\infty$, we have

$$\mathrm{i}\omega Z = -d\omega^2 + \mathrm{i}\omega(R + v) + O(\omega \exp\{2\,\mathrm{Im}(\omega)\}).$$

This is the same as for the MSD impedance with R replaced by $R + v$. We therefore have the same behaviour, namely $\mathrm{Im}(\omega) \to -\infty$ as $|k| \to \infty$, so that the EHR model is also illposed.

We have therefore demonstrated that the MSD and EHR models are illposed for uniform mean flow $0 < U < 1$, for any $d \geqslant 0$. Moreover, exactly the same argument shows that this illposedness holds true for *any* locally reacting linear lining such that, as $\mathrm{Im}(\omega) \to -\infty$, either $\mathrm{i}\omega Z \sim A\omega^\mu$ for $\mu > 1$ or $\mathrm{i}\omega Z \sim A\omega$ for $\mathrm{Im}(A) > 0$. This covers a very general class of lining models.

4. Conclusion

The debate about the possible existence of an instability of uniform slipping mean flow over a linearly reacting acoustic lining has been explained by the discovery that the problem is illposed, at least for the lining models considered here. This explains the tentativeness of previous stability analyses [2,9,24], the confusion over the direction of propagation of "strange", or surface, modes [20,26,28,31–35,37], and the apparent numerical instability of time-domain simulations of flow over acoustic linings [4,14–19]. While the term illposed has been used with varying meaning in the acoustics literature (e.g. [4]), here it is used in its formal functional-analysis sense [30].

The stability criterion of [24,26,27], as described by Rienstra [28] as the "Crighton–Leppington" criterion, has no rigorous mathematical derivation, and has been shown to incorrectly predict the stability of the wave-like differential equation example 1. Its use should therefore be strongly discouraged. This criterion differs from the one described by Crighton and Leppington [29], which is a similar, although less general, version of the Briggs–Bers criterion; moreover, Crighton and Leppington never relied on their time-harmonic analysis, but instead gave explicit causal solutions in terms of ultradistributions (which would be impossible to simulate numerically).

The problem of acoustics in lined ducts with uniform mean flow has been shown to be illposed for a large class of lining models, including the mass–spring–damper and enhanced Helmholtz resonator models. Being illposed implies that there is no regular mathematical solution to the problem, essentially because the equation permits exponential growth $\exp\{\eta t\}$ for arbitrarily large η, and so the solution at time t in the limit $t \to 0$ does not necessarily coincide with the initial conditions. Not having a regular mathematical solution makes it impossible to analyse stability. Since there is no regular mathematical solution to expand in terms of duct modes of single frequencies, single-frequency analyses are at best contradictory, and at worst invalid, causing problems with frequency-domain simulations, mode-matching, and scattering. Being illposed with $\mathrm{Im}(\omega) \to -\infty$ as $|k| \to \infty$ also means that numerical simulations in the time domain become unstable at the grid scale for all sufficiently fine grids, since the problem possesses arbitrarily quick growth at arbitrarily short wavelengths. Simply filtering out the instability is not satisfactory, since there is no reason the filtered numerical solution should be connected to the underlying physical system being modelled. We would therefore strongly discourage the use of illposed models.

The illposed problem of perturbations to a vortex sheet was regularized by Jones [38] by considering a boundary layer of finite thickness h, with the previous illposed behaviour [29,39] recovered in the limit $h \to 0$. This is comparable to the regularization of the mass–spring–damper lining model by considering the boundary as a thin shell [36], where the problem is wellposed for thin shell thicknesses $h>0$, and the mass–spring–damper model is recovered in the limit $h \to 0$. Of course, the thin-shell regularization is not a correct model for the actual physics of an acoustic lining; a wellposed model of the actual physics would obviously be preferable. It has been suggested [6,17,40–42] that modelling the shear layer over the lining is important, rather than the assumption of a vortex sheet [5,6], particularly for upstream-propagating sound, and this would seem to agree with Jones' regularization of the Kelvin–Helmholtz instability. However, a shear layer model brings with it its own problems, such as the hydrodynamic continuous spectrum and thereby the lack of solutions expressible as a sum of duct modes.

Appendix A

A.1. A detailed description of the Briggs– Bers stability criterion

The Briggs–Bers criterion [21,22] is effectively a Fourier–Laplace transform method. Consider the linear partial differential equation

$$\Delta\left(\mathrm{i}\frac{\partial}{\partial x}, -\mathrm{i}\frac{\partial}{\partial t}\right)G(x,t) = 0, \tag{A.1}$$

where Δ is the differential operator. We wish to analyse the stability of this system at a frequency $\omega_f \in \mathbb{R}$. As described by Briggs, we introduce a harmonic point-forcing term $\delta(x)H(t)\exp\{\mathrm{i}\omega_f t\}$ to the right-hand side of Eq. (A.1), and then require that $G \equiv 0$ for $t<0$ in order to satisfy causality. To solve this, we consider the transformation

$$\tilde{G}(k,\omega) = \int_0^\infty \int_{-\infty}^\infty G(x,t)\exp\{\mathrm{i}kx - \mathrm{i}\omega t\}\,\mathrm{d}x\,\mathrm{d}t.$$

Note that this is a Fourier transform $x \to k$ and a Laplace transform $t \to \mathrm{i}\omega$, the usual Laplace transform variable being $s = \mathrm{i}\omega$. These transformations are only valid provided the integrals converge.[1] Transforming the differential equation with the added forcing term then gives

$$\Delta(k,\omega)\tilde{G}(k,\omega) = \frac{-\mathrm{i}}{\omega - \omega_f},$$

where $\Delta(k,\omega)$ is just a polynomial in k and ω. Inverting this transform gives the exact analytic solution

$$G(x,t) = \frac{1}{4\pi^2}\int_{\mathscr{C}_\omega}\int_{\mathscr{C}_k} \frac{-\mathrm{i}\,\exp\{\mathrm{i}\omega t - \mathrm{i}kx\}}{(\omega - \omega_f)\Delta(k,\omega)}\,\mathrm{d}k\,\mathrm{d}\omega, \tag{A.2}$$

[1] For the t-integral, convergence requires that $\mathrm{Im}(\omega)$ be sufficiently negative (so that $\mathrm{Re}(s)$ is sufficiently large) that the integrand tends to zero as $t \to \infty$. For the x-integral, we require that $|G(x,t)| \to 0$ sufficiently fast as $|x| \to \infty$, where sufficiently fast means fast enough that we may differentiate under the integral the number of times required by the differential equation (exponential decay is sufficient, but not necessary). These conditions will be seen to be satisfied in what follows.

where $\mathscr{C}_\omega$ and $\mathscr{C}_k$ are the inversion contours. Since the k-integral is a Fourier inversion, the $\mathscr{C}_k$ contour is along the real-k-axis. For $x<0$ the contour may be closed in the upper-half k-plane and Jordan's Lemma applied, showing that for $x<0$ the solution is given as a sum of residues of poles (i.e. modes) in the upper-half k-plane, and similarly for $x>0$ and modes in the lower-half k-plane. The $\mathscr{C}_\omega$ contour will be chosen to ensure that there are no poles for real k, which ensures that $G(x,t)$ has suitable behaviour as $|x| \to \infty$, as required for the convergence of the integrals above, since all modes will then decay exponentially quickly as $|x| \to \infty$. For the ω-integral, which is a Laplace inversion contour, the inversion contour for $s = \mathrm{i}\omega$ must be taken to the right of all poles of the integrand, so that $\mathscr{C}_\omega$ must be taken below all poles of the integrand for any real k. This requirement ensures that there are no poles for real k, and also that the solution is causal, since for $t<0$ the inversion contour may be closed in the lower-half ω-plane and Jordan's Lemma applied to give $G \equiv 0$ for $t<0$. It is also equivalent to the requirement of taking $\mathrm{Im}(\omega)$ sufficiently negative that the forward-transform integral converges.

Now that we have the full analytic solution (in integral form), we may assess the stability of the original differential equation by looking at the long-time (large-t) limit. The large-t limit is found by deforming the $\mathscr{C}_\omega$ contour upwards in the ω-plane, maintaining analyticity by deforming the contour around any poles of the integrand. The exponentially dominating large-t contribution is then from the lowest pole in the ω-plane, with all other contributions from the remainder of the contour being exponentially small in comparison.

During this process, the $\mathscr{C}_k$ contour may need to be deformed in order that no poles cross this contour in the k-plane, thereby maintaining the correct analytic continuation. In certain cases, two modes occurring on either side of the $\mathscr{C}_k$ contour may coincide as the $\mathscr{C}_\omega$ contour is deformed and *pinch* the $\mathscr{C}_k$ contour. Overall, there are three possibilities:

(a) The $\mathscr{C}_\omega$ contour may be deformed into the upper-half ω-plane, with the only deformation necessary being around the pole at ω_f on the real axis, and no deformation of the $\mathscr{C}_k$ contour being necessary. The dominant large-t contribution therefore comes from the pole at ω_f. The solution for $x<0$ is hence a sum of modes of the form $\exp\{\mathrm{i}\omega_f t - \mathrm{i}kx\}$, where k satisfies the dispersion relation $\Delta(k,\omega_f) = 0$ and k lies in the upper-half k-plane. Similarly, the solution for $x>0$ is the sum of poles occurring in the lower-half k-plane. Since all of these modes decay exponentially as $|x| \to \infty$, the system is *stable*.
If there are poles on the real-k-axis for ω_f, say $\Delta(k_0,\omega_f) = 0$ with $k_0 \in \mathbb{R}$, care must be taken to assign these to the correct side of the point forcing. For $\omega = \omega_f - \mathrm{i}\varepsilon$, we have $k = k_0 - \mathrm{i}\varepsilon/(\mathrm{d}\omega/\mathrm{d}k) + O(\varepsilon^2)$, so that if the group velocity $\mathrm{d}\omega/\mathrm{d}k$ has a positive real part the mode will have originated from below the $\mathscr{C}_k$ contour and will therefore be present for $x>0$, whereas if the group velocity has a negative real part the mode will be present for $x<0$. In both of these cases the system is referred to as being *neutrally stable*, and modes with $k \in \mathbb{R}$ are called *propagating*. If the group velocity is purely imaginary, either higher order terms must be consulted or the trajectory of the mode plotted to determine if it is left- or right-propagating.

(b) If the $\mathscr{C}_\omega$ contour deformation into the upper-half k-plane (and around the pole at ω_f) is possible, but the $\mathscr{C}_k$ contour needed to be deformed to do so because of poles in the k-plane crossing the real-k-axis, the large-t solution is still a sum of modes of the form $\exp\{\mathrm{i}\omega_f t - \mathrm{i}kx\}$, as in (a), with k being solutions of $\Delta(k,\omega_f)$ and modes below the $\mathscr{C}_k$ contour occurring for $x>0$ and modes above for $x<0$. However, it is now possible that a mode that started in one half of the k-plane finishes in the other, and therefore grows exponentially in space as $|x| \to \infty$; in this case, the system is referred to as being *convectively unstable*. The sign of the group velocity is also no longer conclusive in determining the direction of propagation of the propagating modes, and the trajectories of these modes must now be plotted.

(c) Finally, if two modes in the k-plane from opposite sides of the $\mathscr{C}_k$ contour coincide and pinch the $\mathscr{C}_k$ contour, say when $\omega = \omega_p$ and $k = k_p$, the $\mathscr{C}_\omega$ contour cannot be deformed further through this point while maintaining analytic continuity. In fact, locally about this point, $\Delta(k_p + \delta k, \omega_p + \delta\omega) = a\delta\omega + b\delta k^2 + O(\delta\omega^2, \delta k^3)$, so that the k-poles locally about $\omega = \omega_p + \delta\omega$ are given by $k = k_p \pm \sqrt{-a\delta\omega/b}$, and a branch cut in the ω-plane is necessary. Taking this branch cut to be along $\omega = \omega_p + \mathrm{i}y$ for $y>0$, the $\mathscr{C}_\omega$ contour must be deformed around the singularity at $\omega = \omega_p$ and along the branch cut. The large-t contribution from this part of the integral can be shown to be of the form $\exp\{\mathrm{i}\omega_p t - \mathrm{i}k_p x\}/\sqrt{t}$ for both $x<0$ and $x>0$, and since $\mathrm{Im}(\omega_p)<0$, it is this that dominates the large-t solution, rather than the pole at ω_f. The system therefore chooses its own frequency at which to be unstable, and the solution grows exponentially in t for any position x; this is referred to as *absolute instability*.

This forms the Briggs–Bers stability criterion. First, we look for any values of ω with $\mathrm{Im}(\omega) < 0$ which leads to a double root in the k-plane, and investigate whether any such ω involve the collision of modes from opposite sides of $\mathscr{C}_k$. If so, the system is absolutely unstable, irrespective of the frequency we attempt to force it at. Otherwise, the system may be stable or convectively unstable, with the long-time solution being time-harmonic at the forcing frequency ω_f. The direction of propagation of modes may be found by tracking their location as ω is varied as $\omega = \omega_f + \mathrm{i}y$, with y starting suitably negative (we require $y < \mathrm{Im}(\omega)$ for any solution for ω of $\Delta(k, \omega) = 0$ for $k \in \mathbb{R}$, as this means we start below the original $\mathscr{C}_\omega$ contour). The direction of propagation of a mode $k(\omega_f)$ is given by which side of the real-k-axis it starts on, so that modes that start on one-side and end on the other as y goes from suitably negative to zero correspond to convective instabilities. We look at some examples of how this criterion is applied in Section 2.3.

For the Briggs–Bers method to be valid, we required that the initial $\mathscr{C}_\omega$ contour be chosen below all values of ω for which $\Delta(k, \omega) = 0$ for any value of $k \in \mathbb{R}$, or looked at another way, we require the Laplace transform to be defined for at least some values of s. However, in some problems, such as for the Kelvin–Helmholtz instability of a vortex sheet, $\mathrm{Im}(\omega)$ is not bounded below, and so it is not possible to choose such an inversion contour; such problems are illposed, as described in Section 3.

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