Available online at www.sciencedirect.com
ScienceDirect
Journal of Sound and Vibration 322 (2009) 1038–1047
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Free vibration of rotating hollow spheres containing acoustic media

Michael Y. Shatalov[a,b], Stephan V. Joubert[a,*], Charlotta E. Coetzee[a], Igor A. Fedotov[a]

[a]Department of Mathematics and Statistics, Tshwane University of Technology, P.O. Box X680, Pretoria 0001, South Africa
[b]CSIR, Material Science and Manufacturing, Sensor Science and Technology, P.O. Box 395, Pretoria 0001, South Africa

Received 16 April 2008; received in revised form 7 November 2008; accepted 12 November 2008
Handling Editor: M.P. Cartmell
Available online 23 December 2008

Abstract

When a vibrating structure is rotated with respect to inertial space, the vibrating pattern rotates at a rate proportional to the inertial rate of rotation. Bryan first observed this effect in 1890. The effect, called Bryan's effect in the sequel, has numerous navigational applications and could be useful in understanding the dynamics of pulsating stars and earthquake series in astrophysics and seismology. Bryan's factor (the coefficient of proportionality between the inertial and vibrating pattern rotation rates) depends on the geometry of the structure and the vibration mode number. The ''gyroscopic effects'' of a hollow isotropic solid sphere filled with an inviscid acoustic medium are considered here, but the theory is readily adapted to a hollow isotropic solid cylinder filled with an inviscid acoustic medium. A linear theory is developed assuming, among other mild conditions, that the rotation rate is constant and much smaller than the lowest eigenfrequency of the vibrating system. Thus centrifugal forces are considered to be negligible. Before calculating solutions for the displacement of a particle in the isotropic, spherical, distributed body, Bryan's factor is interpreted using a complex function. Here it is demonstrated that neither Bryan's effect nor Bryan's factor is influenced by including light, isotropic, viscous damping in the mathematical model. Hence damping is neglected in the sequel. Two scenarios are then identified. Firstly, we may assume that the acoustic medium is completely involved in the rotation (the spheroidal mode). Secondly, we may assume that the acoustic medium remains static with respect to the inertial reference frame (the torsional mode). We investigate the spheroidal mode using a numerical experiment that compares the rotational angular rate of a sphere (filled with an inviscid acoustic medium) with those of its vibrating patterns at both high and low vibration frequency.

1. Introduction

When a vibrating structure is subjected to a rotation, the vibrating pattern rotates at a rate proportional to the inertial angular rate. This effect, known as ''Bryan's effect'' in the sequel, was first observed by Bryan in 1890 [1]. For the constant of proportionality, Bryan made the following calculation for a body consisting of a

*Corresponding author. Tel.: +27 123826358; fax: +27 123826114.
E-mail address: joubertsv@tut.ac.za (S.V. Joubert).

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2008.11.020

ring or cylinder:

$$\mathrm{BF} = \frac{\text{Angular velocity of the vibrating pattern}}{\text{Inertial angular velocity of the vibrating body}} \tag{1}$$

for various modes of vibration. This constant of proportionality BF has come to be known as ‘‘Bryan's factor’’. In 1988, Zhuravlev and Klimov [2] investigated Bryan's factor for an elastic, isotropic, spherically symmetric body, rotating in three-dimensional space. Among other results, Zhuravlev and Klimov demonstrated that Bryan's factor depends on the vibration mode. Zhuravlev and Klimov's concise formulation is given in general terms without computational detail or assumptions on the magnitude of rotation or illustrative examples. We assume ‘‘slow rotation’’ (explained below) of spherical bodies consisting of concentric layers of elastic and/or acoustic media, we supply some detail of computations and we present an illustrative example. We do not assume a ‘‘thin shell theory’’, as Loveday and Rogers [3] do, where they considered Bryan's effect in a thin cylindrical shell for both high and low rotational rates. The effect has numerous navigational applications, and Loveday and Rogers list some papers dealing with this. Apart from navigational applications, the theory presented below could be useful in understanding the dynamics of pulsating stars and earthquake series in astrophysics and seismology. What we discuss below is also applicable to an isotropic solid sphere with distributed parameters in the form of concentric isotropic solid spherical layers. The theory is readily adapted to an isotropic solid cylinder consisting of concentric layers or a hollow isotropic solid cylinder containing an inviscid acoustic medium. Situations where damping can be neglected exist (see for instance the patent of Loper and Lynch [4]). If damping is present in the media we expect to encounter, it will be light in the sense that the ‘‘damping factor’’ will be substantially smaller than the lowest eigenvalue of the system. Using Rayleigh's dissipation function [5], we demonstrate that light, isotropic, viscous damping does not influence Bryan's effect or Bryan's factor (1). Hence, in the sequel, we assume that the body is subjected to non-decaying vibrations in one of its natural modes. The introduction of prestress and mass-stiffness imperfections (anisotropic damping effects) into the calculations is important for real-life situations (as opposed to ideal situations) and has been earmarked by us for further study.

Consider a coordinate system $Oxyz$ and a composite spherical body, with its centre at the origin O, consisting of concentric solid and or acoustic layers. Let N be the number of concentric spherical media in the system and a_{i-1} and a_i the inner and outer radii of the ith body, respectively, $i = 1, \ldots, N$. We convert to spherical coordinates $Or\theta\phi$ as depicted in Fig. 1, where we have adopted the notation of Spiegel [6].

Consider the position of rest $P(r, \theta, \phi)$ of a vibrating particle in the ith body, $a_{i-1} \leqslant r \leqslant a_i$. Let $\hat{\mathbf{r}}$ be the unit vector in the direction of increasing r. Hence the position vector of the point $P(r, \theta, \phi)$ is $\mathbf{r} = r\hat{\mathbf{r}}$. Consider the usual unit vectors $\hat{\boldsymbol{\phi}} = (\partial\mathbf{r}/\partial\phi)/|\partial\mathbf{r}/\partial\phi|$ (in the direction of increasing ϕ) and $\hat{\boldsymbol{\theta}} = (\partial\mathbf{r}/\partial\theta)/|\partial\mathbf{r}/\partial\theta|$ (in the direction of increasing θ). Let $\mathbf{w}_i + \mathbf{u}_i + \mathbf{v}_i$ (where $\mathbf{w}_i = w_i\hat{\mathbf{r}}$, $\mathbf{u}_i = u_i\hat{\boldsymbol{\theta}}$ and $\mathbf{v}_i = v_i\hat{\boldsymbol{\phi}}$) represent the displacement from the position of rest of the vibrating particle in the ith body (see Fig. 1). For the sake of simplicity we

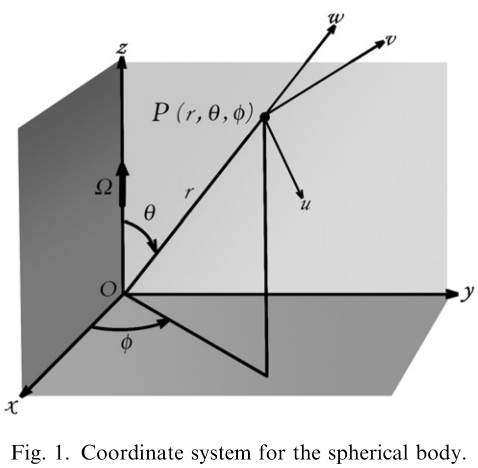

Fig. 1. Coordinate system for the spherical body.

suppress subscripts i when no confusion is expected. The position vector of the vibrating particle is thus

$$\mathbf{R} = (r+w)\hat{\mathbf{r}} + u\hat{\boldsymbol{\theta}} + v\hat{\boldsymbol{\phi}}. \tag{2}$$

Now consider an inertial coordinate system $OXYZ$ with origin O where initially the X, Y, Z-axes correspond to the x, y, z-axes, respectively. Let the spherical body (the $Or\theta\phi \equiv Oxyz$ system) rotate about the z-axis with a small constant angular rate Ω with respect to inertial space $OXYZ$. If $\hat{\mathbf{k}}$ is the unit vector in the direction of increasing z, then the angular velocity $\boldsymbol{\Omega}$ of the body is

$$\boldsymbol{\Omega} = \Omega\hat{\mathbf{k}} = \Omega(\hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta). \tag{3}$$

By ''smallness'' of the angular rate of rotation Ω we mean that this rate is substantially smaller than the lowest eigenvalue of the system. Consequently, we will neglect centrifugal effects and all other terms of $O(\Omega^2)$.

2. Gyroscopic effects in distributed bodies

The mathematical formulation given below (in spherical coordinates) is presented within the framework of the linearized, three-dimensional theory of elasticity and is similar to that presented by Berliner and Solecki [7] for cylindrical coordinates. With Lagrange's equations (see Ref. [6]) in mind, we formulate expressions for the (approximate) kinetic and potential energies of the system of concentric spherical bodies. The absolute linear velocity of the vibrating particle is

$$\mathbf{V} = \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} + \boldsymbol{\Omega}\times\mathbf{R} = (\dot{w} - \Omega v\sin\theta)\hat{\mathbf{r}} + (\dot{u} - \Omega v\cos\theta)\hat{\boldsymbol{\theta}} + (\dot{v} + \Omega(u\cos\theta + (r+w)\sin\theta))\hat{\boldsymbol{\phi}}. \tag{4}$$

The approximate kinetic energy of the system of concentric spherical bodies is given by

$$K = \frac{1}{2}\sum_{i=1}^{N}\rho_i\int_0^{2\pi}\int_0^{\pi}\int_{a_{i-1}}^{a_i}\{(\dot{u}_i^2 + \dot{v}_i^2 + \dot{w}_i^2) + 2\Omega[(u_i\dot{v}_i - \dot{u}_i v_i)\cos\theta + (\dot{v}_i(r+w_i) - v_i\dot{w}_i)\sin\theta]\}r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi. \tag{5}$$

The potential energy of the system of concentric spheres is

$$P = \frac{1}{2}\sum_{i=1}^{N}\int_0^{2\pi}\int_0^{\pi}\int_{a_{i-1}}^{a_i}\{\sigma_{i,rr}\varepsilon_{i,rr} + \sigma_{i,\theta\theta}\varepsilon_{i,\theta\theta} + \sigma_{i,\phi\phi}\varepsilon_{i,\phi\phi} + \sigma_{i,r\theta}\varepsilon_{i,r\theta} + \sigma_{i,\theta\phi}\varepsilon_{i,\theta\phi} + \sigma_{i,r\phi}\varepsilon_{i,r\phi}\}r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi, \tag{6}$$

where we use "ρ" for mass density, "σ" for stress and "ε" for strain. We indicate Lamé's constants (from the theory of elasticity) by λ_i and μ_i for the ith body. In a spherical coordinate system, with the subscript i suppressed, a standard reference such as Redwood [8] yields stresses

$$\sigma_{rr} = \lambda(\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{\phi\phi}) + 2\mu\varepsilon_{rr}, \quad \sigma_{\theta\theta} = \lambda(\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{\phi\phi}) + 2\mu\varepsilon_{\theta\theta},$$
$$\sigma_{\phi\phi} = \lambda(\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{\phi\phi}) + 2\mu\varepsilon_{\phi\phi},$$
$$\sigma_{r\theta} = \mu\varepsilon_{r\theta}, \quad \sigma_{\theta\phi} = \mu\varepsilon_{\theta\phi}, \quad \sigma_{r\phi} = \mu\varepsilon_{r\phi} \tag{7}$$

and strains

$$\varepsilon_{rr} = \frac{\partial w}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{1}{r}\left(\frac{\partial u}{\partial\theta} + w\right), \quad \varepsilon_{\phi\phi} = \frac{1}{r}\left(u\cot\theta + \frac{1}{\sin\theta}\frac{\partial v}{\partial\phi} + w\right),$$
$$\varepsilon_{r\theta} = \frac{\partial u}{\partial r} + \frac{1}{r}\left(\frac{\partial w}{\partial\theta} - u\right), \quad \varepsilon_{\theta\phi} = \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial u}{\partial\phi} + \frac{\partial v}{\partial\theta} - v\cot\theta\right),$$
$$\varepsilon_{r\phi} = \frac{\partial v}{\partial r} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial w}{\partial\phi} - v\right). \tag{8}$$

Assume that we can express the magnitude of the displacements u_i, v_i and w_i of the vibrating particle P in the ith body as follows:

$$u_i(r,\theta,\phi,t) = U_i(r,\theta)[C(t)\cos m\phi + S(t)\sin m\phi], \tag{9a}$$

$$v_i(r,\theta,\phi,t) = V_i(r,\theta)[C(t)\sin m\phi - S(t)\cos m\phi], \tag{9b}$$

$$w_i(r,\theta,\phi,t) = W_i(r,\theta)[C(t)\cos m\phi + S(t)\sin m\phi], \tag{9c}$$

where the nature of the functions $C(t)$ and $S(t)$ is still to be determined, $U_i(r,\theta)$, $V_i(r,\theta)$ and $W_i(r,\theta)$ are eigenfunctions of the system and m is the circumferential wavenumber.

Substituting Eq. (9) into Eqs. (8), (7), (6) and (5) involves a long algebraic calculation. A computer algebra system is handy for checking the calculation that yields

$$K = \pi[I_0(\dot{C}^2 + \dot{S}^2) + 2\Omega I_1(\dot{C}S - C\dot{S}))] \tag{10}$$

and

$$P = \pi I_2(C^2 + S^2). \tag{11}$$

Here

$$I_0 = \frac{1}{2}\sum_{i=1}^{N}\rho_i\int_0^{\pi}\int_{a_{i-1}}^{a_i}(U_i^2 + V_i^2 + W_i^2)r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta \tag{12}$$

and

$$I_1 = \sum_{i=1}^{N}\rho_i\int_0^{\pi}\int_{a_{i-1}}^{a_i}(U_i\cos\theta + W_i\sin\theta)V_i r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta. \tag{13}$$

The detail of the technically complex, positive integral I_2 is unnecessary for what follows. However, I_2 can be computed readily with the aid of a computer algebra system. Because $K = K(C,S,\dot{C},\dot{S})$ and $P = P(C,S)$, the Lagrangian

$$L(C,S,\dot{C},\dot{S}) = K - P \tag{14}$$

yields two equations of motion from Lagrange's equations (see Ref. [5]):

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{C}} - \frac{\partial L}{\partial C} = -\frac{\partial\mathscr{F}}{\partial\dot{C}}, \tag{15a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{S}} - \frac{\partial L}{\partial S} = -\frac{\partial\mathscr{F}}{\partial\dot{S}}, \tag{15b}$$

where

$$\mathscr{F} = \tfrac{1}{2}(c\dot{C}^2 + s\dot{S}^2) \tag{16}$$

is Rayleigh's dissipation function, c and s are viscous damping constants. Anisotropic damping effects caused by imperfections in the media are beyond the scope of this paper and are left for future study. We assume isotropic damping, that is, $c = s = \pi D$, say. We further assume that for the media we will encounter, the "damping factor"

$$\delta = \frac{D}{2I_0} \tag{17}$$

is substantially smaller than the lowest eigenvalue of the vibrating system. Computation of Eqs. (15) yields a coupled system of second-order, linear, ordinary differential equations (ODE):

$$\ddot{C} + 2\eta\Omega\dot{S} + \omega^2 C + 2\delta\dot{C} = 0, \tag{18a}$$

$$\ddot{S} - 2\eta\Omega\dot{C} + \omega^2 S + 2\delta\dot{S} = 0, \tag{18b}$$

where

$$-1 \leqslant \eta = \frac{I_1}{I_0} \leqslant 1 \tag{19}$$

and

$$\omega^2 = \frac{I_2}{I_0}. \tag{20}$$

We now show that η in Eq. (19) is Bryan's factor as given by Eq. (1) and that ω in Eq. (20) is an eigenvalue for the vibrating system.

In order to interpret what the system of ODE (18) represents, combine the two equations by considering the complex function $Z = C + iS$ to obtain the single equation

$$\ddot{Z} + 2(\delta - i\eta\Omega)\dot{Z} + \omega^2 Z = 0. \tag{21}$$

Writing Z in polar form

$$Z(t) = Y(t)e^{i\beta(t)} \tag{22}$$

and assuming that $\beta(t)$ has the linear form

$$\beta(t) = at, \tag{23}$$

while $Y(t)$ decays according to

$$Y(t) = X(t)e^{-bt}, \tag{24}$$

then

$$Z = X(t)e^{(ia-b)t} \tag{25}$$

and substituting into Eq. (21), we obtain the ODE

$$\ddot{X} + 2[(ia - b) + (\delta - i\eta\Omega)]\dot{X} + [2(\delta - i\eta\Omega)(ia - b) + (ia - b)^2 + \omega^2]X = 0. \tag{26}$$

If we choose $a = \eta\Omega$ and $b = \delta$, then the coefficient of $\dot{X}$ vanishes in Eq. (26) and we obtain the ODE

$$\ddot{X} + \lambda^2 X = 0, \tag{27}$$

where

$$\begin{aligned} \lambda^2 &= \omega^2 - (\delta - i\eta\Omega)^2 \\ &= \omega^2 + \eta^2\Omega^2 - \delta^2 + 2i\eta\Omega\delta. \end{aligned} \tag{28}$$

Neglecting $O(\Omega^2)$, $O(\Omega\delta)$ and $O(\delta^2)$ we obtain

$$\lambda \approx \omega. \tag{29}$$

Consequently Eq. (27) approximates the equation of motion of an harmonic oscillator. Eqs. (18) can now be viewed in the form

$$Z(t) = [e^{-\delta t}X(t)]e^{i\eta\Omega t}. \tag{30}$$

These equations represent a "vector" in the complex plane with its size varying like a damped harmonic oscillator and its position varying at a rate $\eta\Omega$ (in the rotating reference frame $Oxyz$). Thus, according to Eq. (1), η is Bryan's factor for the system. Consequently, neither Bryan's effect nor the value of Bryan's factor η depend on the inclusion of light, isotropic, viscous damping in the model and so we will neglect damping in the sequel. The rotation of the vibrating pattern is in the direction of rotation of the system if $\eta > 0$ and in the opposite direction if $\eta < 0$. Eqs. (27) and (28) show that ω is an eigenvalue of the vibrating system.

3. Equations of motion and their solutions

Using Redwood [8] and our notation for stresses, the equations of motion of an isotropic solid body in spherical coordinates are

$$\rho\frac{\partial^2 u}{\partial t^2} = \frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\sigma_{r\phi}}{\partial\phi} + \frac{2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi} + \cot\theta\,\sigma_{r\theta}}{r}, \tag{31a}$$

$$\rho\frac{\partial^2 v}{\partial t^2}=\frac{\partial\sigma_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial\sigma_{\theta\phi}}{\partial\phi}+\frac{3\sigma_{r\theta}+\cot\theta(\sigma_{\theta\theta}-\sigma_{\phi\phi})}{r}, \tag{31b}$$

$$\rho\frac{\partial^2 w}{\partial t^2}=\frac{\partial\sigma_{r\phi}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta\phi}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial\sigma_{\phi\phi}}{\partial\phi}+\frac{3\sigma_{r\phi}+2\cot\theta\sigma_{\theta\phi}}{r}. \tag{31c}$$

The stresses are given by Eq. (7).

Eqs. (31) are coupled second-order partial differential equations (PDE) of the three displacement components u, v and w. They can be uncoupled directly, but this leads to a sixth-order PDE. Hence, in a manner similar to that explained in Eringen and Suhubi [9] (and Ref. [7] for cylindrical coordinates), we express the displacement components in terms of derivatives of potentials $\Phi=\Phi(r,\theta,\phi)$, $\chi=\chi(r,\theta,\phi)$ and $\Psi=\Psi(r,\theta,\phi)$ as follows:

$$u=\left\{\frac{1}{r}\frac{\partial}{\partial\theta}\left[\Phi+\frac{\partial(r\chi)}{\partial r}\right]+\frac{1}{\ell\sin\theta}\frac{\partial\Psi}{\partial\phi}\right\}e^{i\omega t}, \tag{32a}$$

$$v=\left\{\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left[\Phi+\frac{\partial(r\chi)}{\partial r}\right]-\frac{1}{\ell}\frac{\partial\Psi}{\partial\theta}\right\}e^{i\omega t}, \tag{32b}$$

$$w=\left\{\frac{\partial\Phi}{\partial r}+\left[\frac{\partial^2(r\chi)}{\partial r^2}+r\frac{\rho\omega^2}{\mu}\chi\right]\right\}e^{i\omega t}, \tag{32c}$$

where ω is the eigenvalue mentioned in Eq. (20) and ℓ is a non-zero constant with the dimension of length. When Eqs. (32) are substituted into Eqs. (31) and the resulting equations of motion are uncoupled, it is found that each potential Φ, χ and Ψ satisfies the Helmholtz equations

$$\nabla^2\Phi+k_1^2(\omega)\Phi=0,\quad \nabla^2\chi+k_2^2(\omega)\chi=0,\quad \nabla^2\Psi+k_2^2(\omega)\Psi=0 \tag{33}$$

with $k_1^2(\omega)=\rho\omega^2/(\lambda+2\mu)$, $k_2^2(\omega)=\rho\omega^2/\mu$ and ∇^2 the Laplace operator in spherical coordinates.

The solutions to Eqs. (33) are

$$\Phi_{m,n}(r,\theta,\phi,\omega)=[B_1 j_n(k_1(\omega)r)+B_2 y_n(k_1(\omega)r)]P_n^m(\cos\theta)\cos(m\phi), \tag{34a}$$

$$\chi_{m,n}(r,\theta,\phi,\omega)=[B_3 j_n(k_2(\omega)r)+B_4 y_n(k_2(\omega)r)]P_n^m(\cos\theta)\cos(m\phi), \tag{34b}$$

$$\Psi_{m,n}(r,\theta,\phi,\omega)=[B_5 j_n(k_2(\omega)r)+B_6 y_n(k_2(\omega)r)]P_n^m(\cos\theta)\sin(m\phi), \tag{34c}$$

where, as usual, $j_n(kr)=\sqrt{(\pi/(2kr))}J_{n+1/2}(kr)$ and $y_n(kr)=\sqrt{(\pi/(2kr))}Y_{n+1/2}(kr)$, where $J_{n+1/2}$ and $Y_{n+1/2}$ represent Bessel and Neumann functions, respectively, while P_n^m is the associated Legendre polynomial. The symbols $B_1,B_2,\ldots,B_6$ are arbitrary constants (if the body contains the centre O, then the constants $B_2=B_4=B_6=0$).

The motion of a compressible inviscid acoustic medium is represented by the following wave equation:

$$\nabla^2 p+k_3^2(\omega)p=0 \tag{35}$$

with $k_3^2(\omega)=E^{(f)}/\rho^{(f)}$, where $E^{(f)}$ is the bulk modulus and $\rho^{(f)}$ the mass density of the acoustic medium. The solution to this equation is

$$p_{m,n}(r,\theta,\phi,t)=\{[B_7 j_n(k_3(\omega)r)+B_8 y_n(k_3(\omega)r)]P_n^m(\cos\theta)\cos(m\phi)\}e^{i\omega t} \tag{36}$$

with $p=p_{m,n}(r,\theta,\phi,t)$ the pressure in the acoustic medium. Particle displacement of the acoustic medium in the radial direction is

$$w^{(f)}=\left(\frac{1}{\rho^{(f)}\omega^2}\right)\frac{\partial p}{\partial r}. \tag{37}$$

4. Boundary conditions and eigenfunctions

Observing Eqs. (32), it is possible to distinguish between spheroidal and torsional modes. For instance, if the potentials $\Phi \equiv \chi \equiv 0$, then radial displacement $w = 0$ and only tangential vibration occurs. This is called the torsional mode. For the spheroidal mode we assume that $\Psi \equiv 0$ and radial vibration also occurs. For the spheroidal mode the stress components of the solid are

$$\sigma_{rr} = \left[\mu\frac{\partial^2\Phi}{\partial r^2} - \lambda k_1^2(\omega)\Phi\right] + 2\mu\frac{\partial}{\partial r}\left[\frac{\partial^2(r\chi)}{\partial r^2} + rk_2^2(\omega)\chi\right], \tag{38a}$$

$$\sigma_{r\theta} = \frac{2\mu}{r}\frac{\partial}{\partial\theta}\left\{\left(\frac{\partial\Phi}{\partial r} - \frac{\Phi}{r}\right) + \left[r\frac{\partial^2\chi}{\partial r^2} + \frac{\partial\chi}{\partial r} + \left(\frac{rk_2^2(\omega)}{2} - \frac{1}{r}\right)\chi\right]\right\}, \tag{38b}$$

$$\sigma_{r\phi} = \frac{2\mu}{r\sin\theta}\frac{\partial}{\partial\phi}\left\{\left(\frac{\partial\Phi}{\partial r} - \frac{\Phi}{r}\right) + \left[r\frac{\partial^2\chi}{\partial r^2} + \frac{\partial\chi}{\partial r} + \left(\frac{rk_2^2(\omega)}{2} - \frac{1}{r}\right)\chi\right]\right\}. \tag{38c}$$

For the torsional mode the corresponding stress components are

$$\sigma_{r\theta} = \frac{\mu}{a\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial\Psi}{\partial r} - \frac{\Psi}{r}\right), \tag{39a}$$

$$\sigma_{r\phi} = -\frac{\mu}{a}\frac{\partial}{\partial\theta}\left(\frac{\partial\Psi}{\partial r} - \frac{\Psi}{r}\right). \tag{39b}$$

Because, in the ideal case, we consider the acoustic medium to be inviscid, there will be no interaction between the medium and the solid on the boundary (there are no shear forces). Hence we do not discuss the torsional mode further in this paper.

Let us model a hollow sphere consisting of an outer layer of solid phase substance for $a = a_1 \leqslant r \leqslant a_2 = b$ that is filled with an inviscid acoustic medium for $0 = a_0 \leqslant r \leqslant a_1$. Considering the spheroidal mode, the following solutions are obtained:

$$p_{m,n}(r,\theta,\phi,t) = [A_1 j_n(k_3(\omega)r)]P_n^m(\cos\theta)\cos(m\phi)e^{i\omega t}, \tag{40}$$

$$\Phi_{m,n}(r,\theta,\phi,\omega) = [A_2 j_n(k_1(\omega)r) + A_3 y_n(k_1(\omega)r)]P_n^m(\cos\theta)\cos(m\phi), \tag{41a}$$

$$\chi_{m,n}(r,\theta,\phi,\omega) = [A_4 j_n(k_2(\omega)r) + A_5 y_n(k_2(\omega)r)]P_n^m(\cos\theta)\cos(m\phi). \tag{41b}$$

Boundary conditions of the system define the eigenvalues ω. The boundary conditions below express the balance between the radial components of stress and the pressure between the solid and acoustic medium and the equality of their radial displacements at $r = a$. They also describe the absence of stresses at the outer surface of the solid spherical layer at $r = b$:

$$\left\{\left[\mu\frac{\partial^2\Phi}{\partial r^2} - \lambda k_1^2(\omega)\Phi\right] + 2\mu\frac{\partial}{\partial r}\left[\frac{\partial^2(r\chi)}{\partial r^2} + rk_2^2(\omega)\chi\right]\right\}\Bigg|_{r=a} = -p\big|_{r=a}, \tag{42a}$$

$$\left\{\frac{\partial\Phi}{\partial r} + \left[\frac{\partial^2(r\chi)}{\partial r^2} + rk_2^2(\omega)\chi\right]\right\}\Bigg|_{r=a} = \frac{1}{\rho^{(f)}\omega^2}\left\{\frac{\partial p}{\partial r}\right\}\Bigg|_{r=a}, \tag{42b}$$

$$\left\{\left(\frac{\partial\Phi}{\partial r} - \frac{\Phi}{r}\right) + \left[r\frac{\partial^2\chi}{\partial r^2} + \frac{\partial\chi}{\partial r} + \left(\frac{rk_2^2(\omega)}{2} - \frac{1}{r}\right)\chi\right]\right\}\Bigg|_{r=a} = 0 \tag{42c}$$

and

$$\left\{\left[\mu\frac{\partial^2\Phi}{\partial r^2} - \lambda k_1^2(\omega)\Phi\right] + 2\mu\frac{\partial}{\partial r}\left[\frac{\partial^2(r\chi)}{\partial r^2} + rk_2^2(\omega)\chi\right]\right\}\Bigg|_{r=b} = 0, \tag{43a}$$

$$\left\{\left(\frac{\partial\Phi}{\partial r}-\frac{\Phi}{r}\right)+\left[r\frac{\partial^2\chi}{\partial r^2}+\frac{\partial\chi}{\partial r}+\left(\frac{rk_2^2(\omega)}{2}-\frac{1}{r}\right)\chi\right]\right\}\bigg|_{r=b}=0. \tag{43b}$$

By substituting $\Psi\equiv 0$ and Eqs. (41) into Eqs. (32) and simplifying, we obtain the following eigenfunctions (where superscript (f) indicates the quantities for the acoustic medium):

$$\begin{aligned}U(r,\theta)=&\frac{1}{r}\{A_2 j_n(k_1r)+A_3 y_n(k_1r)+A_4[(n+1)j_n(k_2r)-k_2rj_{n+1}(k_2r)]\\&+A_5[(n+1)y_n(k_2r)-k_2ry_{n+1}(k_2r)]\}\\&\times\left\{-(n+1)\cot\theta\,P_n^m(\cos\theta)+\frac{n-m+1}{\sin\theta}P_{n+1}^m(\cos\theta)\right\},\end{aligned} \tag{44}$$

$$\begin{aligned}V(r,\theta)=&-\frac{m}{r\sin\theta}\{A_2 j_n(k_1r)+A_3 y_n(k_1r)+A_4[(n+1)j_n(k_2r)-k_2rj_{n+1}(k_2r)]\\&+A_5[(n+1)y_n(k_2r)-k_2ry_{n+1}(k_2r)]\}P_n^m(\cos\theta),\end{aligned} \tag{45}$$

$$\begin{aligned}W(r,\theta)=&\left\{A_2\left[\frac{n}{r}j_n(k_1r)-k_1j_{n+1}(k_1r)\right]+A_3\left[\frac{n}{r}y_n(k_1r)-k_1y_{n+1}(k_1r)\right]\right.\\&\left.+A_4\left[\frac{n(n+1)}{r}j_n(k_2r)\right]+A_5\left[\frac{n(n+1)}{r}y_n(k_2r)\right]\right\}P_n^m(\cos\theta),\end{aligned} \tag{46}$$

$$U^{(f)}(r,\theta)=\frac{1}{r}A_1j_n(k_3r)\left\{-(n+1)\cot\theta P_n^m(\cos\theta)+\frac{n-m+1}{\sin\theta}P_{n+1}^m(\cos\theta)\right\}, \tag{47}$$

$$V^{(f)}(r,\theta)=-\frac{m}{r\sin\theta}A_1j_n(k_3r)P_n^m(\cos\theta), \tag{48}$$

$$W^{(f)}(r,\theta)=\left\{A_1\left[\frac{n}{r}j_n(k_3r)-k_3j_{n+1}(k_3r)\right]\right\}P_n^m(\cos\theta). \tag{49}$$

4.1. Example

Let us consider the spheroidal vibrations of a spherical layer (with inner radius $a=0.4\,\mathrm{m}$, outer radius $b=0.5\,\mathrm{m}$), made from brass ($E=100\,\mathrm{MPa}$, $\rho=8500\,\mathrm{kg\,m^{-3}}$, and Poisson's ratio $\nu=0.34$) and filled with water ($E^{(f)}=2.2\,\mathrm{MPa}$, $\rho^{(f)}=1000\,\mathrm{kg\,m^{-3}}$) that is rotating at a constant rate.

Suppressing the mode number subscripts m,n, from Eq. (19), Bryan's factor for this structure is given by

$$\eta=\frac{2\left\{\int_0^\pi\left[\int_0^a\rho^{(f)}(U^{(f)}\cos\theta+W^{(f)}\sin\theta)V^{(f)}r^2\,\mathrm{d}r+\int_a^b\rho(U\cos\theta+W\sin\theta)V\,r^2\,\mathrm{d}r\right]\sin\theta\,\mathrm{d}\theta\right\}}{\int_0^\pi\left[\int_0^a\rho^{(f)}(U^{(f)2}+V^{(f)2}+W^{(f)2})r^2\,\mathrm{d}r+\int_a^b\rho(U^2+V^2+W^2)r^2\,\mathrm{d}r\right]\sin\theta\,\mathrm{d}\theta}. \tag{50}$$

Calculations of eigenvalues and the corresponding Bryan's factors are given in Table 1.

5. Conclusions and discussions

Gyroscopic effects in rotating symmetrically distributed spherical bodies were considered and the dependence of the rate of rotation of the vibrating pattern on the inertial angular rate of the system determined. This dependence is described by the so-called "Bryan's factor" which is calculated in spherical coordinates without using "thin shell theory". It is pointed out that the theory is readily adaptable to symmetrically distributed cylindrical bodies. It was demonstrated that neither Bryan's effect nor Bryan's factor depend on light, isotropic, viscous damping and consequently damping was neglected in the mathematical models. The introduction of prestress and mass-stiffness imperfections (anisotropic damping

Table 1
Eigenvalues and corresponding Bryan's factor.

n	m	ω_1 (Hz) η_1	ω_2 (Hz) η_2	ω_3 (Hz) η_3	ω_4 (Hz) η_4	ω_5 (Hz) η_5
2	2	1791	1972	2094	2989	4302
		0.9107	−0.7738	−0.7952	0.4319	−0.0848
3	2	2664	2916	3605	4101	5066
		−0.4775	−0.547	−0.5559	0.2860	−0.0648
3	3	2664	2916	3605	4101	5066
		−0.7162	−0.8204	−0.8339	0.4290	−0.0973
4	2	3332	3981	5071	5194	5807
		−0.3366	−0.4397	0.1735	0.1635	−0.0521
4	3	3332	3981	5071	5194	5807
		−0.5048	−0.6596	0.2602	0.2453	−0.0781
4	4	3332	3981	5071	5194	5807
		−0.6731	−0.8795	0.3470	0.3270	−0.1041

effects) into the calculations is important for real-life situations (as opposed to ideal situations) and has been earmarked by us for further study.

Solutions to the dynamic equations of elastic solid and inviscid acoustic bodies composed of concentric spherical layers were obtained and boundary conditions formulated for calculating eigenvalues and eigenfunctions for the system.

The results of the general theory were applied to an example of a rotating elastic, spherical layer (brass) filled with an acoustic medium (water, assumed to be inviscid). Eigenvalues and Bryan's factors were calculated and tabulated for various vibration modes. It was observed that negative Bryan's factors predominate in the table. However, no discernible pattern for the sign of Bryan's factor is obvious from the table. Furthermore, for low eigenvalues and circumferential wavenumbers, the difference between the rotational angular rates of the hollow sphere filled with an acoustic medium and those of its vibrating patterns is small ($|\eta| \approx 1$). However, this difference is large for higher modes and eigenvalues of the system ($|\eta| \approx 0$).

It is pointed out that earthquakes and pulsating stars might be better understood by introducing Bryan's effect into appropriate models.

We believe that we now know more about the operation of a hemispherical resonator gyroscope [4]. Roughly speaking, suppose that a vibrating hemisphere is fixed to a vehicle moving through three-dimensional space and that an electronic sensor is set to observe the position of a node of the fundamental vibration of the hemisphere (such vibration can be observed in the excellent holographic interferograms of a vibrating wineglass given in Ref. [10]). Suppose the vehicle undergoes a slow rate of rotation Ω with respect to inertial space and that this rotation rate is too small for the human vestibular system to observe. The electronic sensor will register that the node rotates away from its inertial position. The rotation rate of the node, say α, can then be calculated and, using Bryan's factor η for the hemisphere for the fundamental mode of vibration, the rate of rotation of the vehicle $\Omega = \alpha/\eta$ with respect to inertial space can be calculated.

The introduction of prestress and mass-stiffness imperfections into the calculations is important for real-life situations (as opposed to ideal situations) and has been earmarked by us for further study.

Acknowledgements

The authors would like to thank the Tshwane University of Technology, the National Research Foundation of South Africa (NRF Grant reference number ICD200607110000608) and the South African CSIR for supporting this work.

References

[1] G.H. Bryan, On the beats in the vibrations of a revolving cylinder or bell, *Proceedings of the Cambridge Philosophical Society* 7 (1890) 101–111.

[2] V. Zhuravlev, D. Klimov, *Applied Methods in the Theory of Oscillations*. Nauka, Moscow, 1988 (in Russian).

[3] P.W. Loveday, C.A. Rogers, Free vibrations of elastically supported thin cylinders including gyroscopic effects, *Journal of Sound and Vibration* 217 (3) (1998) 547–562.

[4] E.J. Loper Jr., D.D. Lynch, Hemispherical resonator gyroscope, USA Patent No. 4,951,508, August 1990.

[5] H. Goldstein, *Classical Mechanics Reading*, Addison-Wesley, Reading, MA, 1950.

[6] M.R. Spiegel, *Theoretical Mechanics*, McGraw-Hill, New York, 1967.

[7] M.J. Berliner, R. Solecki, Wave propagation in fluid-loaded, transversely isotropic cylinders. Part I. Analytical formulation, *Journal of the Acoustical Society of America* 99 (4) (1996) 1841–1847.

[8] M. Redwood, *Mechanical Waveguides*, Oxford Pergamon Press, Oxford, 1960.

[9] A.E. Eringen, E.S. Suhubi, Elastodynamics, Vol. 1, Academic Press, New York, 1975.

[10] T.D. Rossing, Acoustics of the glass harmonica, *Journal of Acoustical Society of America* 95 (2) (1994) 1106–1111.