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The free vibration of symmetrically angle-ply laminated fully clamped skew plates

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Abstract

In this paper, the analysis of the free vibration of thin laminated skew plates with fully clamped edges is investigated. The governing differential equation for skew plate is obtained by transforming the differential equation in Cartesian coordinates into skew coordinates. The natural frequencies of the plate are then calculated by using the finite strip transition matrix method. The numerical results are obtained for different values of skew angles, fiber orientation angles and for different composites laminates. Comparisons have been made with the available results in the literature which show the accuracy and efficiency of the method.

1. Introduction

Skew plates are widely used in many engineering applications such as aircraft wings, marine industries, and as intersection elements in many bridges and highways. In the recent decades, composite structures are used widely in aerospace and other industries due to their high strength-to-weight ratios. In order to have an efficient and reliable design, it is essential to predict the vibration frequencies of such structural elements. Rectangular composite plates have been investigated extensively in the literature on the other hand skew composite plates have received relatively less interest. A useful and extensive survey has been provided by Leiw and Wang [1] on the vibration of isotropic and orthotropic skew plates. In general, the exact solutions of laminated skew plates are somehow complicated or impossible to obtain. Hence the analysis of the vibration of laminated skew plates has been carried out by different numerical techniques. For example Krishnan and Deshpanda [2] used the finite element method to study the free vibration of skew isotropic plates, single layer laminas and three-layered cross-ply laminates. Kapania and Singhvi [3] used Rayleigh–Ritz method to analyze the vibration of tapered thickness, skew laminated plates. The free vibration of fully clamped symmetrically laminated skew plate is studied by Hosokawa et al. [4] using the Green's function approach. Wang [5] has applied the B-spline Rayleigh–Ritz method to investigate the free vibration of thin skew fiber-reinforced

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composite laminates. Also Han and Dickinson [6] applied Ritz approach for analyzing the vibration of symmetrically laminated composite skew plates. The free vibration analysis of skew laminated composite plates with simply supported and clamped edges is obtained by Reddy and Palaninathan [7] using the finite element method. The p-Ritz method is adopted by Wang et al. [8] for analyzing the free vibration of skew sandwich plates with laminated facing. The vibration of skew laminated composite plates with simply supported and clamped edges has been studied by Anlas and Goker [9] using orthogonal polynomials with Ritz method. Recently, Park [10] performed a structural dynamic analysis of skew sandwich plate with laminated composite faces based on the high-order shear deformation plate theory (HSDT). Using a differential quadrature (DQ) method, large amplitude free vibration analysis of laminated composite skew thin plates have been considered by Malekzadeh [11].

In the present paper, the finite strip transition matrix method [12–15], is employed to investigate the free vibration of thin symmetrically skew laminated plate with fully clamped edges. The finite strip transition matrix is a semi-analytical approach which uses the finite strip method (also known as the Kantorovich method) to transform the partial differential equation into a system of coupled fourth-order differential equations. The transition matrix can be obtained for each strip by transforming the prescribed fourth-order differential equations of the system into a system of four first-order ordinary differential equations which can be represented exactly for the strip. The natural frequencies of the plates are obtained iteratively, as the values that cause the transition matrix to be singular after imposing the particular boundary conditions across the plates. The frequency parameters for such plates are obtained for different skew angles, fiber orientation angles and laminated composites. The results are compared with the available results in the literature which showed the accuracy and efficiency of the method.

2. Formulation of the problem and analysis

The governing equation of free flexural vibration of symmetrically laminated plate in rectangular coordinates x, y is given by

$$D_{11}\frac{\partial^4 w}{\partial x^4}+4D_{16}\frac{\partial^4 w}{\partial x^3\partial y}+2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial x^2\partial y^2}+4D_{26}\frac{\partial^4 w}{\partial x\partial y^3}+D_{22}\frac{\partial^4 w}{\partial y^4}-\rho h\omega^2 w=0 \tag{1}$$

where $w(x,y)$ is the flexural deflection, ρ is the density of the plate, ω is the radian frequency, h is the thickness of the plate and $D_{i,j}$ $(i,j = 1,2,6)$ are the flexural rigidities of the plate given by

$$D_{ij}=\int_{-h/2}^{h/2}\bar{Q}_{ij}^{k}z^2\,\mathrm{d}z \quad i,j=1,2,6 \tag{2}$$

where $\bar{Q}_{ij}^{k}$ are the plane stress transformed reduced stiffness coefficients of the lamina in the laminate coordinate system $oxyz$. They are related to the reduced stiffness coefficients of the lamina in the material axes of the lamina Q_{ij}^{k} by proper coordinate relationships that are available in many references e.g. Whitney [16] and can be expressed in terms of the engineering notations as

$$Q_{11}=\frac{E_{11}}{(1-v_{12}v_{21})},\quad Q_{22}=\frac{E_{22}}{(1-v_{21}v_{12})},\quad Q_{12}=\frac{v_{21}E_{11}}{(1-v_{12}v_{21})},\quad Q_{21}=Q_{12},\quad \text{and}\quad Q_{66}=G_{12} \tag{3}$$

where E_{11}, E_{22} are the longitudinal and transverse Young's moduli parallel and perpendicular to the fiber orientation, respectively and G_{12} is the in plane shear modulus of elasticity, v_{12} and v_{21} are the Poisson's ratios in material coordinate system $v_{12}E_2 = v_{21}E_1$. Using the skew coordinate system (u,v) as shown in Fig. 1

$$u = x\sec\phi \quad \text{and} \quad v = y - x\tan\phi \tag{4}$$

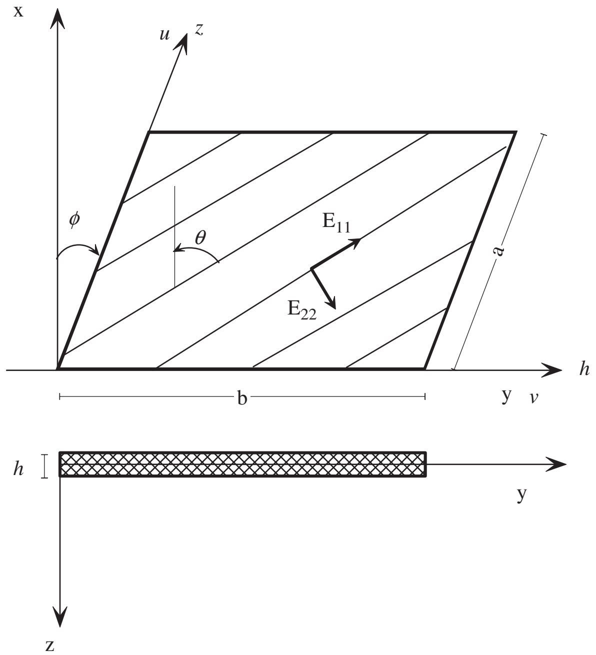

Fig. 1. The skew plate and non-dimensional oblique coordinate system.

where ϕ is the skew angle, then Eq. (1) can be re-written in term of the skew coordinates as

$$\begin{aligned}&D_{11}\sec^4\phi\frac{\partial^4 w}{\partial u^4}+4\sec^3\phi[D_{16}-D_{11}\tan\phi]\frac{\partial^4 w}{\partial u^3\partial v}\\&+2\sec^2\phi[(D_{12}+2D_{66})-6D_{16}\tan\phi+3D_{11}\tan^2\phi]\frac{\partial^4 w}{\partial u^2\partial v^2}\\&+4\sec\phi[D_{26}-(D_{12}+2D_{66})\tan\phi+3D_{16}\tan^2\phi-D_{11}\tan^3\phi]\frac{\partial^4 w}{\partial u\partial v^3}\\&+[D_{22}-4D_{26}\tan\phi+2(D_{12}+2D_{66})\tan^2\phi-4D_{16}\tan^3\phi+D_{11}\tan^4\phi]\frac{\partial^4 w}{\partial v^4}\\&-\rho h\omega^2 w=0\end{aligned}\tag{5}$$

For convenience, the following non-dimensional oblique coordinates are used

$$\zeta=\frac{u}{a}\quad\text{and}\quad\eta=\frac{v}{b}\tag{6}$$

where a, b are the oblique dimensions of the plate. Assume the deflection function be in the form

$$w(\zeta,\eta,t)=\sum_{m=1}^{R}U_m(\zeta)V_m(\eta)\,\mathrm{e}^{\mathrm{i}\omega t}\tag{7}$$

where ω is the natural frequency of the skew plate, $U_m(\zeta)$ the basic function along ζ, $V_m(\zeta)$ the unknown function along the η direction, and R the harmonic number of the truncated series. The most commonly used basic function $U_m(\zeta)$ is the eigen function derived from the solution of the differential equation of a beam vibration under the prescribed boundary conditions of the plate at $\zeta=0$,1. Substituting Eq. (7) into Eq. (6) after normalization, multiplying both sides by $U_n(\zeta)\,\mathrm{d}\zeta$, integrating them from 0 to 1, the equation of motion may be then reduced to

$$\sum_{m=1}^{R}\left[a_{nm}A_{ps}\frac{d^4V_m}{\partial\eta^4}+b_{nm}B_{ps}\frac{d^3V_m}{\partial\eta^3}+(c_{nm}C_{ps})\frac{d^2V_m}{\partial\eta^2}+(d_{nm}D_{ps})\frac{dV_m}{\partial\eta}+e_{nm}(E_{ps}-\lambda^2)V_m\right]=0$$
$$n=1,2,3,\ldots,R \tag{8}$$

where the frequency parameter $\lambda = a^2\omega\sqrt{\rho h/D_{22}}$ and all other constants are defined in Appendix A.

The boundary conditions for clamped skew plate are

$$\frac{\partial w}{\partial n}=\sin\,\phi\frac{\partial w}{\partial x}-\cos\,\phi\frac{\partial w}{\partial y}=0 \tag{9}$$

and since

$$w=0 \tag{10}$$

Using Eq. (7), one can find that the boundary conditions at $\eta = 0$, $\eta = 1$ are hence

$$V(\eta)=0 \quad \text{and} \quad \frac{dV}{d\eta}=0 \tag{11}$$

The orthogonality conditions yield $a_{nm} = e_{nm} = 0$ for $m \neq n$. The fourth-order differential Eqs. (8) are transformed into a $4R$-number of first-order ordinary differential equations, and the following relation will be then introduced at any nodal line j of the divided plate:

$$\frac{d}{d\eta}\{v\}_j=[A]_j\{v\}_j;\quad j=1,2,3,\ldots,N \tag{12}$$

where

$$\{v\}_j=\{\mathbf{v}_1\ \ \mathbf{v}_2\ \ \cdots\ \ \mathbf{v}_k\ \ \cdots\ \ \mathbf{v}_m\}_j^{\mathrm{T}} \tag{13}$$

and

$$\mathbf{v}_k=\left\{V_k\frac{dV_k}{d\eta}\frac{d^2V_k}{d\eta^2}\frac{d^3V_k}{d\eta^3}\right\} \tag{14}$$

The solution of the system of first-order differential equations can be thus carried out using the finite strip transition matrix technique [12–16].

3. Results and discussion

The natural frequencies of the free vibration of skew laminated plates with fully clamped edges plates are calculated. First, a convergence analysis is carried out and the results are compared with the available results in the literature. A five-layer symmetric angle-ply [45°/−45°/45°/−45°/45°] skew laminates with different skew angles ($\phi = 0°, 30°, 45°$) are considered for this convergence analysis. The material properties of each layer are given by $E_{11}/E_{22} = 40$, $G_{12}/E_{22} = 0.6$, $\upsilon_{12} = 0.25$. Also, it is assumed that the laminates have a rhombic geometry ($a = b$). Table 1 shows the results of the natural frequencies $\Omega = \lambda/\pi^2\sqrt{D_{22}/E_{22}h^3}$ of the first six modes of vibration with skew angles $\phi = 0°, 30°, 45°$ for different values of R varying from 2 to 7. These presented results are demonstrated and compared with Wang [5] and Anlas and Goker [9]. It can be observed that the finite strip transition matrix method presents good results and fast convergence. Then, three layer (60°/−60°/60°) E-glass/epoxy and graphite/epoxy laminates are considered in Tables 2 and 3, respectively for different skew angles 0°, 15°, 45°, and 60°. The material properties of the graphite/epoxy are $E_{11} = 138.0$ GPa, $E_{22} = 8.96$ GPa, $G_{12} = 7.1$ GPa, $\upsilon_{12} = 0.3$ and that of the E-glass/epoxy are $E_{11} = 60.7$ GPa, $E_{22} = 24.8$ GPa, $G_{12} = 11.99$ GPa, $\upsilon_{12} = 0.23$. The results are calculated using 6 and 7 terms to demonstrate the convergence of the solution and compared with Anlas and Goker [9] which are in very good agreement of Han and Dickinson [6] and Hosokawa et al. [4]. It is important to point out that the date in Ref. [9] is presented for

Table 1
Convergence and comparison of the first five frequency parameters for skew laminates with five layers.

R	ϕ (°)	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
2	0	3.929	7.259	8.806	12.069	14.818	18.597
3		3.925	7.178	8.558	11.301	13.678	15.132
4		3.907	7.164	8.502	11.270	13.376	14.896
5		3.906	7.155	8.485	11.233	13.367	14.821
6		3.903	7.152	8.472	11.230	13.338	14.792
7		3.902	7.160	8.409	11.354	13.304	13.320
Wang [5]		3.901	7.146	8.458	11.21	13.322	14.742
2	30	4.557	8.424	9.998	13.177	16.180	19.796
3		4.551	8.422	9.896	12.920	15.776	17.584
4		4.546	8.393	9.891	12.887	15.740	17.501
5		4.545	8.391	9.884	12.874	15.715	17.498
6		4.544	8.385	9.883	12.863	15.704	17.492
7		4.544	8.385	9.882	12.861	15.698	17.491
Wang [5]		4.543	8.382	9.881	12.853	15.691	17.489
2	45	6.411	11.381	15.269	18.192	24.837	28.367
3		6.334	10.868	14.673	15.838	22.678	22.854
4		6.318	10.857	14.560	15.553	21.218	22.154
5		6.310	10.831	14.528	15.518	21.193	22.130
6		6.308	10.838	14.511	15.490	21.093	22.088
7		6.306	10.823	14.504	15.482	21.091	22.082
Wang [5]		6.305	10.819	14.495	15.470	21.062	22.076

(45°/−45°/45°/−45°/45°) $\Omega = \lambda/\pi^2\sqrt{D_{22}/E_{22}h^3}$.

Table 2
Convergence and comparison of the first five frequency parameters for skew laminates with three layers.

R	ϕ (°)	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
6	0	1.320	2.460	2.902	3.881	4.405	5.284
7	0	1.320	2.459	2.901	3.879	4.405	5.283
Ref. [9]	0	1.32	2.46	2.90	3.88	4.40	5.28
6	15	1.342	2.619	2.834	4.071	4.635	5.105
7	15	1.342	2.619	2.834	4.070	4.634	5.103
6	30	1.554	2.890	3.410	4.378	5.566	5.987
7	30	1.553	2.890	3.409	4.376	5.565	5.984
Ref. [9]	30	1.55	2.89	3.41	4.37	5.56	5.98
6	45	2.145	3.641	5.003	5.259	7.184	7.544
7	45	2.144	3.640	4.997	5.258	7.168	7.535
6	60	3.912	5.919	7.981	9.637	10.345	12.671
7	60	3.905	5.910	7.945	9.609	10.300	12.657

(60°/−60°/60°) $\Omega = \lambda/\pi^2\sqrt{D_{22}/E_{22}h^3}$.

30°/−30°/30° laminates and skew angle is 90° which is equivalent in our presentation to 60°/−60°/60° with a skew angle of 0° and similarly for Skew angle 30° and 60°/−60°/60° which is equivalent to skew angle 60 and 30°/−30°/30°.

It is observed that that the numerical accuracy and stability of the present method are insensitive to stacking sequence of laminates, the degree of orthotropy of each layer, and the skew angles, even for such high skew angle as 60°, however the matrix becomes ill-conditioned when the number of terms used becomes higher, especially for strong orthotropic material.

Table 3
Convergence and comparison of the first five frequency parameters for skew graphite-epoxy laminates with three layers.

R	ϕ (°)	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
6	0	2.471	3.780	5.714	6.044	7.601	8.053
7	0	2.470	3.777	5.713	6.039	7.584	8.029
Ref. [9]	0	2.47	3.77	5.70	6.03	7.56	8.03
6	15	2.273	3.927	5.207	6.206	7.303	8.882
7	15	2.273	3.925	5.206	6.204	7.295	8.880
6	30	2.374	4.651	4.949	7.352	8.155	8.786
7	30	2.374	4.651	4.949	7.352	8.155	8.786
Ref. [9]	30	2.37	4.65	4.95	7.35	8.15	8.76
6	45	3.017	5.581	6.611	8.606	10.440	11.759
7	45	3.017	5.581	6.609	8.603	10.437	11.758
5	60	5.232	8.410	11.965	12.483	16.096	17.561
7	60	5.230	8.405	11.963	12.472	16.065	17.544

(60°/−60°/60°) $\Omega = \lambda/\pi^2\sqrt{D_{22}/E_{22}h^3}$.

4. Conclusions

The finite strip method is used to study the free vibration of skew laminated plates with fully clamped edges. The accuracy and the efficiency of the method are examined by comparing the calculated results with the available results in the literature for five symmetric laminates angle-ply layers problem. Then the effect of the skew angle on the natural frequencies is investigated for three and five-layer symmetric angle-ply laminates problems with different materials. As a result, we can conclude that the Finite Strip Transition Matrix can provide efficient, accurate and fast convergence for the skew laminates.

Appendix A

$$
\begin{aligned}
a_{nm} &= \int_0^1 U_n U_m \, \mathrm{d}\zeta \\
b_{nm} &= \int_0^1 U_n \frac{\mathrm{d}U_m}{\mathrm{d}\zeta} \, \mathrm{d}\zeta \\
c_{nm} &= \int_0^1 U_n \frac{\mathrm{d}^2 U_m}{\mathrm{d}\zeta^2} \, \mathrm{d}\zeta \\
d_{nm} &= \int_0^1 U_n \frac{\mathrm{d}^3 U_m}{\mathrm{d}\zeta^3} \, \mathrm{d}\zeta \\
e_{nm} &= \int_0^1 U_n \frac{\mathrm{d}^4 U_m}{\mathrm{d}\zeta^4} \, \mathrm{d}\zeta \\
E_{ps} &= \psi_1 \sec^4 \phi \\
D_{ps} &= 4\beta \sec^3 \phi[\psi_3 - \psi_1 \tan \phi] \\
C_{ps} &= 2\beta^2 \sec^2 \phi[\psi_2 - 6\psi_3 \tan \phi + 3\psi_1 \tan^2 \phi] \\
B_{ps} &= 4\beta^3 \sec \phi[\psi_4 - \psi_2 \tan \phi + 3\psi_3 \tan^2 \phi - \psi_1 \tan^3 \phi] \\
A_{ps} &= \beta^4[1 - 4\psi_4 \tan \phi + 2\psi_2 \tan^2 \phi - 4\psi_3 \tan^3 \phi + \psi_1 \tan^4 \phi]
\end{aligned}
$$

where

$$
\psi_1 = \frac{D_{11}}{D_{22}}, \quad \psi_2 = \frac{D_{12} + 2D_{66}}{D_{22}}, \quad \psi_3 = \frac{D_{16}}{D_{22}}, \quad \psi_4 = \frac{D_{26}}{D_{22}}, \quad \psi_5 = \frac{D_{12}}{D_{22}}, \quad \text{and} \quad \beta = a/b
$$

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