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Breaking of symmetries for stabilization of rotating continua in frictional contact

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Abstract

Rotating structures subject to frictional contact are susceptible to self-excited vibrations that are responsible for noise problems. In previous work the underlying mechanism has been explained through mathematical–mechanical models. From practical experience it is known that breaking the symmetry of a rotor can have a stabilizing effect. The present paper is devoted to a mathematical justification of this phenomenon. At the same time a method for a quantitative investigation of the influence of asymmetries on the stability behavior is outlined. As an example a rotating annular Kirchhoff plate in contact with friction pads is studied serving as a minimal model for brake squeal.

A possible application of the results is the support of the design process for squeal free brake rotors where currently only experimental methods yield information about the tendency of an asymmetric brake rotor to squeal.

1. Introduction

Rotating components are important in many technical applications. If they are thin walled they can be modeled as shells using two-dimensional curvilinear surface coordinates describing the displacements of the neutral surface. In many cases the undeformed shells are symmetric with respect to the axis of rotation. Important examples are hollow shafts occurring in rotor systems, paper calenders and many more. Other systems, such as bicycle rims, can be modeled as rings with only one curvilinear coordinate. The dynamics of shells loaded by conservative forces has been under intensive investigation also for the rotating case. The occurrence of nonconservative forcing due to friction can yield instabilities and has especially been studied in the context of brake squeal [1–5]. It is known that in some cases breaking of symmetry of the rotor tends to stabilize the system.

The goal of this paper is to provide a mathematical justification for this phenomenon using perturbation theory for eigenvalues in the context of Floquet theory developed by Seyranian et al. in Refs. [6–8].

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For the systems of interest the equations of motion are most conveniently set up either in an inertial frame or in a frame rotating with the rotor. In case of a purely symmetric rotor the former is most convenient since linear equations with constant coefficients arise. For asymmetric rotors the latter is the more straightforward approach and yields linear equations with periodic coefficients. The equations of motion resulting from a Ritz discretization using the most important modes of the nonrotating system as shape functions represent a general conservative homogeneous system perturbed by small arbitrary parametric excitation. Apart from the well known instability mechanism of parametric resonance the influence of the symmetries of the occurring matrices on the stability behavior is discussed. In Section 3 as an example we study a minimal model for a disk brake discussing the structure of the equations.

2. Linear conservative systems perturbed by periodic linear forcing

The equations of motion for the systems under consideration in a frame rotating with the rotor are of the form

$$\mathbf{M}\ddot{\mathbf{q}} + \varepsilon\,\Delta\mathbf{D}(t)\dot{\mathbf{q}} + (\mathbf{K} + \varepsilon\,\Delta\mathbf{K}(t))\mathbf{q} = \mathbf{0}, \tag{1}$$

where $\mathbf{M} = \mathbf{M}^{\mathrm{T}}$ and $\mathbf{K} = \mathbf{K}^{\mathrm{T}}$ are positive definite and $\Delta\mathbf{D}(t) = \Delta\mathbf{D}(t + T)$, $\Delta\mathbf{K}(t) = \Delta\mathbf{K}(t + T)$. The parameter $\varepsilon \ll 1$ can be interpreted as a norm of multiple parameters depending linearly on ε and vanishing for $\varepsilon = 0$. In case the dependence is nonlinear, Eq. (1) contains only the first term of the Taylor series of the parameters. Without loss of generality we assume $T = 2\pi$, $\mathbf{M} = \mathrm{diag}(1, 1, \ldots, 1)$ and $\mathbf{K} = \mathrm{diag}(\omega_1^2, \ldots, \omega_N^2)$ since the system can always be brought into this form by an orthogonal transformation using the eigenvectors corresponding to the unperturbed problem. It is convenient to work with the equivalent first-order system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \tag{2}$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{K} - \varepsilon\,\Delta\mathbf{K}(t) & -\varepsilon\,\Delta\mathbf{D}(t) \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix}. \tag{3}$$

According to Floquet theory, stability of the system is determined by the eigenvalues ρ_j of the monodromy matrix, which is the fundamental matrix $\mathbf{X}(t, \varepsilon)$ evaluated at $t = T$ with $\mathbf{X}(0, \varepsilon) = \mathbf{I}_{2N \times 2N}$. If the modulus of $|\rho_j| \leqslant 1$ for all j and all ρ_j are simple or semi-simple then the system is stable. Since the $\mathbf{X}(t, \varepsilon)$ is a smooth function of ε we expand it in a Taylor series with respect to ε, i.e.

$$\mathbf{X}(t, \varepsilon) = \mathbf{X}(t, 0) + \left.\frac{\partial\mathbf{X}(t, \varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0} \varepsilon + \cdots . \tag{4}$$

With the help of the adjoint problem

$$\dot{\mathbf{y}} = -\mathbf{A}^{\mathrm{T}}\mathbf{y} \tag{5}$$

one can calculate the derivatives of the $\mathbf{X}(t, \varepsilon)$ with respect to ε evaluated at $\varepsilon = 0$ in terms of the derivatives of $\mathbf{A}$ with respect to ε [9].

The formula for the first derivative reads

$$\left.\frac{\partial\mathbf{X}(T, \varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0} = \mathbf{X}(T, 0)\int_0^{2\pi} \mathbf{Y}(t, 0)^{\mathrm{T}}\left.\frac{\partial\mathbf{A}(t, \varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0}\mathbf{X}(t, 0)\,\mathrm{d}t, \tag{6}$$

where in the following we use the notation $\mathbf{H} = \int_0^{2\pi} \mathbf{Y}(t, 0)^{\mathrm{T}}\partial\mathbf{A}(t, \varepsilon)/\partial\varepsilon|_{\varepsilon=0}\mathbf{X}(t, 0)\,\mathrm{d}t$. Since in our case the unperturbed problem has constant coefficients and is decoupled we have very simple analytic expressions

for $\mathbf{X}(t,0)$ and $\mathbf{Y}(t,0)$ namely

$$\mathbf{X} = \begin{bmatrix} \cos\omega_1 t & 0 & \cdots & 0 & \frac{1}{\omega_1}\sin\omega_1 t & 0 & \cdots & 0 \\ 0 & \cos\omega_2 t & \cdots & 0 & 0 & \frac{1}{\omega_2}\sin\omega_2 t & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & 0 & & \cos\omega_N t & 0 & 0 & & \frac{1}{\omega_N}\sin\omega_N t \\ -\omega_1\sin\omega_1 t & 0 & \cdots & 0 & \cos\omega_1 t & 0 & \cdots & 0 \\ 0 & -\omega_2\sin\omega_2 t & \cdots & 0 & 0 & \cos\omega_2 t & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & 0 & & -\omega_N\sin\omega_N t & 0 & 0 & & \cos\omega_N t \end{bmatrix} \quad (7)$$

and the matrix $\mathbf{Y} = \mathbf{X}^{-1}$ simply reads

$$\mathbf{Y} = \begin{bmatrix} \cos\omega_1 t & 0 & \cdots & 0 & \omega_1\sin\omega_1 t & 0 & \cdots & 0 \\ 0 & \cos\omega_2 t & \cdots & 0 & 0 & \omega_2\sin\omega_2 t & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & 0 & & \cos\omega_N t & 0 & 0 & & \omega_N\sin\omega_N t \\ -\frac{1}{\omega_1}\sin\omega_1 t & 0 & \cdots & 0 & \cos\omega_1 t & 0 & \cdots & 0 \\ 0 & -\frac{1}{\omega_2}\sin\omega_2 t & \cdots & 0 & 0 & \cos\omega_2 t & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & 0 & & -\frac{1}{\omega_N}\sin\omega_N t & 0 & 0 & & \cos\omega_N t \end{bmatrix}. \quad (8)$$

We can now calculate how the eigenvalues of $\mathbf{X}(T,0)$ change for small ε using bifurcation theory. In our case the eigenvalues of the unperturbed monodromy matrix $\mathbf{X}(T,0)$ are of the form

$$\rho_{0j} = \cos 2\pi\omega_j + \mathrm{i}\sin 2\pi\omega_j \quad (9)$$

and obviously appear in complex conjugate pairs. The corresponding eigenvectors read

$$\mathbf{u}_j = \left(0,\ldots,0,\frac{\mathrm{i}}{\omega_j},0,\ldots,0,1,0,\ldots,0\right)^{\mathrm{T}}$$

and

$$\mathbf{v}_j = (0,\ldots,0,-\mathrm{i}\omega_j,0,\ldots,0,1,0,\ldots,0)^{\mathrm{T}}$$

for the adjoint problem. Since the unperturbed problem of Eq. (1) is self-adjoint there are always $2N$ linearly independent eigenvectors; therefore the spectrum of the unperturbed problem is either simple or semi-simple. Following Vishik and Lyusternik [10] there is an expansion for the Floquet multipliers and corresponding eigenvectors of the form

$$\rho_j = \rho_{0j} + \varepsilon\frac{\partial\rho_j}{\partial\varepsilon}\bigg|_{\varepsilon=0} + \cdots, \quad (10)$$

$$\mathbf{w}_j = \mathbf{u}_j + \varepsilon\mathbf{w}_{1j}^{\varepsilon} + \cdots, \quad (11)$$

where ρ_{0j} and $\mathbf{u}_j$, respectively, correspond to the j-th eigenvalue and eigenvector of the unperturbed problem. In order to derive the perturbation formulas we have to consider the cases of simple and semi-simple eigenvalues separately.

2.1. Simple eigenvalues

For a simple Floquet multiplier $\partial\rho_j/\partial\varepsilon|_{\varepsilon=0}$ in the following for simplicity denoted by $\partial\rho_j/\partial\varepsilon$ is given by [9]

$$\frac{\partial\rho_j}{\partial\varepsilon} = \frac{\mathbf{v}_j^{\mathrm{T}}\dfrac{\partial\mathbf{X}(2\pi,0)}{\partial\varepsilon}\bigg|_{\varepsilon=0}\mathbf{u}_j}{\mathbf{v}_j^{\mathrm{T}}\mathbf{u}_j} = \frac{\rho_{0j}\mathbf{v}_j^{\mathrm{T}}\mathbf{H}\mathbf{u}_j}{\mathbf{v}_j^{\mathrm{T}}\mathbf{u}_j}. \tag{12}$$

Due to the simple structure of $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{u}_j$ and $\mathbf{v}_j$ and using

$$\mathbf{v}_j^{\mathrm{T}}\int_0^{2\pi}\mathbf{Y}(t,0)^{\mathrm{T}}\frac{\partial\mathbf{A}(t,\varepsilon)}{\partial\varepsilon}\bigg|_{\varepsilon=0}\mathbf{X}(t,0)\,\mathrm{d}t\,\mathbf{u}_k = \int_0^{2\pi}\mathbf{v}_j^{\mathrm{T}}\mathbf{Y}(t,0)^{\mathrm{T}}\frac{\partial\mathbf{A}(t,\varepsilon)}{\partial\varepsilon}\bigg|_{\varepsilon=0}\mathbf{X}(t,0)\mathbf{u}_k\,\mathrm{d}t \tag{13}$$

the derivative of ρ_j with respect to ε can be calculated as

$$\frac{\partial\rho_j}{\partial\varepsilon} = \left(-\frac{\mathrm{i}}{\omega_j}\int_0^{2\pi}\Delta k_{jj}\,\mathrm{d}t - \int_0^{2\pi}\Delta d_{jj}\,\mathrm{d}t\right)\rho_{0j}, \tag{14}$$

where Δk_{jj}, Δd_{jj} are the matrix entries of $\Delta\mathbf{K}$, $\Delta\mathbf{D}$. It is interesting to note that on the main diagonal of the matrix $\int_0^{2\pi}\mathbf{V}^{\mathrm{T}}\mathbf{Y}(t,0)^{\mathrm{T}}\partial\mathbf{A}(t,\varepsilon)/\partial\varepsilon|_{\varepsilon=0}\mathbf{X}(t,0)\mathbf{U}\,\mathrm{d}t$ where $\mathbf{U}$ ($\mathbf{V}$) is build up form the eigenvectors of the (adjoint) unperturbed problem, the entries are proportional to the mean value of the perturbation matrices. Since

$$\frac{\partial|\rho_j|}{\partial\varepsilon} = \frac{1}{|\rho_{0j}|}\mathrm{Re}\left(\bar{\rho}_{0j}\frac{\partial\rho_j}{\partial\varepsilon}\right), \tag{15}$$

we see that in the first approximation for simple eigenvalues the modulus of the eigenvalues is not influenced by restoring terms Δk_{jj} and is decreased by dissipative forces Δd_{jj}.

2.2. Semi-simple eigenvalues

For two semi-simple Floquet multipliers ρ_j and ρ_k the first term in Eq. (10) is calculated from Ref. [9] as

$$\det\begin{bmatrix} \rho_{0j}\mathbf{v}_j^{\mathrm{T}}\mathbf{H}\mathbf{u}_j - \dfrac{\partial\rho_j}{\partial\varepsilon} & \rho_{0j}\mathbf{v}_j^{\mathrm{T}}\mathbf{H}\mathbf{u}_k \\ \rho_{0k}\mathbf{v}_j^{\mathrm{T}}\mathbf{H}\mathbf{u}_j & \rho_{0k}\mathbf{v}_k^{\mathrm{T}}\mathbf{H}\mathbf{u}_k - \dfrac{\partial\rho_j}{\partial\varepsilon} \end{bmatrix} = 0 \tag{16}$$

using again the adjoint problem. The reason for occurrence of a semi-simple eigenvalue can be twofold. Either we have multiple eigenfrequencies or we have internal resonance. Let us first consider the case of two equal eigenfrequencies of the unperturbed problem. The corresponding expressions in Eq. (16) read

$$\mathbf{v}_j^{\mathrm{T}}\mathbf{H}\mathbf{u}_k = -\frac{\mathrm{i}}{\omega_j}\int_0^{2\pi}\Delta k_{jk}\,\mathrm{d}t - \int_0^{2\pi}\Delta d_{jk}\,\mathrm{d}t \tag{17}$$

and are proportional to the mean of the perturbation matrices as in the case of simple eigenvalues. From Eq. (16) $\partial\rho_j/\partial\varepsilon$ can be calculated analytically. We now investigate the case $\Delta\mathbf{D}_{jk} = 0$, i.e. a perturbation only of the stiffness matrix. It is particularly interesting to see how $\partial\rho_j/\partial\varepsilon$ depends on the symmetries of $\Delta\mathbf{K}$. We therefore split k_{kj} into a symmetric and a skew-symmetric part

$$\Delta\mathbf{K}_{jk} = \begin{bmatrix} \Delta\tilde{k}_{jj} & \Delta\tilde{k}_{jk} \\ \Delta\tilde{k}_{jk} & \Delta\tilde{k}_{kk} \end{bmatrix} + \begin{bmatrix} 0 & \tilde{n} \\ -\tilde{n} & 0 \end{bmatrix}, \tag{18}$$

where $\tilde{n} = \int_0^{2\pi} \frac{1}{2}(\Delta k_{jk} - \Delta k_{kj}) \, dt$, $\Delta \tilde{k}_{jk} = \int_0^{2\pi} \frac{1}{2}(\Delta k_{jk} + \Delta k_{kj}) \, dt$. We obtain

$$\frac{\partial \rho_j}{\partial \varepsilon} = \frac{i}{2\omega_j} \left(-i \, \Delta \tilde{k}_{jj} - i \, \Delta \tilde{k}_{kk} \pm \sqrt{-(\Delta \tilde{k}_{jj} - \Delta \tilde{k}_{kk})^2 - 4(\Delta \tilde{k}_{jk}^2 - \tilde{n}^2)} \right) \rho_{0j}, \tag{19}$$

which yields

$$\frac{\partial |\rho_j|}{\partial \varepsilon} = \text{Re} \left(\pm \frac{1}{2} \sqrt{-(\Delta \tilde{k}_{jj} - \Delta \tilde{k}_{kk})^2 - 4(\Delta \tilde{k}_{jk}^2 - \tilde{n}^2)} \right). \tag{20}$$

It follows for a purely symmetric $\Delta \mathbf{K}_{jk}$ that $\partial |\rho_j|/\partial \varepsilon = 0$ and for a purely skew symmetric $\Delta \mathbf{K}_{jk}$ that $\partial |\rho_j|/\partial \varepsilon = \pm \tilde{n}$, i.e. one of the Floquet multipliers goes into the other one out of the unit circle and the system gets unstable. At the same time one can observe from that a skew symmetric perturbation in the first approximation can be compensated by a perturbation splitting the multiple eigenfrequencies of the unperturbed system. Given a skew symmetric perturbation it is seen from Eq. (20) that in order to have stability in the first approximation the condition

$$\tilde{n} \leqslant \frac{1}{4}(\Delta \tilde{k}_{jj} - \Delta \tilde{k}_{kk})^2 + \Delta \tilde{k}_{jk}^2 = \frac{1}{4}(\lambda_1 - \lambda_2)^2 \tag{21}$$

has to hold, where λ_1 and λ_2 are the eigenvalues of the matrix $\Delta \mathbf{K}_{jk}$. Since perturbations with constant coefficients are contained in Eq. (20) as a special case at this point it can be seen that breaking the symmetry of a rotor, which can be regarded as a constant symmetric perturbation of the stiffness matrix of the unperturbed problem, splits up multiple eigenfrequencies and has a stabilizing effect.

Analogously we investigate the case of a pure perturbation of the damping matrix, i.e. $\Delta \mathbf{K}_{jk} = \mathbf{0}$. From

$$\frac{\partial \rho_j}{\partial \varepsilon} = \frac{i\omega_j}{2} \left(-\Delta \tilde{d}_{jj} - \Delta \tilde{d}_{kk} \pm \sqrt{(\Delta \tilde{d}_{jj} + \Delta \tilde{d}_{kk})^2 - 4(\Delta \tilde{d}_{jj} \Delta \tilde{d}_{kk} - \Delta \tilde{d}_{jk} \Delta \tilde{d}_{kj})} \right) \tag{22}$$

with $\Delta \tilde{d}_{jk} = \int_0^{2\pi} \Delta d_{jk} \, dt$ we see that for $\det \Delta \mathbf{D}_{jk} > 0$

$$\text{Re} \left(\sqrt{(\Delta \tilde{d}_{jj} + \Delta \tilde{d}_{kk})^2 - 4(\Delta \tilde{d}_{jj} \Delta \tilde{d}_{kk} - \Delta \tilde{d}_{jk} \Delta \tilde{d}_{kj})} \right) < \Delta \tilde{d}_{jj} + \Delta \tilde{d}_{kk} \tag{23}$$

and therefore we get $\partial |\rho_j|/\partial \varepsilon < 0$, hence the system is stabilized in the first approximation.

Let us now consider the case of an internal resonance for the j-th eigenvalue, i.e. we have $\omega_j = k/2$ such that the imaginary part of the Floquet multiplier in Eq. (9) vanishes. Instead of a pair of complex conjugate Floquet multipliers we now have a double eigenvalue $\rho_0 = (-1)^k$ which is semi-simple. The first derivative of the Floquet multipliers can again be analyzed from Eq. (16). Examining the corresponding eigenvectors we observe that the formulas for the first derivative of the Floquet multipliers coincide with the ones for a damped version of Hill's equation

$$\ddot{q} + \varepsilon \, \Delta d_{jj}(t) \dot{q} + (\omega_j^2 + \varepsilon \, \Delta k_{jj}(t)) q = 0, \tag{24}$$

which have been calculated in Ref. [11] for a constant damping term. For a periodic damping term the first derivatives of the corresponding multipliers read

$$\frac{\partial \rho}{\partial \varepsilon} = (-1)^k \left(-c_d k \pm \pi \sqrt{(2a_k - b_d k)^2 + (2b_k - a_d k)^2 - (2c_k)^2} \right), \tag{25}$$

where

$$a_k = \frac{1}{2\pi k} \int_0^{2\pi} \sin(k\tau) \, \Delta k_{jj}(\tau) \, d\tau, \quad a_d = \frac{1}{2\pi k} \int_0^{2\pi} \sin(k\tau) \, \Delta d_{jj}(\tau) \, d\tau,$$

$$b_k = \frac{1}{2\pi k} \int_0^{2\pi} \cos(k\tau) \, \Delta k_{jj}(\tau) \, d\tau, \quad b_d = \frac{1}{2\pi k} \int_0^{2\pi} \cos(k\tau) \, \Delta d_{jj}(\tau) \, d\tau,$$

$$c_k = \frac{1}{2\pi k} \int_0^{2\pi} \Delta k_{jj}(\tau) \, d\tau, \quad c_d = \frac{1}{2\pi k} \int_0^{2\pi} \Delta d_{jj}(\tau) \, d\tau.$$

From the truncated expansion

$$\rho_j = \rho_{0j} + \frac{\partial \rho_j}{\partial \varepsilon} + \mathcal{O}(\varepsilon) = (-1)^k (1 + (-c_d k \pm \pi \sqrt{D})) + \mathcal{O}(\varepsilon), \tag{26}$$

$$D = (2a_k - b_d k)^2 + (2b_k - a_d k)^2 - (2c_k)^2 \tag{27}$$

of the Floquet multiplier we observe that the system tends to get unstable if the term under the square root is larger than $|c_d k|$. In particular this is the case when damping is absent and describes vertices of the instability regions for the Hill and the Mathieu equation [9,11]. From Eq. (25) we see that damping has a stabilizing effect provided

$$c_d > \sqrt{a_d^2 + b_d^2}.$$

For increasing k the damping terms dominate, since they are multiplied by k.

2.3. Discussion

The results obtained in the previous sections obviously only have a local character. They show that in the vicinity of a semi-simple eigenvalue a skew symmetric perturbation has a particularly strong destabilizing effect. At the same time it has to be noted that for $\partial|\rho|/\partial\varepsilon = 0$, ρ moves along the unit circle in the complex plane in the first approximation. The system can still be destabilized by higher order terms. In particular this is the case for constant coefficient $\mathbf{M}$, $\Delta\mathbf{G}$, $\mathbf{K}$, $\Delta\mathbf{N}$ systems with very small gyroscopic terms that are almost always unstable [1,12,13] although neither $\Delta\mathbf{G}$ nor $\Delta\mathbf{N}$ appear in the first derivative of the expansion for the Floquet multipliers provided the spectrum of the unperturbed problem is simple.

Only in specific cases, for example for systems with reversible symmetry (i.e. invariant under time reversal) we can conclude that in the absence of damping the system has a double eigenvalue at the stability boundary [14]. In Ref. [14, p. 133] it is shown that if $\Delta\mathbf{D}(-t) = -\Delta\mathbf{D}(t)$ and $\Delta\mathbf{K}(-t) = \Delta\mathbf{K}(t)$, then the system is invariant under time reversal.

Despite of having only local character, the perturbation formulas indicate that splitting up the eigenfrequencies of the unperturbed problem tends to stabilize the system, especially when additional damping is present.

2.4. Numerical example

Consider equation (1) with

$$\Delta\mathbf{D} = d\cos^2 t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Delta\mathbf{K} = n\cos^2 t \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}. \tag{28}$$
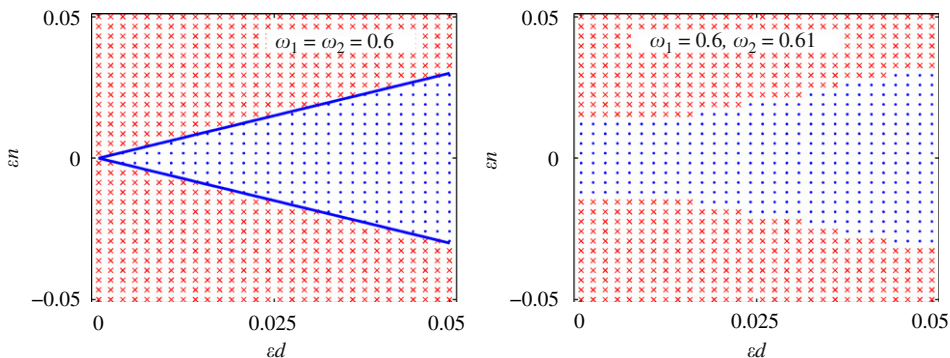

Fig. 1. Stable regions for a symmetric and an asymmetric system (dot: stable, cross: unstable).

In case we chose a symmetric system, i.e. $\omega_1 = \omega_2$ we obtain from Eq. (16)

$$\frac{\partial \rho_1}{\partial \varepsilon} = \left(-d\pi \pm \frac{n\pi}{\omega_1}\right)\rho_0 \tag{29}$$

and hence

$$\frac{\partial |\rho_1|}{\partial \varepsilon} = -\omega_1 \pi d \pm n\pi \tag{30}$$

i.e. we have instable regions near $\varepsilon = 0$. For the asymmetric system $\omega_1 \neq \omega_2$ there is no instable region for small ε and $\delta > 0$ as discussed in the previous section. For the symmetric and the asymmetric case the stable and instable regions in the parameter space εn, εd are shown in Fig. 1 using Floquet theory. The lines in the left plot stand for approximations of the stability boundaries that can be calculated from Eq. (30) as $n = \pm\omega_1 d$ for the symmetric case $\omega_1 = \omega_2 = 0.6$.

3. Application to a disk brake model

In this section as an example of a rotating continuum with equations of motion of the type (1) we consider the disk brake model shown in Fig. 2, which for a completely symmetric brake rotor has been previously studied in Ref. [5]. The brake rotor is modeled as a rotating Kirchhoff plate which is in contact with pointwise elastic friction pads consisting of massless pins which are pressed on the surface of the brake disk by prestressed springs (cf. Fig. 3). Since the analysis in Ref. [5] showed that in- and out-of-plane vibrations are
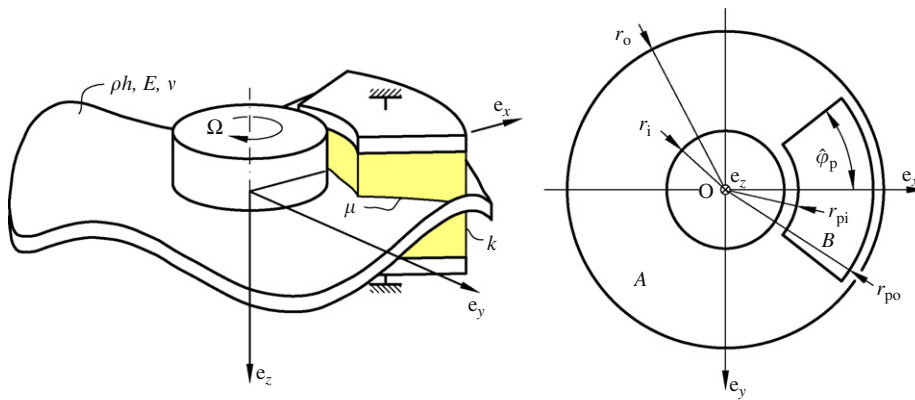

Fig. 2. Kirchhoff plate in distributed frictional contact.
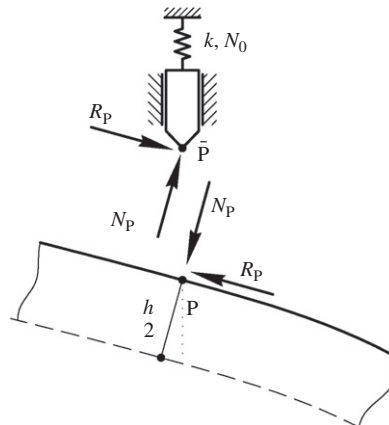

Fig. 3. Pin in contact with the plate surface.

decoupled and the equations for the out-of-plane vibrations decide the stability behavior, we concentrate on their investigation. Whereas in Ref. [5] a purely symmetric disk with infinitely many double eigenfrequencies was investigated setting up the equations of motion with respect to a stationary frame, we now consider the consequences of breaking the rotational symmetry.

For an asymmetric disk, the direct solution of the equations of motion via a matrix eigenvalue problem is no longer possible, since the current orientation of the plate has to be taken into account. Nevertheless, eigenfunctions of the corresponding nonrotating asymmetric plate can be conveniently used to describe the vibrations in a rotating coordinate system. The equations of motion can be derived from the principle of virtual work where the position vectors of the material points of the plate and the brake pads are expressed in coordinates of the rotating frame. A point M on the neutral plane of the plate has the position vector

$$\mathbf{p_M} = r\mathbf{e}_r + w(r, \varphi, t)\mathbf{e}_z, \tag{31}$$

where r is a (constant) radius and $w(r, \varphi, t)$ describes the displacement of the plate's neutral plane. The unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\varphi$ depend on time, and their time derivatives are

$$\dot{\mathbf{e}}_r = \Omega\mathbf{e}_z \times \mathbf{e}_r = \Omega\mathbf{e}_\varphi, \tag{32}$$

$$\dot{\mathbf{e}}_\varphi = \Omega\mathbf{e}_z \times \mathbf{e}_\varphi = -\Omega\mathbf{e}_r, \tag{33}$$

which has to be considered when taking derivatives of vectors expressed in the rotating frame. We therefore have

$$\dot{\mathbf{p}}_M = r\Omega\mathbf{e}_\varphi + \dot{w}(r, \varphi, t)\mathbf{e}_z. \tag{34}$$

Using Kirchhoff's kinematical assumptions, the position vector of a point on the surface of the plate is

$$\mathbf{p}(r, \varphi, t) = r\mathbf{e}_r + w(r, \varphi, t)\mathbf{e}_z - \frac{h}{2}\mathbf{e}_\nabla(r, \varphi, t), \tag{35}$$

where $\mathbf{e}_\nabla(r, \varphi, t)$ is the vector normal to the plate's neutral plane.

We now exemplarily concentrate on the upper contact point P of the plate currently coinciding with $\bar{\text{P}}$, the corresponding point on the brake pad (cf. Fig. 3), and calculate its velocity. The position vector from O to P is given by

$$\mathbf{p_P} = \mathbf{p}(r_P, \varphi_P, t)$$
$$= r_P\mathbf{e}_r(\varphi_P) + w(r_P, \varphi_P, t)\mathbf{e}_z - \frac{h}{2}\mathbf{e}_\nabla(r_P, \varphi_P, t), \tag{36}$$

where r_P and φ_P is in the linear approximation equal to the position of the pad in the undeformed configuration [5]. The velocity $\mathbf{v_P}$ of P (the material point on the surface of the plate) follows from time differentiation of $\mathbf{p_P}$, noting that r_P and φ_P are constants in this case. On the other hand, the velocity $\mathbf{v_{\bar{P}}}$ of $\bar{\text{P}}$ (the material point of the pad) follows from time differentiation of $\mathbf{p_P}$ with time dependent r_P and φ_P. The contact forces between each of the pins belonging to a pad and the disk are calculated from a force balance on each pin in vertical direction and using Coulomb's law of friction, i.e. $R_P = \mu N_P$. The direction of the friction force is always opposite to the relative velocity of the contact point on the pad $\bar{\text{P}}$ and the contact point on the disk P, the normal force is perpendicular to the surface of the plate. It is assumed that the rotational speed of the disk is sufficiently high such that the relative velocity of between the contact points $\bar{\text{P}}$ and P does not change sign. The discretized linear equations follow from the principle of virtual work and in the linearized version read

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{D}(t)\dot{\mathbf{q}} + (\mathbf{K} + \mathbf{N}(t))\mathbf{q} = \mathbf{0}, \tag{37}$$

where $\mathbf{M} = \text{diag}(M_i)$, $\mathbf{K} = \text{diag}(\omega_i^2 M_i)$ and M_i is the normalization factor to the i-th eigenform corresponding to eigenfrequency ω_i of the nonrotating disk. The matrices $\mathbf{D}(t) = \mathbf{D}(t + 2\pi/\Omega)$ and $\mathbf{N}(t) = \mathbf{N}(t + 2\pi/\Omega)$ are periodic matrices and the plate is discretized by

$$w(r, \varphi, t) = \sum_{i=1}^{N} W_i(r, \varphi)\, q_i(t) \tag{38}$$

using adequate shape functions $W_i(r, \varphi)$. The generic element of the matrices $\mathbf{N}$ and $\mathbf{D}$ are

$$
N_{ij} = \int_B \left[2kW^iW^j - \frac{hk\mu}{r}W^i_{,\varphi}W^j + (1+\mu^2)\frac{hN_0}{r^2}W^i_{,\varphi}W^j_{,\varphi} - \frac{h^2N_0\mu}{2r^3}W^i_{,\varphi\varphi}W^j_{,\varphi} \right.
$$
$$
+ \frac{h^2N_0\mu}{2r^2}(W^i_{,\varphi}W^j_{,r} - W^i_{,r}W^j_{,\varphi}) + hN_0W^i_{,r}W^j_{,r} - \frac{h^2N_0\mu}{2r}W^i_{,\varphi r}W^j_{,r}
$$
$$
\left. + \frac{\mathrm{d}\Omega\,h\mu}{r}W^i_{,\varphi}W^j_{,\varphi} - 2\,\mathrm{d}\Omega\,W^iW^j_{,\varphi} \right] \mathrm{d}B \tag{39}
$$

and

$$
D_{ij} = \int_B \left[2\mathrm{d}W^iW^j - \frac{\mathrm{d}h\,\mu}{r}W^i_{,\varphi}W^j + \frac{h^2N_0\mu}{2r\Omega}W^i_{,r}W^j_{,r} \right]\mathrm{d}B, \tag{40}
$$

respectively, where

$$
\int_B (*)\,\mathrm{d}B = \int_{r_{pi}}^{r_{pa}} \int_{-\hat{\varphi}_\mathrm{p}-\Omega t}^{\hat{\varphi}_\mathrm{p}-\Omega t} (*)\,r\,\mathrm{d}r\,\mathrm{d}\varphi \tag{41}
$$

is the integration over the area of the brake pads. In contrast to the equations obtained in the inertial frame [5] we now have periodic coefficients in the matrices. For parameters corresponding to a disk brake, the norms of the matrices $\mathbf{N}_{ij}$ and $\mathbf{D}_{ij}$ are small compared to the mass and stiffness matrices. Therefore they can be interpreted as perturbation matrices and the equations of motion can be written in form (1) such that the observations made in the previous sections apply. Inspecting (40) we observe that the entries on the main diagonal are strictly positive since $h/r \ll 1$. This remains true if $d = 0$, which means that no additional damping is introduced in the pads. We therefore conclude that for a simple spectrum of the unperturbed problem and provided that no internal resonances occur, the trivial solution of Eq. (37) is asymptotically stable independently of the particular values of the parameters. As to the case of internal resonances, we saw from Eq. (25) that in the presence of damping with a sufficiently high mean value, for higher order resonances no instabilities occur. Since the rotational speed of the disk is low compared to its eigenfrequencies and the diagonal terms in Eq. (40) are nonnegative these should not be important for squeal problems. It is interesting to note that the setting investigated in Ref. [5] is a special case of the investigation performed here. Due to the symmetry of the plate studied in Ref. [5], in the rotating frame semi-simple eigenvalues arise and, depending on the parameters, stable and unstable configurations are possible. In a two parameter space in the vicinity of the unperturbed problem the stability region was described by an angle. In conclusion we see that braking the symmetry of the rotor avoids semi-simple eigenvalues and therefore stabilizes the system. Due to the presence of damping, the higher order internal resonances occurring in the brake example cannot destabilize the system.

In addition to these qualitative predictions, Eq. (37) can be also used for quantitative studies on asymmetric disks. A promising approach is to perform a finite element analysis on an asymmetric disk and to study the rotating disk with pads using the shape functions in Eq. (37). Due to the small rotational frequencies of the disk compared to the eigenfrequencies, the integration times using Floquet theory are however long and may yield numerical uncertainties. Nevertheless this method has been successfully used in Ref. [15] to study asymmetries. Considering the high uncertainty in parameter values, another approach could be the investigation of structured pseudospectra of the monodromy matrix.

4. Conclusion

In this paper we give a mathematical justification for the approach of breaking symmetries in order to stabilize rotating continua. Mathematically we study a general linear conservative system perturbed by small arbitrary parametric excitation. In addition to the well know instability mechanism of parametric resonance we clarify the influence of the symmetries of the perturbation matrices on the stability behavior of the system. The results are relevant for any kind of rotating shells or plates with small nonconservative position forces. As an important example a minimal model for an asymmetric disk brake was discussed.

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