Available online at www.sciencedirect.com
ScienceDirect
Journal of Sound and Vibration 323 (2009) 352–365
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Theoretical study of the effects of nonlinear viscous damping on vibration isolation of sdof systems

Z.Q. Lang[a,*], X.J. Jing[a], S.A. Billings[a], G.R. Tomlinson[b], Z.K. Peng[a]

[a]*Department of Automatic Control and Systems Engineering, University of Sheffield, Mappin Street, Sheffield S1 3JD, UK*
[b]*Department of Mechanical Engineering, University of Sheffield, Mappin Street, Sheffield S1 3JD, UK*

Received 12 July 2008; received in revised form 13 December 2008; accepted 3 January 2009
Handling Editor: A.V. Metrikine
Available online 6 February 2009

Abstract

The present study is concerned with the theoretical analysis of the effects of nonlinear viscous damping on vibration isolation of single degree of freedom (sdof) systems. The concept of the output frequency response function (OFRF) recently proposed by the authors is applied to study how the transmissibility of a sdof vibration isolator depends on the parameter of a cubic viscous damping characteristic. The theoretical analysis reveals that the cubic nonlinear viscous damping can produce an ideal vibration isolation such that only the resonant region is modified by the damping and the non-resonant regions remain unaffected, regardless of the levels of damping applied to the system. Simulation study results demonstrate the validity and engineering significance of the analysis. This research work has significant implications for the analysis and design of viscously damped vibration isolators for a wide range of practical applications.

1. Introduction

The purpose of vibration isolation is to control unwanted vibrations so that the adverse effects are kept within acceptable limits. Transmissibility is a concept widely used in the design of vibration isolators to measure the vibration transmission at different frequencies.

In a vibration isolation device, viscous damping is often introduced to reduce vibration amplitude at resonance. However, if the effect of the introduced viscous damping is basically linear, as the damping level is increased to reduce the transmissibility in the resonant region, the transmissibility is increased in the region where isolation is required. As a result, the range of frequencies where vibration isolation can be achieved is reduced. This is a well-known dilemma associated with viscously damped vibration isolator design [1]. In order to resolve this problem, engineers have developed many techniques such as the "sky hook" and other active vibration isolators where, for example, the linear viscous damping effect is automatically switched off to

*Corresponding author.
E-mail address: z.lang@sheffield.ac.uk (Z.Q. Lang).

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2009.01.001

minimize the transmissibility levels in the isolation region [2,3]. However, as far as we are aware, there is no approach which can systematically address this important problem using a passive viscously damped vibration isolation device.

In order to resolve this problem, the authors have recently proposed the introduction of nonlinear viscous damping to provide a vibration isolation solution and have conducted a series of studies on this subject [4–8].

In Ref. [4], based on the Volterra series theory of nonlinear systems, an analytical relationship between the output frequency response and the nonlinear viscous damping characteristic parameters was derived for a single degree of freedom (sdof) spring damper system to provide a theoretical basis for this study. In Ref. [5], the result in Ref. [4] was extended to nonlinear systems that can be described by a general polynomial form differential equation model, and a concept known as the output frequency response function (OFRF) was proposed.

It is well known that exact closed-form analytic solutions of nonlinear differential equations are possible only for a limited number of special classes of differential equations. Consequently, in traditional nonlinear structural analysis, different approaches such as harmonic balance [9], perturbation and averaging methods [10,11] were often used to produce approximate solutions. The harmonic balance method is based on the assumption that the system time domain response can be expressed in the form of a Fourier series, and is usually used to study nonlinear systems whose output responses are periodic in time. Perturbation and averaging methods are used to obtain an approximate solution such that the approximation error is small and the approximate solution is expressed in terms of equations simpler than the original equation. The advantage of the recently developed OFRF and associated results is that these new methods can reveal how the output frequency response of a wide class of nonlinear systems depends on the model parameters that define the system nonlinearity. This can considerably facilitate the frequency domain analysis and design of nonlinear structural systems including systems with nonlinear viscous damping devices.

By using an OFRF based approach, a nonlinear engine mount was analyzed in Ref. [6]. The nonlinear damping design issue was addressed from the perspective of control system design using the OFRF approach and nonlinear control system theory in Ref. [7], where some advantages of nonlinear viscous damping were also demonstrated by simulation studies. The study in Ref. [8] applied the basic idea in Ref. [4] but focused on the case of sdof vibrating systems with a cubic viscous damping characteristic. Simulation examples were used to demonstrate how nonlinear viscous damping affects the system behavior at the resonant frequency.

In one respect, these previous studies establish an OFRF based theory for the study of the effect of nonlinearities on the output frequency response of nonlinear structures including nonlinearly damped structural systems. In another respect, certain beneficial effects of nonlinear viscous damping are demonstrated by numerical simulation studies. In order to further develop these research results, the authors realized that there is a need to apply the OFRF based theory to nonlinear viscous damping systems to comprehensively prove significant effects of nonlinear viscous damping on vibration isolation and theoretically confirm conclusions that have so far only been obtained by specific simulation studies. In our opinion, this is a starting point for establishing solid theoretical results for the analysis and design of nonlinear viscously damped structures.

To achieve this objective, a novel theoretical analysis is introduced in the present study. Based on the OFRF concept, the study first derives an analytical relationship between the transmissibility and the nonlinear characteristic parameter of sdof viscously damped vibration isolators with a cubic damping curve. Then it is rigorously proved that the introduction of a cubic viscous damping characteristic can produce an ideal vibration isolation such that ‘‘There is little damping in the isolation region but considerable damping around the isolator's natural frequency'' [12] so as to achieve the required transmissibility and reduced amplification at resonance at the same time. Finally, the results of numerical simulations are used to demonstrate the validity and engineering significance of the conclusions reached by the theoretical analysis. This research study is of significance in solving the problems with linear viscous damping model based analysis and design and, therefore, has significant implications for the analysis and design of viscously damped vibration isolators for a wide range of practical applications.

2. Sdof vibration isolators with a nonlinear damping characteristic

Consider the sdof vibration isolator system shown in Fig. 1, where

$$f_{IN}(t) = A\ \sin(\Omega t) \tag{1}$$

is the harmonic force acting on the system with frequency Ω and magnitude A, $f_{OUT}(t)$ is the force transmitted to the supporting base, which is assumed to be perfectly immobile, i.e., has infinite impedance, and $z(t)$ is the displacement of the mass. For simplicity of analysis, assume that the vibration isolator has a linear spring and a cubic damping characteristic as indicated in the figure so that the equations of motion of the sdof vibration isolator system are given by

$$\begin{cases} M\ddot{z}(t) + C_1\dot{z}(t) + C_2[\dot{z}(t)]^3 + Kz(t) = f_{IN}(t) = A\ \sin(\Omega t) \\ f_{OUT}(t) = Kz(t) + C_1\dot{z}(t) + C_2[\dot{z}(t)]^3 \end{cases} \tag{2}$$

where K and C_1, C_2 are the spring and viscous damping characteristic parameters of the system, respectively.

In order to conduct an analysis which is not specific to particular choices of M and K, denote

$$\tau = \Omega_0 t \tag{3}$$

where $\Omega_0 = \sqrt{K/M}$ is the resonant frequency of the system,

$$\bar{\Omega} = \frac{\Omega}{\Omega_0} \tag{4}$$

and

$$x(\tau) = z(t) = z\left(\frac{\tau}{\Omega_0}\right) \tag{5}$$

and write the first equation in Eq. (2) into a dimensionless form as

$$\ddot{y}(\tau) + \xi_1\dot{y}(\tau) + \xi_2[\dot{y}(\tau)]^3 + y(\tau) = \sin(\bar{\Omega}\tau) \tag{6}$$

where

$$y(\tau) = \frac{Kx(\tau)}{A} \tag{7}$$

$$\xi_1 = \frac{C_1}{\sqrt{KM}} \tag{8}$$

$$\xi_2 = \frac{C_2 A^2}{\sqrt{(KM)^3}} \tag{9}$$

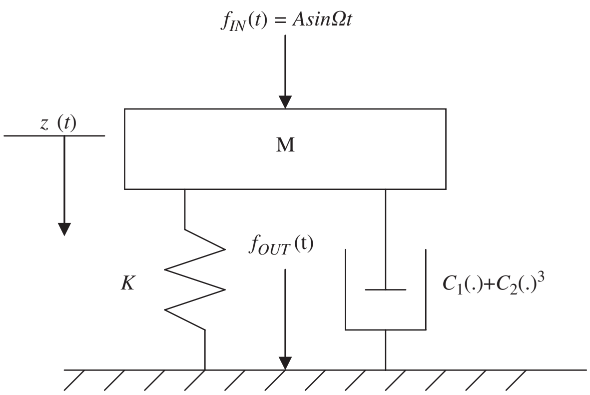

Fig. 1. Single degree of freedom vibration isolator system with a cubic nonlinear viscous damping characteristic.

Let

$$u(\tau) = \sin(\bar{\Omega}\tau) \tag{10}$$

$$y_1(\tau) = y(\tau) \tag{11}$$

$$y_2(\tau) = y_1(\tau) + \xi_1 \dot{y}_1(\tau) + \xi_2[\dot{y}_1(\tau)]^3 \tag{12}$$

Then, the sdof vibration isolator system (2) can be described as a dimensionless, one input two output system as

$$\begin{cases} \ddot{y}_1(\tau) + y_2(\tau) = u(\tau) \\ y_2(\tau) = y_1(\tau) + \xi_1 \dot{y}_1(\tau) + \xi_2[\dot{y}_1(\tau)]^3 \end{cases} \tag{13}$$

From Eqs. (2), (5), (7)–(9), and (11), it can be shown that

$$\frac{f_{OUT}(t)}{A} = \frac{Kz(t) + C_1\dot{z}(t) + C_2[\dot{z}(t)]^3}{A} = y_1(\tau) + \xi_1\dot{y}_1(\tau) + \xi_2[\dot{y}_1(\tau)]^3 = y_2(\tau) \tag{14}$$

Therefore, denote $T(\bar{\Omega})$ as the force transmissibility of the sdof isolator system (2) in terms of the normalized frequency $\bar{\Omega}$. Then

$$T(\bar{\Omega}) = |Y_2(j\bar{\Omega})| \tag{15}$$

where $Y_2(j\bar{\Omega})$ is the spectrum $Y_2(j\omega)$ of the second output of system (13) evaluated at frequency $\omega = \bar{\Omega}$. Therefore the transmissibility of the sdof isolator system (2) can be studied by investigating the spectrum of the second output of system (13).

In the next section, the OFRF of system (13) which provides an explicit, analytical relationship between $Y_2(j\omega)$ and the system nonlinear viscous damping characteristic parameter ξ_2 is derived to facilitate the analysis of the force transmissibility of the sdof isolator system (2). This result is then used in Section 4 to reveal the significant effects of nonlinear viscous damping on the performance of vibration isolating systems.

3. Representation of the force transmissibility using the OFRF

The OFRF is a concept recently proposed by the authors in Ref. [5] for the study of the output frequency response of nonlinear Volterra systems.

Nonlinear Volterra systems represent a wide class of nonlinear systems whose input output relationship can be described by a Volterra series model over a regime around a stable equilibrium [13,14]. For nonlinear Volterra systems which can equally be described by a polynomial type nonlinear differential equation model which has been widely used for the modeling of practical physical systems, it has been shown in Ref. [5] that the system output spectrum can be represented by an explicit polynomial function of the model parameters which define the system nonlinearity. This result is referred to as the OFRF, which provides a significant analytical link between the output frequency response and the nonlinear characteristic parameters for a wide range of practical nonlinear systems.

In the following, the OFRF concept will be applied to the case of the one input two output system (13) to produce an analytical polynomial relationship between the spectrum of the system's second output $Y_2(j\omega)$ and the nonlinear characteristic parameter ξ_2. Because $Y_2(j\omega)$ is related to the force transmissibility $T(\bar{\Omega})$ of system (2) via Eq. (15), the result, in fact, provides an OFRF based analytical expression for $T(\bar{\Omega})$.

According to Ref. [15], it is known that when subject to a sinusoidal input

$$u(\tau) = \sin(\bar{\Omega}\tau) = \cos(\bar{\Omega}\tau - \pi/2) \tag{16}$$

the spectra of the outputs of system (13) are given by

$$Y_J(j\omega)=\sum_{n=1}^{N}\frac{1}{2^n}\sum_{\omega_1+\cdots+\omega_n=\omega}H_n^{(J)}(j\omega_1,\ldots,j\omega_n)\bar{A}(\omega_1),\ldots,\bar{A}(\omega_n)\quad J=1,2 \tag{17}$$

In Eq. (17)

$$\bar{A}(\omega_i)=\begin{cases}\mathrm{e}^{-j\pi/2} & \text{when } \omega_i=\bar{\Omega}\\ \mathrm{e}^{j\pi/2} & \text{when } \omega_i=-\bar{\Omega}\quad i=1,\ldots,n\\ 0 & \text{otherwise}\end{cases} \tag{18}$$

N is the maximum order of nonlinearity in the Volterra series expansion of the system outputs given by

$$y_J(\tau)=\sum_{n=1}^{N}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n^{(J)}(\tau_1,\ldots,\tau_n)\prod_{i=1}^{n}u(\tau-\tau_i)\,\mathrm{d}\tau_i\quad J=1,2 \tag{19}$$

with $h_n^{(J)}(\tau_1,\ldots,\tau_n)$, $J=1,2$, denoting the nth order Volterra kernel, and

$$H_n^{(J)}(j\omega_1,\ldots,j\omega_n)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n^{(J)}(\tau_1,\ldots,\tau_n)\,\mathrm{e}^{-(\omega_1\tau_1+\cdots+\omega_n\tau_n)j}\,\mathrm{d}\tau_1,\ldots,\mathrm{d}\tau_n\quad J=1,2 \tag{20}$$

are the definition of the nth order generalized frequency response function (GFRF) [16] between the input and the first and second outputs of system (13), respectively.

The specific expression of $H_n^{(J)}(j\omega_1,\ldots,j\omega_n)$, $J=1,2$, can be obtained by applying the results in Ref. [17] to the case of the one input two output nonlinear differential model (13) to yield

$$H_1^{(2)}(j\omega_1)=(1+j\xi_1\omega_1)H_1^{(1)}(j\omega_1) \tag{21}$$

$$H_n^{(2)}(j\omega_1,\ldots,j\omega_n)=-(j\omega_1+\cdots+j\omega_n)^2H_n^{(1)}(j\omega_1,\ldots,j\omega_n)\quad n=2,\ldots,N \tag{22}$$

$$H_1^{(1)}(j\omega_1)=-\frac{1}{L[j\omega_1]} \tag{23}$$

$$H_3^{(1)}(j\omega_1,j\omega_2,j\omega_3)=\xi_2\frac{\prod_{i=1}^{3}H_1^{(1)}(j\omega_i)(j\omega_i)}{L[j(\omega_1+\cdots+\omega_3)]} \tag{24}$$

$$H_{2\bar{n}+1}^{(1)}(j\omega_1,\ldots,j\omega_{2\bar{n}+1})=\xi_2^{\bar{n}}\frac{\prod_{i=1}^{2\bar{n}+1}H_1^{(1)}(j\omega_i)(j\omega_i)}{L[j(\omega_1+\cdots+\omega_{2\bar{n}+1})]}\sum_{Z=1}^{N_{\bar{n}}}\prod_{i=1}^{\bar{n}-1}\frac{(j\omega^Z_{l_i(1)}+\cdots+j\omega^Z_{l_i(j_i^{\bar{n}})})}{L[j\omega^Z_{l_i(1)}+\cdots+j\omega^Z_{l_i(j_i^{\bar{n}})}]}$$
$$\bar{n}=2,\ldots,\lfloor(N-1)/2\rfloor \tag{25}$$

$$H_{2\bar{n}}^{(1)}(j\omega_1,\ldots,j\omega_{2\bar{n}+1})=0\quad \bar{n}=1,2,\ldots,\quad\lfloor(N-1)/2\rfloor \tag{26}$$

where

$$\begin{aligned}&L[j(\omega_1+\cdots+\omega_n)]=-\{(j\omega_1+\cdots+j\omega_n)^2+(j\omega_1+\cdots+j\omega_n)\xi_1+1\}\\ &j_i^{\bar{n}},\quad i=1,\ldots,\bar{n}-1,\ \in\{3,5,\ldots,2\bar{n}-1\}\quad \text{for } \bar{n}\geqslant 2\\ &\omega^Z_{l_i(\bar{j})},\quad i=1,\ldots,\bar{n}-1,\ \bar{j}=1,\ldots,j_i^{\bar{n}},\ \in\{\omega_1,\ldots,\omega_{2\bar{n}+1}\}\quad \text{for } \bar{n}\geqslant 2\end{aligned} \tag{27}$$

$N_{\bar{n}}$ is an $\bar{n}$ dependent integer, and $\lfloor(N-1)/2\rfloor$ is the floor function indicating the largest integer less than or equal to $(N-1)/2$.

From (17) and the expression for $H_n^{(2)}(j\omega_1,\ldots,j\omega_n)$ given by Eqs. (21)–(25), the spectrum $Y_2(j\omega)$ of the second output of system (13) can be written as

$$Y_2(j\omega)=P_0(j\omega)+P_1(j\omega)\xi_2+P_2(j\omega)\xi_2^2+\cdots+P_{\lfloor(N-1)/2\rfloor}(j\omega)\xi_2^{\lfloor(N-1)/2\rfloor} \tag{28}$$

where

$$P_0(j\omega) = H_1^{(2)}(j\omega)\bar{A}(\omega) \tag{29}$$

$$P_{\bar{n}}(j\omega) = \frac{(-1)(j\omega)^2}{2^{2\bar{n}+1}L[j\omega]} \sum_{\omega_1+\cdots+\omega_{2\bar{n}+1}=\omega} \left[\prod_{i=1}^{2\bar{n}+1} H_1^{(1)}(j\omega_i)(j\omega_i)\bar{A}(\omega_i)\right] \sum_{Z=1}^{N_{\bar{n}}} \prod_{i=1}^{\bar{n}-1} \frac{(j\omega_{l_i(1)}^Z + \cdots + j\omega_{l_i(j_i^{\bar{n}})}^Z)}{L[j\omega_{l_i(1)}^Z + \cdots + j\omega_{l_i(j_i^{\bar{n}})}^Z]}$$
$$\bar{n} = 1, \ldots, \lfloor (N-1)/2 \rfloor \tag{30}$$

$$\sum_{Z=1}^{N_1} \prod_{i=1}^{1-1} \frac{(j\omega_{l_i(1)}^Z + \cdots + j\omega_{l_i(j_i^{\bar{n}})}^Z)}{L[j\omega_{l_i(1)}^Z + \cdots + j\omega_{l_i(j_i^{\bar{n}})}^Z]} = 1 \tag{31}$$

Eq. (28) is the OFRF of system (13), which represents the spectrum of the system's second output as an explicit polynomial function of the nonlinear characteristic parameter ξ_2. Obviously this explicit analytical expression for the output spectrum can considerably facilitate the analysis of the effect of the system nonlinearity on the system output frequency response.

By using the OFRF Eq. (28), the transmissibility of the sdof isolator system (2) as given by Eq. (15) can further be expressed as

$$T(\bar{\Omega}) = \left| P_0(j\bar{\Omega}) + \sum_{\bar{n}=1}^{\lfloor (N-1)/2 \rfloor} P_{\bar{n}}(j\bar{\Omega})\xi_2^{\bar{n}} \right| \tag{32}$$

where

$$P_0(j\bar{\Omega}) = -\frac{(1 + j\xi_1\bar{\Omega})}{L(j\bar{\Omega})} \tag{33}$$

$$P_{\bar{n}}(j\bar{\Omega}) = \frac{(\bar{\Omega})^{2\bar{n}+3} H_1^{(1)}(j\bar{\Omega}) \left| H_1^{(1)}(j\bar{\Omega}) \right|^{2\bar{n}}}{2^{2\bar{n}+1}L[j\bar{\Omega}]} \sum_{\omega_1+\cdots+\omega_{2\bar{n}+1}=\bar{\Omega}} \sum_{Z=1}^{N_{\bar{n}}} \prod_{i=1}^{\bar{n}-1} \frac{(j\omega_{l_i(1)}^Z + \cdots + j\omega_{l_i(j_i^{\bar{n}})}^Z)}{L[j\omega_{l_i(1)}^Z + \cdots + j\omega_{l_i(j_i^{\bar{n}})}^Z]}$$
$$= -\frac{(\bar{\Omega})^{2\bar{n}+3}}{2^{2\bar{n}+1}\{L[j\bar{\Omega}]\}^2 |L[j\bar{\Omega}]|^{2\bar{n}}} \sum_{\omega_1+\cdots+\omega_{2\bar{n}+1}=\bar{\Omega}} \sum_{Z=1}^{N_{\bar{n}}} \prod_{i=1}^{\bar{n}-1} \frac{(j\omega_{l_i(1)}^Z + \cdots + j\omega_{l_i(j_i^{\bar{n}})}^Z)}{L[j\omega_{l_i(1)}^Z + \cdots + j\omega_{l_i(j_i^{\bar{n}})}^Z]}$$
$$\bar{n} = 1, 2, \ldots, \lfloor (N-1)/2 \rfloor \tag{34}$$

and $\omega_k \in \{-\bar{\Omega}, \bar{\Omega}\}$, $k = 1, \ldots, 2\bar{n}+1$.

From (32), it is known that when $\xi_2 = 0$, i.e., there is no nonlinear viscous damping,

$$T(\bar{\Omega}) = |P_0(j\bar{\Omega})| = \left| \frac{(1 + j\xi_1\bar{\Omega})}{[1 + j\xi_1\bar{\Omega} + (j\bar{\Omega})^2]} \right| = \sqrt{\frac{1 + (\xi_1\bar{\Omega})^2}{(1 - \bar{\Omega}^2)^2 + (\xi_1\bar{\Omega})^2}} \tag{35}$$

which is the expression of transmissibility widely used in engineering practice for the design of linear sdof vibration isolators.

When nonlinear viscous damping is introduced, i.e., $\xi_2 \neq 0$, Eq. (32) indicates that the transmissibility will be different from the well-known result given by Eq. (35) and, given the linear viscous damping characteristic parameter ξ_1, the difference as described by the second term in Eq. (32) is a function of both the nonlinear viscous damping characteristic parameter ξ_2 and $\bar{\Omega}$. In the next section, $T(\bar{\Omega})$ given by Eq. (32) over the frequency ranges of $\bar{\Omega} \ll 1$ and $\bar{\Omega} \gg 1$, and the effect of ξ_2 on the value of $T(\bar{\Omega})$ over the frequency range of $\bar{\Omega} \approx 1$ will be analyzed theoretically to comprehensively reveal the significant benefits of nonlinear viscous damping on vibration isolation.

4. The effects of nonlinear viscous damping on vibration isolation

Consider the sdof vibration isolators subject to a sinusoidal force excitation as described by Eq. (2). Assume that the outputs of the isolator's dimensionless, one input two output system representation given by Eq. (13) can be described by the nonlinear Volterra series model (19) around zero equilibrium. Then the following proposition regarding the force transmissibility $T(\bar{\Omega})$ of the sdof vibration isolators holds.

Proposition 1. (i) *When* $\bar{\Omega} \ll 1$ *or* $\bar{\Omega} \gg 1$,

$$T(\bar{\Omega}) \approx |P_0(j\bar{\Omega})| = \sqrt{\frac{1+(\xi_1\bar{\Omega})^2}{(1-\bar{\Omega}^2)^2+(\xi_1\bar{\Omega})^2}} \tag{36}$$

(ii) *When* $\bar{\Omega} \approx 1$, *there exists a* $\bar{\xi} > 0$ *such that*

$$\frac{\mathrm{d}[T(\bar{\Omega})]^2}{\mathrm{d}\xi_2} < 0 \tag{37}$$

if $0 < \xi_2 < \bar{\xi}$.

Proof. See Appendix A.

The two conclusions of Proposition 1 reveal considerable beneficial effects of nonlinear viscous damping on vibration isolation, which, as far as we are aware of, have never been realized before. Conclusion (i) indicates that a cubic nonlinear viscous damping characteristic has almost no effect on the transmissibility of sdof vibration isolators over both low and high frequency ranges where the frequencies are much lower or much higher than the isolator's resonant frequency. Conclusion (ii) indicates that an increase in the cubic nonlinear viscous damping effect can reduce the transmissibility over the resonant frequency range. These are very ideal effects for vibration isolation. An effective exploitation of these effects can provide a novel solution to the aforementioned well-known dilemma associated with the design of passive linear viscously damped vibration isolators.

5. Simulation studies and discussion

In order to demonstrate the effects of nonlinear viscous damping on vibration isolation, which have been theoretically analyzed above, numerical simulation studies were conducted for the dimensionless, one input

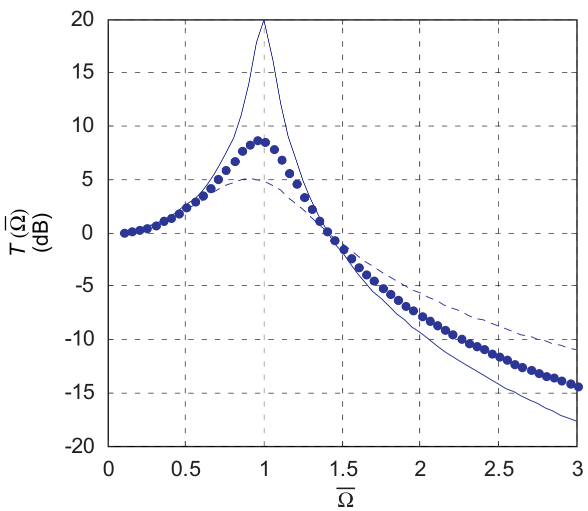

Fig. 2. The force transmissibility of system (2) under linear viscous damping characteristics where $\xi_2 = 0$. Solid: $\xi_1 = 0.1$; dotted: $\xi_1 = 0.4$; dashed: $\xi_1 = 0.7$.

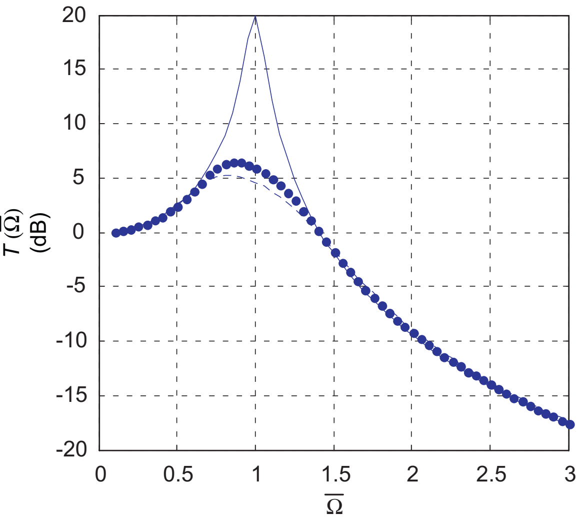

Fig. 3. The force transmissibility of system (2) under nonlinear viscous damping characteristics where $\xi_1 = 0.1$. Solid: $\xi_2 = 0$; dotted: $\xi_2 = 0.2$; dashed: $\xi_2 = 0.4$.

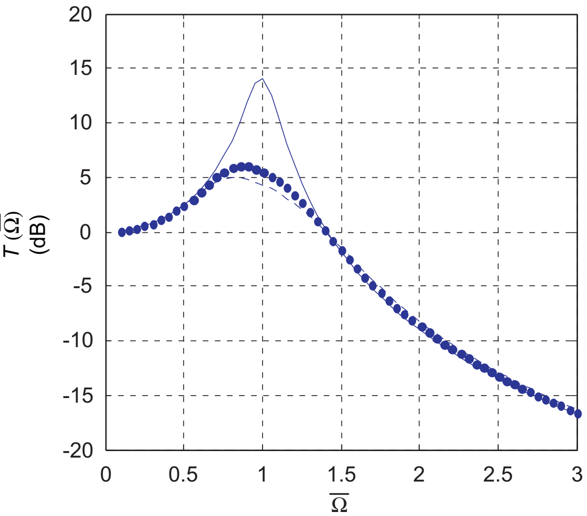

Fig. 4. The force transmissibility of system (2) under nonlinear viscous damping characteristics where $\xi_1 = 0.2$. Solid: $\xi_2 = 0$; dotted: $\xi_2 = 0.2$; dashed: $\xi_2 = 0.4$.

two output system (13) to evaluate how the transmissibility $T(\bar{\Omega})$ changes with the linear and nonlinear viscous damping characteristic parameters ξ_1 and ξ_2. The results are given in Figs. 2–5.

Fig. 2 shows the transmissibility $T(\bar{\Omega})$ in the three linear viscous damping cases of $\xi_1 = 0.1$, 0.4 and 0.7. This is basically a well-known result and indicates that an increase of the linear viscous damping characteristic parameter ξ_1 can reduce $T(\bar{\Omega})$ and consequently suppress the vibration at the resonant frequency where $\bar{\Omega} \approx 1$. However, the increase of ξ_1 is detrimental for vibration isolation over the isolation frequency range ($\bar{\Omega} \gg 1$).

Figs. 3–5 all show the transmissibility $T(\bar{\Omega})$ in the three cubic nonlinear viscous damping cases of $\xi_2 = 0$, 0.2 and 0.4. However, the linear viscous damping characteristic parameter ξ_1 for the results presented in the three figures is different. $\xi_1 = 0.1$, 0.2 and 0.4 are used for the results in Figs. 3–5, respectively to demonstrate the effect of the linear viscous damping characteristic parameter ξ_1 on the analysis result. All the results clearly indicate that the increase of the nonlinear viscous damping characteristic parameter ξ_2 cannot only reduce $T(\bar{\Omega})$ and suppress the vibration at the resonant frequency $\bar{\Omega} \approx 1$, but also keep $T(\bar{\Omega})$ almost unchanged over

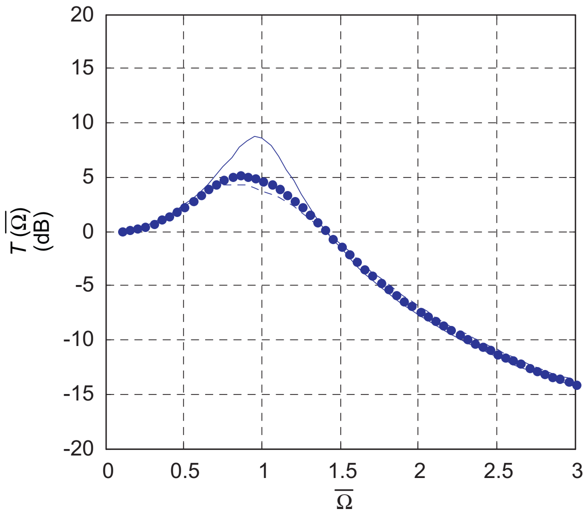

Fig. 5. The force transmissibility of system (2) under nonlinear viscous damping characteristics where $\xi_1 = 0.4$. Solid: $\xi_2 = 0$; dotted: $\xi_2 = 0.2$; dashed: $\xi_2 = 0.4$.

the frequency ranges of $\bar{\Omega} \gg 1$ and $\bar{\Omega} \ll 1$ compared with the case where no nonlinear viscous damping characteristic is introduced. In addition, the beneficial effects of ξ_2 on $T(\bar{\Omega})$ do not change with the change of the linear viscous damping parameter ξ_1. These are exactly the conclusions of Proposition 1.

The effects of nonlinear viscous damping on transmissibility $T(\bar{\Omega})$ as theoretically proved in the last section and numerically demonstrated in Figs. 3–5 can be qualitatively interpreted using Eq. (32) from the perspective of the system resonance and resonant frequency.

Eq. (32) shows that when cubic nonlinear viscous damping is introduced into the sdof system (2), the transmissibility $T(\bar{\Omega})$ is determined by a sum of many terms. The first of these terms is $P_0(j\bar{\Omega})$ as given by Eq. (33), which is the transmissibility $T(\bar{\Omega})$ when $\xi_2 = 0$, i.e., no nonlinear viscous damping is introduced into the system. A general expression for the other terms is given by Eq. (34), and it can be observed from this equation that $|1/L(j\bar{\Omega})|^{2\bar{n}+2}$ basically dominates the effects of these terms on $T(\bar{\Omega})$. According to Eq. (23), $|1/L(j\bar{\Omega})|^{2\bar{n}+2}$ is the magnitude of the first linear FRF of the dimensionless system representation (13) evaluated at frequency $\omega = \bar{\Omega}$ raised to power $(2\bar{n}+2)$. Therefore, over the frequency ranges of $\bar{\Omega} \gg 1$ and $\bar{\Omega} \ll 1$ where the original system (2) works far below or beyond the resonant frequency Ω_0, $|1/L(j\bar{\Omega})| < 1$ and $|1/L(j\bar{\Omega})|^{2\bar{n}+2}$ is considerably less than $|1/L(j\bar{\Omega})|$ for $\bar{n} \geqslant 1$. This implies that apart from the first term $P_0(j\bar{\Omega}) = -(1 + j\xi_1\bar{\Omega})/L(j\bar{\Omega})$, the effect of all the terms in Eq. (32) on the transmissibility $T(\bar{\Omega})$ can be neglected in this case. Consequently $T(\bar{\Omega}) \approx |P_0(j\bar{\Omega})|$, i.e., the first conclusion of Proposition 1 holds.

Over the frequency range of $\bar{\Omega} \approx 1$, the magnitude of the first FRF of system (13) reaches its resonance. In this case, $|1/L(j\bar{\Omega})| > 1$, $|1/L(j\bar{\Omega})|^{2\bar{n}+2}$ is considerably larger than $|1/L(j\bar{\Omega})|$ for $\bar{n} \geqslant 1$, and the transmissibility $T(\bar{\Omega})$ is determined by the contributions of all terms in Eq. (32). Consequently, the cubic nonlinear viscous damping parameter ξ_2 has a significant effect on the transmissibility $T(\bar{\Omega})$. Because $\xi_2 > 0$ is determined by coefficient C_2 of the cubic damping term of the original system, the physical impact of the increase of ξ_2 is that a larger damping force is produced. Therefore, the increase of ξ_2 will reduce the transmissibility $T(\bar{\Omega})$ when $\bar{\Omega} \approx 1$. This is just the second conclusion of Proposition 1.

The beneficial effects of nonlinear viscous damping can also be interpreted from a physical perspective as follows. It is obvious that a cubic viscous damping term actually produces a very small damping force at small velocities and a considerable damping force at large velocities. Clearly, at the system resonance where $\bar{\Omega} \approx 1$, the relative velocity between the mass and supporting base is largest; but over the isolation range where $\bar{\Omega} \gg 1$, the relative velocity is much smaller. Therefore, the effect of a cubic viscous damping term on vibration isolation is significant over the region of resonant frequency $\bar{\Omega} \approx 1$ but is much less and even negligible over the isolation range ($\bar{\Omega} \gg 1$). These are again the conclusion of Proposition 1.

The effects of nonlinear viscous damping on vibration isolation that have been revealed in the present study have considerable significance in engineering practice. In order to demonstrate this significance, consider, for example, the design of vibration isolators made of metal springs. According to Ref. [18], the engineering design procedure for the isolators basically involves:

(1) Find the weight W of the machine to be isolated and the lowest forcing frequency f_L of the machine.
(2) Determine the required degree of isolation I_R or transmissibility $T_R = 1 - I_R$ at frequency f_L. For example, $I_R = 90\%$ implies that 90% of the original vibration at frequency f_L should be suppressed by the isolator, or the transmissibility of the vibrating system at frequency f_L should be $T_R = 1 - I_R = 10\%$.
(3) Assume there is little damping with springs, i.e., $\xi_1 \approx 0$, and determine the system resonant frequency f_0 from Eq. (35) with $T = T_R$, $\bar{\Omega} = \Omega_L/\Omega_0 = 2\pi f_L/2\pi f_0 = f_L/f_0$, i.e., $T_R = 1/(f_L^2/f_0^2 - 1)$ in this case to yield $f_0 = f_L\sqrt{T_R/(T_R+1)}$.
(4) Assume that the spring is linear, determine its stiffness K from $f_0 = \sqrt{K/M}/2\pi = \sqrt{Kg/W}/2\pi$ to yield $K = 4\pi^2 f_0^2 W/g$, and use the result to select a spring from a manufacturer's catalog for the vibration isolator.
(5) Because springs possess very little damping, the transmissibility at the resonance can be very high if a spring designed above is directly used for vibration isolation. So the damping lacking in the designed spring is often obtained in this step by placing a viscous damper in parallel with the spring to reduce the system transmissibility at the resonant frequency to a desired level.

The problem with this traditional design procedure is that the isolation range of the vibrating system where $T \leqslant T_R$ as determined by the designed spring will be impaired by step (5) if a linear viscous damper is used in this step to reduce the system transmissibility at the resonant frequency. This phenomenon can clearly be observed from Fig. 2, if the design steps (1)–(4) literally produce a system with the transmissibility corresponding to $\xi_1 = 0.1$ (this ξ_1 can be, e.g., a linear viscous damping effect inherent with the designed spring), but step (5) increases the system linear viscous damping effect from $\xi_1 = 0.1$ to 0.7. It can be seen from Fig. 2 that if $T_R = -9$ dB, the isolation range as determined by the designed spring is $f/f_0 \geqslant 2$ where $T \leqslant T_R = -9$ dB. However, although step (5) can considerably reduce the transmissibility to about 5 dB at the resonant frequency $f/f_0 = 1$, the actual isolation range where $T \leqslant T_R = -9$ dB is reduced to $f/f_0 \geqslant 2.6$ by the increase of the linear viscous damping effect.

The techniques available for addressing this problem with linear viscous damping include "sky hook" methods and automatic damping switch-off techniques etc., which are almost all based on an active solution. However, the analysis in the present study indicates that the introduction of nonlinear viscous damping can produce a passive solution to the problem so as to considerably save costs and avoid the complexity associated with fully active control. It can be seen from Fig. 3 that if the design steps (1)–(4) are used to produce a system with the transmissibility corresponding to $\xi_1 = 0.1$, but a nonlinear viscous damping term corresponding to $\xi_2 = 0.4$ is then introduced to reduce the system transmissibility to about 5 dB over the frequency range of $f/f_0 \approx 1$, then the actual isolation range of the system where $T \leqslant T_R = -9$ dB is still about $f/f_0 \geqslant 2$, which is the same as in the case where no additional damping effects are introduced. Therefore, the introduction of cubic nonlinear viscous damping can completely overcome the problems associated with the increase of linear viscous damping effects to suppress resonant vibration. This demonstrates that an ideal vibration isolation, which involves *a little damping in the isolation region but a required damping around the resonant frequency,* can be achieved by using a cubic nonlinear viscous damping characteristic.

The implementation of nonlinear viscous damping can be achieved by a proper design of the characteristic of a passive viscous damper. According to our discussions with experts in industry, such designs can be realized by damper manufacturers using available technology. The authors are also currently studying MR damper techniques [19,20] to develop an experimental system to experimentally demonstrate the significant effects of nonlinear viscous damping as theoretically revealed in the present study. The results will be reported in a future publication. In fact, nonlinear viscous damping has already been used in engineering practice. For example, automotive dampers have always been meticulously designed to be nonlinear [21]. Therefore, some practising engineers may have realised the benefits of nonlinear dampers in some specific applications. However, it is in the present study that the significant effects of nonlinear viscous damping on vibration isolation have been clearly pointed out, rigorously proved by theoretical analysis, and comprehensively demonstrated by simulation studies.

6. Conclusions

Vibration isolation is a very important problem for a wide range of engineering practice. The conventional design of viscously damped vibration isolators is often concerned with the determination of the stiffness and damping characteristic parameters in a linear sdof vibration isolator model. A well-known dilemma associated with the design is that although the introduction of a linear viscous damping effect can significantly reduce the transmissibility over the range of the system resonant frequency, the linear damping effect is often detrimental for vibration isolation over the range of the system normal working frequencies. In order to solve this problem, active vibration control solutions often have to be used which may considerably increase the system costs and complexities.

To address the problem with active vibration control devices, a series of studies have been conducted by the authors to investigate the use of nonlinear viscous damping to provide an effective passive solution. The present study is concerned with a further development of these previous results and is mainly focused on a comprehensive theoretical analysis of the beneficial effects of nonlinear viscous damping on vibration isolation. The analysis indicates that a nonlinear viscous damper with a positive cubic damping term has no the detrimental effects in the isolation region but adds considerable damping around the isolator's natural frequency so as to achieve an ideal transmissibility characteristic over the whole frequency range. This implies that a vibration isolator with a nonlinear viscous damping characteristic can also achieve the desired vibration isolation that can be achieved by an active device. Simulation studies demonstrate the validity and engineering significance of these results. The conclusions reached by the present study have significant implications for the engineering design of passive vibration isolators in a wide range of practical applications.

Future publications will focus on the investigation of the effect of more complicated nonlinear viscous damping characteristics on vibration isolation to extend the results achieved in the present study to more general cases.

Finally it is worth mentioning that damping which is not of a viscous type such as damping with rubbers has no the detrimental effects associated with linear viscous dampers [22]. Therefore, it should be emphasized that the results achieved in the present study are significant for the analysis and design of viscously damped vibration isolators. Viscously damped vibration isolators including, for example, shock absorbers for bridges and vehicles have a wide range of applications in engineering practice.

Acknowledgments

The authors gratefully acknowledge that the work was supported by EPSRC (UK), and would like to thank the referees and associate editor for very helpful comments. The authors would also like to thank Dr. L. Zhao for help with the simulation studies.

Appendix A. Proof of Proposition 1

Substituting Eq. (27) into Eq. (34) for $L[.]$ yields

$$\begin{aligned}
|P_{\bar{n}}(j\bar{\Omega})| &= \frac{(\bar{\Omega})^{2\bar{n}+3}}{2^{2\bar{n}+1}\left|(j\bar{\Omega})^2+\xi_1(j\bar{\Omega})+1\right|^{2\bar{n}+2}} \\
&\quad\times\left|\sum_{\omega_1+\cdots+\omega_{2\bar{n}+1}=\bar{\Omega}}\sum_{Z=1}^{N_{\bar{n}}}\prod_{i=1}^{\bar{n}-1}\frac{(j\omega^Z_{l_i(1)}+\cdots+j\omega^Z_{l_i(j_i^{\bar{n}})})}{(j\omega^Z_{l_i(1)}+\cdots+j\omega^Z_{l_i(j_i^{\bar{n}})})^2+\xi_1(j\omega^Z_{l_i(1)}+\cdots+j\omega^Z_{l_i(j_i^{\bar{n}})})+1}\right| \\
&\leqslant \frac{(\bar{\Omega})^{2\bar{n}+3}}{2^{2\bar{n}+1}\left|(j\bar{\Omega})^2+\xi_1(j\bar{\Omega})+1\right|^{2\bar{n}+2}} \\
&\quad\times\sum_{\omega_1+\cdots+\omega_{2\bar{n}+1}=\bar{\Omega}}\sum_{Z=1}^{N_{\bar{n}}}\prod_{i=1}^{\bar{n}-1}\frac{|(j\omega^Z_{l_i(1)}+\cdots+j\omega^Z_{l_i(j_i^{\bar{n}})})|}{|(j\omega^Z_{l_i(1)}+\cdots+j\omega^Z_{l_i(j_i^{\bar{n}})})|^2+\xi_1(j\omega^Z_{l_i(1)}+\cdots+j\omega^Z_{l_i(j_i^{\bar{n}})})+1}
\end{aligned} \tag{A.1}$$

Therefore, when $\bar{\Omega} \ll 1$

$$|P_{\bar{n}}(j\bar{\Omega})| \leqslant \frac{(\bar{\Omega})^{2\bar{n}+3}}{2^{2\bar{n}+1}|(j\bar{\Omega})^2 + \xi_1(j\bar{\Omega}) + 1|^{2\bar{n}+2}} \times \sum_{\omega_1+\cdots+\omega_{2\bar{n}+1}=\bar{\Omega}} \sum_{Z=1}^{N_{\bar{n}}} \prod_{i=1}^{\bar{n}-1} \frac{|(j\omega^Z_{l_i(1)} + \cdots + j\omega^Z_{l_i(j_i^{\bar{n}})})|}{|(j\omega^Z_{l_i(1)} + \cdots + j\omega^Z_{l_i(j_i^{\bar{n}})})^2 + \xi_1(j\omega^Z_{l_i(1)} + \cdots + j\omega^Z_{l_i(j_i^{\bar{n}})}) + 1|}$$

$$\approx \frac{(\bar{\Omega})^{2\bar{n}+3}}{2^{2\bar{n}+1}} \sum_{\omega_1+\cdots+\omega_{2\bar{n}+1}=\bar{\Omega}} \sum_{Z=1}^{N_{\bar{n}}} \prod_{i=1}^{\bar{n}-1} |(j\omega^Z_{l_i(1)} + \cdots + j\omega^Z_{l_i(j_i^{\bar{n}})})| = \frac{(\bar{\Omega})^{3\bar{n}+2}}{2^{2\bar{n}+1}} C1(\bar{n}) \tag{A.2}$$

where

$$C1(\bar{n}) = \sum_{\omega_1+\cdots+\omega_{2\bar{n}+1}=\bar{\Omega}} \sum_{Z=1}^{N_{\bar{n}}} \prod_{i=1}^{\bar{n}-1} \left| \frac{(j\omega^Z_{l_i(1)} + \cdots + j\omega^Z_{l_i(j_i^{\bar{n}})})}{\bar{\Omega}} \right|$$

is a bounded constant which is $\bar{n}$ dependent but independent of $\bar{\Omega}$. So that, when $\bar{\Omega} \ll 1$

$$|P_{\bar{n}}(j\bar{\Omega})| \leqslant \frac{(\bar{\Omega})^{3\bar{n}+2}}{2^{2\bar{n}+1}} C1(\bar{n}) \approx 0 \quad \text{for } \bar{n} = 1, 2, \ldots, \lfloor N/2 - 1 \rfloor \tag{A.3}$$

Consequently, Eq. (36) holds.

When $\bar{\Omega} \gg 1$, it is known from Eq. (38) that

$$|P_{\bar{n}}(j\bar{\Omega})| \leqslant \frac{(\bar{\Omega})^{2\bar{n}+3}}{2^{2\bar{n}+1}|(j\bar{\Omega})^2 + \xi_1(j\bar{\Omega}) + 1|^{2\bar{n}+2}} \times \sum_{\omega_1+\cdots+\omega_{2\bar{n}+1}=\bar{\Omega}} \sum_{Z=1}^{N_{\bar{n}}} \prod_{i=1}^{\bar{n}-1} \frac{|(j\omega^Z_{l_i(1)} + \cdots + j\omega^Z_{l_i(j_i^{\bar{n}})})|}{|(j\omega^Z_{l_i(1)} + \cdots + j\omega^Z_{l_i(j_i^{\bar{n}})})^2 + |\xi_1(j\omega^Z_{l_i(1)} + \cdots + j\omega^Z_{l_i(j_i^{\bar{n}})}) + 1|}$$

$$\approx \frac{1}{2^{2\bar{n}+1}\bar{\Omega}^{2\bar{n}+1}} \sum_{\omega_1+\cdots+\omega_{2\bar{n}+1}=\bar{\Omega}} \sum_{Z=1}^{N_{\bar{n}}} \prod_{i=1}^{\bar{n}-1} \frac{1}{|(j\omega^Z_{l_i(1)} + \cdots + j\omega^Z_{l_i(j_i^{\bar{n}})})|} = \frac{1}{2^{2\bar{n}+1}\bar{\Omega}^{3\bar{n}}} C2(\bar{n}) \tag{A.4}$$

where

$$C2(\bar{n}) = \sum_{\omega_1+\cdots+\omega_{2\bar{n}+1}=\bar{\Omega}} \sum_{Z=1}^{N_{\bar{n}}} \prod_{i=1}^{\bar{n}-1} \left| \frac{\bar{\Omega}}{(j\omega^Z_{l_i(1)} + \cdots + j\omega^Z_{l_i(j_i^{\bar{n}})})} \right|$$

is another bounded constant which is $\bar{n}$ dependent but independent of $\bar{\Omega}$. So that, when $\bar{\Omega} \gg 1$

$$|P_{\bar{n}}(j\bar{\Omega})| \leqslant \frac{1}{2^{2\bar{n}+1}\bar{\Omega}^{3\bar{n}}} C2(\bar{n}) \approx 0 \quad \text{for } \bar{n} = 1, 2, \ldots, \lfloor N/2 - 1 \rfloor \tag{A.5}$$

Consequently, Eq. (36) also holds. Thus conclusion (i) of the proposition is reached.

To prove Eq. (37), express $[T(\bar{\Omega})]^2$ as

$$[T(\bar{\Omega})]^2 = \left[P_0(j\bar{\Omega}) + \sum_{\bar{n}=1}^{\lfloor (N-1)/2 \rfloor} P_{\bar{n}}(j\bar{\Omega})\xi_2^{\bar{n}}\right]\left[P_0(-j\bar{\Omega}) + \sum_{\bar{n}=1}^{\lfloor (N-1)/2 \rfloor} P_{\bar{n}}(-j\bar{\Omega})\xi_2^{\bar{n}}\right] = \sum_{n=0}^{2\lfloor (N-1)/2 \rfloor} \xi_2^n \sum_{q=0}^{n} P_q(j\bar{\Omega})P_{n-q}(-j\bar{\Omega}) \tag{A.6}$$

and evaluate $\mathrm{d}[T(\bar{\Omega})]^2/\mathrm{d}\xi_2$ from Eq. (A.6) to yield

$$\frac{\mathrm{d}[T(\bar{\Omega})]^2}{\mathrm{d}\xi_2} = Re[P_0(j\bar{\Omega})P_1(-j\bar{\Omega})] + \xi_2 \sum_{n=2}^{2\lfloor (N-1)/2 \rfloor} n\xi_2^{n-2} \sum_{q=0}^{n} P_q(j\bar{\Omega})P_{n-q}(-j\bar{\Omega}) \tag{A.7}$$

When $\bar{\Omega} \approx 1$,

$$\frac{d[T(\bar{\Omega})]^2}{d\xi_2} = Re[P_0(j)P_1(-j)] + \xi_2 \sum_{n=2}^{2\lfloor (N-1)/2 \rfloor} n\xi_2^{n-2} \sum_{q=0}^{n} P_q(j)P_{n-q}(-j) \tag{A.8}$$

From Eqs. (33) and (34), it can be obtained that

$$P_0(j) = \frac{1+j\xi_1}{L(j)} = \frac{j-\xi_1}{-(j^2+j\xi_1+1)} = \frac{-\xi_1+j}{-j\xi_1} = \frac{-(1+\xi_1 j)}{\xi_1}$$

$$P_1(-j) = \frac{6}{2^3 L^2(-j)|L(-j)|^2} = \frac{3/4}{[(-j)^2 - j\xi_1 + 1]^2 |(-j)^2 - j\xi_1 + 1|^2} = \frac{3/4}{\xi_1^4}$$

so that

$$Re[P_0(j)P_1(-j)] = -\frac{3}{4\xi_1^5} < 0$$

Therefore, when $\bar{\Omega} \approx 1$,

$$\frac{d[T(\bar{\Omega})]^2}{d\xi_2} = -\frac{3}{4\xi_1^5} + \xi_2 \sum_{n=2}^{2\lfloor (N-1)/2 \rfloor} n\xi_2^{n-2} \sum_{q=0}^{n} P_q(j)P_{n-q}(-j) \tag{A.9}$$

Eq. (A.9) implies that when $\bar{\Omega} \approx 1$, there must exist a $\bar{\xi} > 0$ such that if $0 < \xi_2 < \bar{\xi}$.

$$\frac{d[T(\bar{\Omega})]^2}{d\xi_2} = -\frac{3}{4\xi_1^5} + \xi_2 \sum_{n=2}^{2\lfloor (N-1)/2 \rfloor} n\xi_2^{n-2} \sum_{q=0}^{n} P_q(j)P_{n-q}(-j) < 0 \tag{A.10}$$

that is, conclusion (ii) of the proposition holds. □

References

[1] P. Thureau, D. Lecter, *An Introduction to the Principle of Vibrations of Linear Systems*, Stanley Thornes (Publishers) Ltd., UK, 1981.
[2] S. Daley, J. Hatonen, D.H. Owens, Active vibration isolation in a 'smart spring' mount using a repetitive control approach, *Control Engineering Practice* 14 (2006) 991–997.
[3] D. Hrovat, Survey of advanced suspension developments and related optimal control applications, *Automatica* 33 (1997) 1781–1817.
[4] Z.Q. Lang, S.A. Billings, G.R. Tomlinson, R. Yue, Analytical description of the effects of system nonlinearities on output frequency responses: a case study, *Journal of Sound and Vibration* 295 (2006) 584–601.
[5] Z.Q. Lang, S.A. Billings, R. Yue, J. Li, Output frequency response functions of nonlinear Volterra systems, *Automatica* 143 (2007) 805–816.
[6] Z.K. Peng, Z.Q. Lang, The effects of nonlinearity on the output frequency response of a passive engine mount, *Journal of Sound and Vibration* 318 (2008) 313–328.
[7] X.J. Jing, Z.Q. Lang, S.A. Billings, G.R. Tomlinson, Frequency domain analysis for suppression of output vibration from periodic disturbance using nonlinearities, *Journal of Sound and Vibration* 314 (2008) 536–557.
[8] B. Zhang, S.A. Billings, Z.Q. Lang, G.R. Tomlinson, A novel nonlinear approach to suppress resonant vibrations, *Journal of Sound and Vibration* 317 (2008) 918–936.
[9] G. Schmidt, A. Tondl, *Non-linear Vibrations*, Cambridge University Press, Cambridge, 1986.
[10] A.H. Nayfeh, *Perturbation Methods*, Wiley, New York, 1973.
[11] H.K. Khalil, *Nonlinear Systems*, Prentice Hall, Engelwood Cliffs, NJ, 2002.
[12] Breeka Web Site: 〈http://www.fabreeka.com/tech/isolation_theory.pdf〉.
[13] I.W. Sandberg, A perspective on systems theory, *IEEE Transactions on Circuits and Systems* CAS-31 (1984) 88–103.
[14] S. Boyd, L.O. Chua, Fading memory and the problem of approximating nonlinear operators with Volterra series, *IEEE Transactions on Circuits Systems* CAS-32 (1985) 1150–1161.
[15] Z.Q. Lang, S.A. Billings, Output frequency characteristics of nonlinear systems, *International Journal of Control* 64 (1996) 1049–1067.
[16] D.A. George, *Continuous nonlinear systems*, MIT Research Laboratory Electron, Technical Report 335, 1959.
[17] A.K. Swain, S.A. Billings, Generalized frequency response function matrix for MIMO nonlinear systems, *International Journal of Control* 74 (2001) 829–844.
[18] 〈http://personal.cityu.edu.hk/~bsapplec/control.htm〉.
[19] N.D. Sims, R. Stanway, D.J. Peel, W.A. Bullough, A.R. Johnson, Controllable viscous damping: an experimental study of an electrorheological long-stroke damper under proportional feedback control, *Smart Materials and Structures* 8 (1999) 601–615.

[20] N.D. Sims, R. Stanway, A.R. Johnson, D.J. Peel, W.A. Bullough, Smart fluid damping: shaping the force/velocity response through feedback control, *Journal of Intelligent Material Systems and Structures* 11 (2000) 945–958.
[21] S. Cafferty, J. Giacomin, K. Worden, A nonlinear model for automotive shock absorbers. *IUTAM Symposium on Identification and Mechanical Systems*, Wuppertal, Germany, August 23–27, 1993.
[22] E.E. Ungar, Acoustics from A to Z, *Sound and Vibration*, January 2002 (35th anniversary issue), 8–23.