A recursive approach for the finite element computation of waveguides

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Abstract

The finite element computation of structures such as waveguides can lead to heavy computations when the length of the structure is large compared to the wavelength. Such waveguides can in fact be seen as one-dimensional periodic structures. In this paper a simple recursive method is presented to compute the global dynamic stiffness matrix of finite periodic structures. This allows to get frequency response functions with a small amount of computations. Examples are presented to show that the computing time is of order $\log_2 N$ where N is the number of periods of the waveguide.
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1. Introduction

We study here the computation of structures considered as waveguides, as shown in Fig. 1, with symmetries which can be a translation (a), a rotation (b) or a periodicity (c). Thus these waveguides can be uniform or periodic. The vibration of such waveguides has been the topic of much research. One can find analytical or finite element models of waveguides and people are generally interested by the computation of wave propagations and dispersion curves or by the determination of the frequency response functions. A first approach considers structures with constant cross-sections as the cases (a) and (b) of Fig. 1. For example, Refs. [1,2] used a wave approach to study the vibrations of structural networks composed of simple uniform beams, and solved for the dynamics of individual elements and of the junctions between elements by analytical methods. The efficiency is greatly improved compared to FE methods as a beam can be modelled using only a single element.

The first numerical approaches were proposed by Refs. [3,4] to approximate the cross-sectional deformations by finite elements. The authors of Refs. [5,6] applied similar ideas to the calculation of wave propagations in rails using a finite element model of the cross-section of a rail. They then calculated dispersion relations and accelerances. Dispersion relations for elastic waves in helical waveguides were also considered by Ref. [7]. For general waveguides with a complex cross-section, the displacements in the cross-section can be

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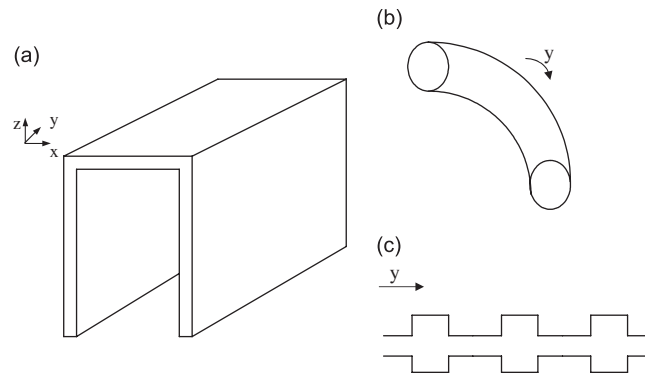E-mail address: duhamel@lami.enpc.fr

Fig. 1. Examples of waveguide structures.

described by the finite element method while the variation along the axis of symmetry is expressed as a wave function. Following these ideas, Refs. [8–13] developed the spectral finite element approach. This leads to efficient computations of dispersion relations and transfer functions but special elements need to be developed for each element type. This makes the connection with the standard use of the finite element method difficult and does not allow the benefits of powerful existing finite element software to be exploited. Similar techniques were also developed by Refs. [14,15] for the computation of dispersion relations in damped waveguides.

More general waveguides can be studied by considering periodic structures. Numerous works provided interesting theoretical insights in the behaviour of these structures, see for instance the work of Ref. [16] and the review paper by Ref. [17]. Mead also presented a general theory for wave propagation in periodic systems in Refs. [18–20]. He showed that the solution can be decomposed into an equal number of positive and negative-going waves. The approach is mainly based on Floquet's principle or the transfer matrix and the objective it to compute propagation constants relating the forces and displacements on the two sides of a cell (a single period) and the waves associated to these constants. For complex structures FE models are used for the computation of the propagation constants and waves. The final objective is to compute dispersion relations to use them in energetic methods, see Refs. [21–25]. In Ref. [26] the general dynamic stiffness matrix for a periodic structure was found from the propagation constants and waves. It leads to a matrix linking the extreme sides of the structure and allows to compute transfer functions in the structure.

The last approach is purely computational and uses rotational and cyclic symmetries to solve the problem by a decomposition of the displacements in cosine and sine functions. It is thus possible to find the transfer functions or modal shapes for periodic structures. A review of the current practises can be found in Ref. [27]. These methods allow computing the frequency response functions in a number of operations proportional to the number of cells in the structure. This paper develops this last approach and presents a recursive method to calculate the forced response of structures such as those illustrated in Fig. 1. A section of the waveguide is modelled using conventional FE methods, using a commercial FE package. The resulting mass, stiffness and damping matrices are then post-processed to give the dynamic stiffness matrix of the cell. Then a recursive method is applied to compute the global dynamic stiffness matrix of the whole waveguide and finally the transfer functions in the structure.

This paper presents a different approach from the previously published paper [26]. Both papers aim at computing the global dynamic stiffness matrix of a *N* cells structure. But in Ref. [26] waves in a period were computed and from these waves the dynamic stiffness matrix of a complete structure was obtained with a computational cost independent of the number of periods in the structure. However, the computation of the waves can be time consuming when the number of degrees of freedom (dofs) in a section is large because this needs the computation of eigenvalues of nonsymmetric matrices. On the contrary, in the present approach no wave needs to be computed and the global dynamic stiffness matrix is obtained by products and inverses of matrices with the same dimensions as the dynamic stiffness matrix of a cell. Then, the frequency response functions can be obtained easily without the computation of any wave.

The paper is divided into two parts. In the first part the recursive method for the finite element analysis of periodic structures is presented. In the second part two examples consisting in a beam and a plate are described before the conclusion.

2. Finite element analysis of periodic structures

Consider a periodic structure, as shown in Fig. 2, which is made of a large number N of cells. We are interested by the computation of the frequency response function for a point force excitation $F = 1$ somewhere in the structure and a response u at another point. We propose here an efficient method to compute this function by using a recursive approach to get the dynamic stiffness matrix of different sets of cells.

2.1. Behaviour of a cell

Consider first the case of only one cell. The discrete dynamic equation of a cell obtained from a FE model at a frequency ω and for the time dependence $e^{-i\omega t}$ is given by

$$(\mathbf{K} - i\omega\mathbf{C} - \omega^2\mathbf{M})\mathbf{q} = \mathbf{f} \tag{1}$$

where $\mathbf{K}$, $\mathbf{M}$ and $\mathbf{C}$ are the stiffness, mass and damping matrices, respectively, $\mathbf{f}$ is the loading vector and $\mathbf{q}$ the vector of the dofs. A viscous damping is considered here but the same results could be obtained with other damping models. Introducing the dynamic stiffness matrix $\widetilde{\mathbf{D}} = \mathbf{K} - i\omega\mathbf{C} - \omega^2\mathbf{M}$, decomposing the dofs into boundary (B) and interior (I) dofs as shown in Fig. 3, and assuming that there are no external forces on the interior nodes, result in the following equation:

$$\begin{bmatrix} \widetilde{\mathbf{D}}_{BB} & \widetilde{\mathbf{D}}_{BI} \\ \widetilde{\mathbf{D}}_{IB} & \widetilde{\mathbf{D}}_{II} \end{bmatrix} \begin{bmatrix} \mathbf{q}_B \\ \mathbf{q}_I \end{bmatrix} = \begin{bmatrix} \mathbf{f}_B \\ \mathbf{0} \end{bmatrix} \tag{2}$$

The interior dofs can be eliminated using the second row of Eq. (2), which results in

$$\mathbf{q}_I = -\widetilde{\mathbf{D}}_{II}^{-1}\widetilde{\mathbf{D}}_{IB}\mathbf{q}_B \tag{3}$$
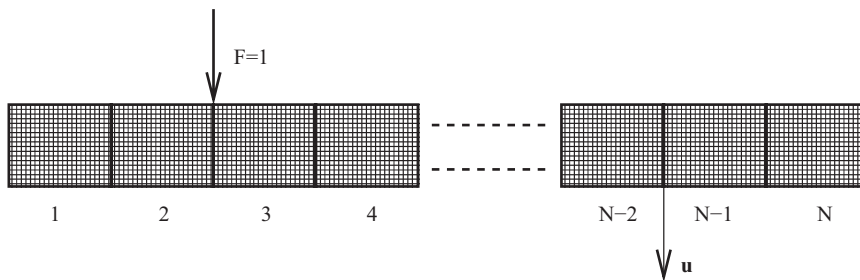

Fig. 2. Periodic structure made of N cells with an excitation by a force F.
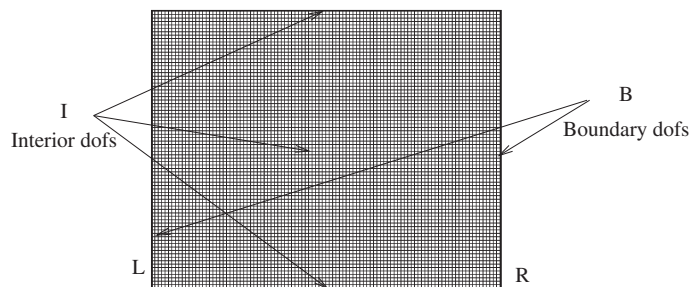

Fig. 3. Interior and boundary dofs for a single cell.

The first row of Eq. (2) becomes

$$\mathbf{f}_B = (\widetilde{\mathbf{D}}_{BB} - \widetilde{\mathbf{D}}_{BI}\widetilde{\mathbf{D}}_{II}^{-1}\widetilde{\mathbf{D}}_{IB})\mathbf{q}_B \tag{4}$$

It should be noted that only boundary dofs are considered in the following. The cell is assumed to be meshed with an equal number of nodes on their opposite sides. The boundary dofs for one cell are decomposed into left (*L*) and right (*R*) dofs as shown in Fig. 3. Thus, Eq. (4) is rewritten as

$$\begin{bmatrix} \mathbf{f}_L \\ \mathbf{f}_R \end{bmatrix} = \begin{bmatrix} \mathbf{D}_{LL}^{(1)} & \mathbf{D}_{LR}^{(1)} \\ \mathbf{D}_{RL}^{(1)} & \mathbf{D}_{RR}^{(1)} \end{bmatrix} \begin{bmatrix} \mathbf{q}_L \\ \mathbf{q}_R \end{bmatrix} = \mathbf{D}^{(1)} \begin{bmatrix} \mathbf{q}_L \\ \mathbf{q}_R \end{bmatrix} \tag{5}$$

where $\mathbf{D}^{(1)}$ is the dynamic stiffness matrix of a single cell. This matrix is symmetric if the matrices $\mathbf{K}$, $\mathbf{M}$ and $\mathbf{C}$ in relation (1) are symmetric.

2.2. Computation of reduced dynamic stiffness matrices

Consider a structure made of two cells with the respective dynamic stiffness matrices denoted by $\mathbf{A}$ and $\mathbf{B}$ as in Fig. 4. We propose to remove the internal dofs at the boundary between the two cells to compute the dynamic stiffness matrix, denoted $\mathbf{D}^{(2)}$, relating the dofs in the first section of *A* and the last section of *B*. The dynamic stiffness matrix of the substructure with two cells is computed by

$$\begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{LL} & \mathbf{A}_{LR} & 0 \\ \mathbf{A}_{RL} & \mathbf{A}_{RR} + \mathbf{B}_{LL} & \mathbf{B}_{LR} \\ 0 & \mathbf{B}_{RL} & \mathbf{B}_{RR} \end{bmatrix} \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \\ \mathbf{q}_3 \end{bmatrix} \tag{6}$$

As there is no load on the interior section, one gets $\mathbf{f}_2 = 0$ and

$$\mathbf{q}_2 = -(\mathbf{A}_{RR} + \mathbf{B}_{LL})^{-1}(\mathbf{A}_{RL}\mathbf{q}_1 + \mathbf{B}_{LR}\mathbf{q}_3) \tag{7}$$

The global dynamic stiffness matrix of the two-cells structure is thus

$$\begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{LL} - \mathbf{A}_{LR}(\mathbf{A}_{RR} + \mathbf{B}_{LL})^{-1}\mathbf{A}_{RL} & -\mathbf{A}_{LR}(\mathbf{A}_{RR} + \mathbf{B}_{LL})^{-1}\mathbf{B}_{LR} \\ -\mathbf{B}_{RL}(\mathbf{A}_{RR} + \mathbf{B}_{LL})^{-1}\mathbf{A}_{RL} & \mathbf{B}_{RR} - \mathbf{B}_{RL}(\mathbf{A}_{RR} + \mathbf{B}_{LL})^{-1}\mathbf{B}_{LR} \end{bmatrix} \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_3 \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{D}_{LL}^{(2)} & \mathbf{D}_{LR}^{(2)} \\ \mathbf{D}_{RL}^{(2)} & \mathbf{D}_{RR}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_3 \end{bmatrix}$$
$$= \mathbf{D}^{(2)} \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_3 \end{bmatrix} \tag{8}$$

This defines the matrix $\mathbf{D}^{(2)}$ which relates the forces and displacements dofs at the extreme sections of the two-cells structure. It can be easily checked that if the matrices $\mathbf{A}$ and $\mathbf{B}$ are symmetric, the resulting matrix $\mathbf{D}^{(2)}$ of
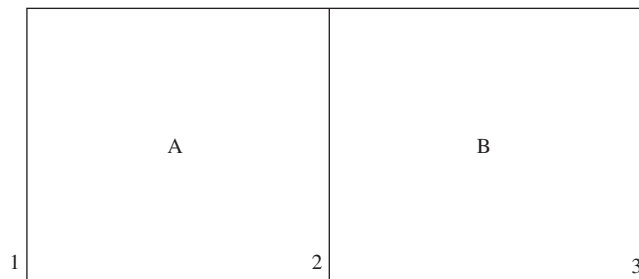

Fig. 4. Structure with two cells and three sections.

relation (8) is also symmetric. The operation of removing the interior dofs is now denoted by {.,.} such that we can write

$$\mathbf{D}^{(2)} = \{\mathbf{A}, \mathbf{B}\} \tag{9}$$

2.3. Case of general structures

Consider now a structure without internal load and made of 2^n cells. We propose to recursively remove the internal dofs between adjacent cells to compute the dynamic stiffness matrix, denoted $\mathbf{D}^{(2^n)}$, relating the dofs in sections 1 and 2^n. Consider firstly a structure with two identical cells. From the precedent analysis, one sees that its dynamic stiffness matrix is given by $\mathbf{D}^{(2)} = \{\mathbf{D}^{(1)}, \mathbf{D}^{(1)}\}$. Repeating the process (see an illustration in Fig. 5), one gets the dynamic stiffness matrix of the structure with 2^n cells in n steps by the recursive relation

$$\mathbf{D}^{(2^n)} = \{\mathbf{D}^{(2^{n-1})}, \mathbf{D}^{(2^{n-1})}\} \tag{10}$$

This matrix is such that

$$\begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_{2^n} \end{bmatrix} = \begin{bmatrix} \mathbf{D}_{LL}^{(2^n)} & \mathbf{D}_{LR}^{(2^n)} \\ \mathbf{D}_{RL}^{(2^n)} & \mathbf{D}_{RR}^{(2^n)} \end{bmatrix} \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_{2^n} \end{bmatrix} = \mathbf{D}^{(2^n)} \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_{2^n} \end{bmatrix} \tag{11}$$

In cases where the structure is not composed of a number of cells which equals a power of two, one can modify the previous procedure using the binary representation of the total number of cells N. Consider the example where $N = 11 = 1011_b$ in binary representation. One calculates first the dynamic stiffness matrix for a structure with eight cells, then this structure is assembled with a structure made of two cells which has been computed during the computation of the eight-cells structure. Finally the resulting matrix is assembled with a one cell matrix. The approach can be resumed by

$$\mathbf{D}^{(11)} = \{\{\mathbf{D}^{(8)}, \mathbf{D}^{(2)}\}, \mathbf{D}^{(1)}\} \tag{12}$$

in which the matrices $\mathbf{D}^{(2^n)}$ are computed by the method presented before. The elimination of the interior dofs gives the matrix linking the forces and displacements dofs in sections 1 and N. One can notice that the final matrix is computed with a number of operations of order $\log_2 N$, thus saving a huge number of computations when we compare to a standard approach in which all the matrices of the cells are assembled into a global matrix.

Using this method it is easy to compute the matrices for the parts of the structure, respectively, on the left and on the right of the force. Assembling these two matrices, applying the appropriate boundary conditions on the first and last sections, one gets the final linear system. This system has a number of dofs which equals approximately three times the number of dofs in a section.
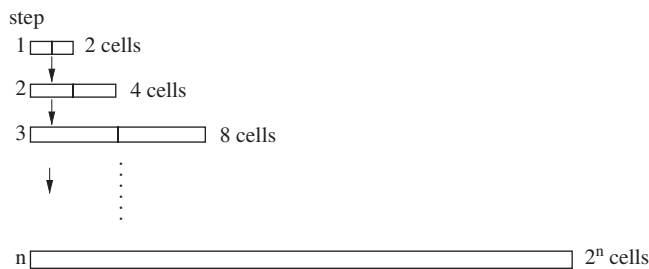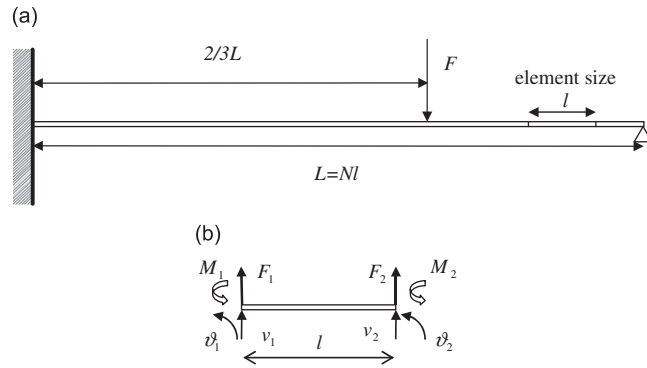

Fig. 5. Structure at each step.
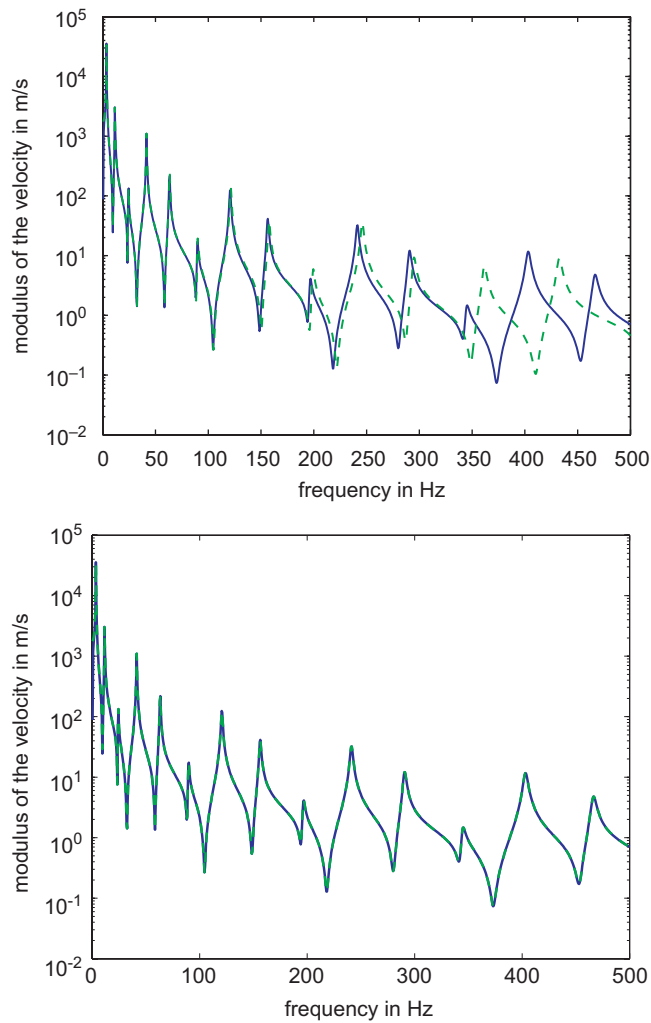
Fig. 6. Beam structure (a) and beam element (b).

Fig. 7. Frequency response functions for a beam with 2×8 elements (upper graph) and 2×1024 elements (lower graph): — analytical solution, − − − finite element solution. The position of the excitation point is shown in Fig. 6 and the response is computed at the same point.

3. Examples

3.1. Beam structure

In the first example, we consider the beam shown in Fig. 6 which is made of elements with four dofs. The stiffness and mass matrices of an element of length l are given by

$$\mathbf{K}_e = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \tag{13}$$

$$\mathbf{M}_e = \frac{\rho S l}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \tag{14}$$

Here, E is Young's modulus, I the second moment of area, ρ the density of the material and S the cross-sectional area of the beam. Using the previous approach, one can compute the dynamic stiffness matrices for the sections on the left and on the right of the force. The damping matrix is obtained by using a complex Young modulus such that $E = E_0(1 + i\eta)$ with $\eta = 0.01$ leading to a hysteretic damping instead of the viscous one such that $\tilde{\mathbf{D}} = (1 + i\eta)\mathbf{K} - \omega^2 \mathbf{M}$. The beam is made of steal such that $E_0 = 2 \times 10^{11}$ Pa, $\rho = 7800 \text{ kg/m}^3$,
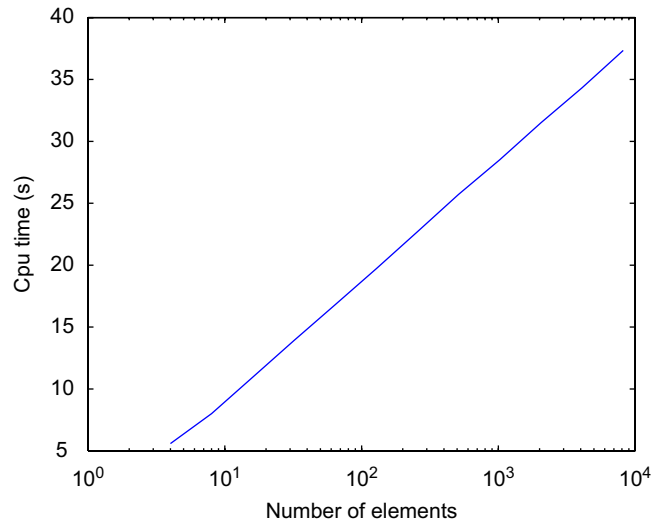

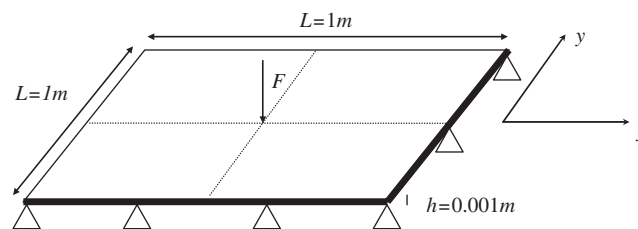Fig. 8. Computing time versus the number of cells in the beam.

Fig. 9. Plate example.

the length of the beam $L = 1\,\text{m}$, $I = 8.33 \times 10^{-14}\,\text{m}^4$ and $S = 10^{-6}\,\text{m}^2$. The matrix for the whole beam is obtained by assembling the matrices of the left and right parts of the beam on each sides of the force. Taking into account the fixed displacement boundary conditions by removing the corresponding dofs in the global matrix results in the final system. Fig. 7 presents the frequency response functions for structures with, respectively, 8 and 1024 elements in each part of the beam. It can be seen that 8 elements are not sufficient to compute accurately the solution while 1024 elements lead to a very good result.

In Fig. 8, the cpu time of the computation is plotted versus the global number of elements in the beam. The computation of 10 000 points in frequency is made for each mesh of the beam and the largest mesh has 2×4196 elements for the complete beam. A linear behaviour of the cpu time versus the logarithm of the number of elements can be seen as expected.

3.2. Plate

Consider now the plate shown in Fig. 9. The mesh of a cell is obtained by Abaqus and consists in 50 elements of size $Ly/50 \times Lx/2^n$ where n is the number of cells along the direction x. The mass and stiffness
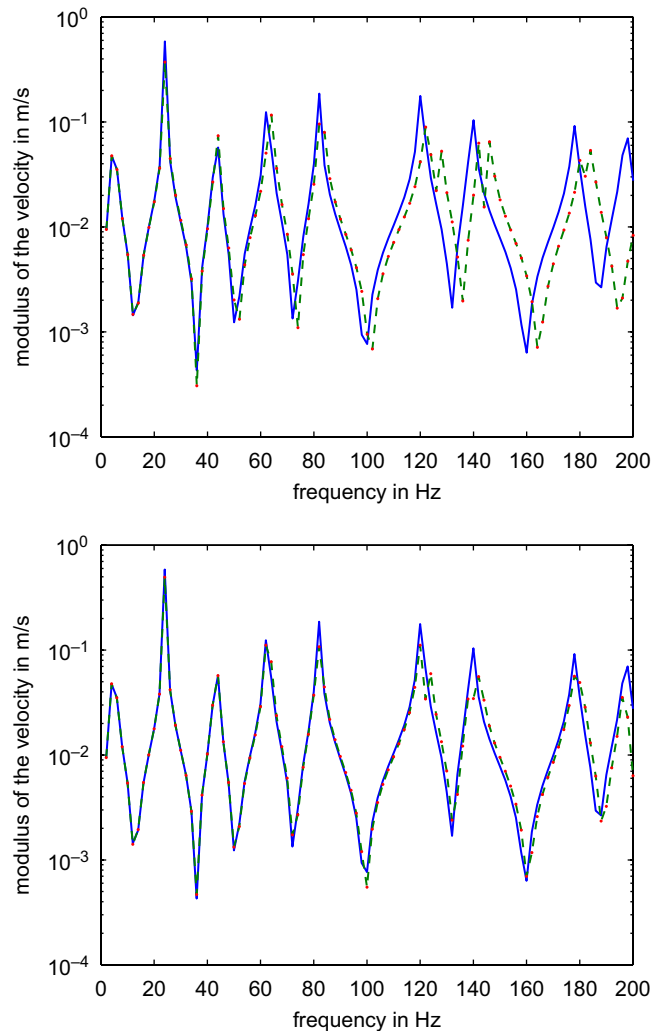

Fig. 10. Frequency response functions for a plate with 2×16 cells (upper graph) and 2×256 cells (lower graph): — analytical solution, $---$ recursive finite element solution and $\cdot\cdot$ standard finite element solution. The excitation is located at the centre of the plate as shown in Fig. 9 and the response is computed at the same point.

matrices are produced by Abaqus then there are loaded in Matlab and the precedent procedure allows computing the displacement for a load at the centre of the plate. Information on the size are given in Fig. 9 and the plate is still made of steal. The boundary conditions are simply supported on all sides. Fig. 10 presents the frequency response functions for structures with globally 32 and 512 cells. It can be seen, as for the beam, that 32 cells are not sufficient to compute accurately the solution while 512 cells lead to much better results. For high frequencies a discrepancy with the analytical solution can still be seen. It has been checked that the result can be considerably improved by taking 100 elements instead of 50 along the direction y. The figures also present the results from the standard finite element approach obtained by assembling the elementary matrices for each period and solving the global linear system. It can be seen that both finite elements computations yield identical results.

In Fig. 11, the cpu time of the computation is plotted versus the global number of cells in the plate. The computation of 100 points in frequency is made for each mesh of the plate and the largest mesh has 2048 cells along x for the complete plate. Once again a linear behaviour of the cpu time versus the logarithm of the number of cells can be observed. The computing times for the recursive and standard finite element methods are compared in Table 1. It can be seen that the recursive method is a little slower than the standard method for a number of periods lower than 16. For a larger number of periods the recursive method tends to be more and more efficient as the number of periods increases.
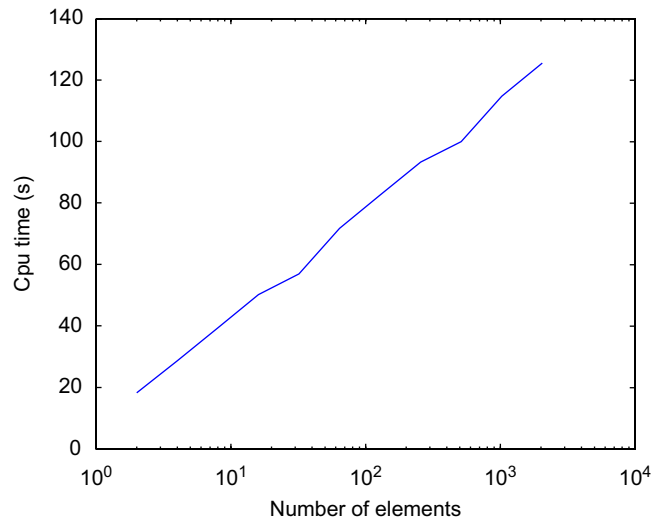

Fig. 11. Computing time versus the number of cells in the plate.

Table 1
Computing times for recursive and standard FEM methods.

Number of periods	Recursive method (s)	Classical FEM (s)
2	18.3	14.4
4	28.8	20.1
8	39.5	33.7
16	50.3	69.2
32	57.0	164.3
64	71.8	435.0
128	82.7	1295.0
256	93.4	4093.1
512	100.0	13 901.4
1024	114.8	49 271.1
2048	125.6	184 397.3

4. Conclusion

A method has been described to compute frequency response functions for waveguide structures with periodic or homogeneous sections. The proposed method allows computing the solution in a time proportional to the logarithm of the numbers of cells in the structure. This simple recursive method can be applied for any type of one-dimensional periodic waveguides. It could be used in the future for the computation of complex structures such as tyres.

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