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# Stabilization of statically unstable systems by parametric excitation

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**Abstract**

A linear vibrational system with multiple degrees of freedom subjected to parametric excitation is considered. It is assumed that the system is statically unstable but close to a critical point, the excitation amplitude and damping are small, and the excitation frequency is arbitrary. A new stabilization condition is derived in terms of integrals depending on eigenfrequencies and modes of the undisturbed conservative system and the symmetric excitation matrix. As a special case, an approximation for high-frequency excitation is deduced from this condition. Influence of damping on stabilization region is shown to be very small. Two examples for systems with one and two degrees of freedom are presented. It is shown that stabilization of statically unstable systems is possible for low, medium and high excitation frequencies.


## 1. Introduction

Perhaps, Stephenson [1] was the first who showed 100 years ago that a pendulum in its inverted, statically unstable position can be stabilized by suitably high-frequency excitation of the pivot. He also confirmed his theoretical prediction by a practical demonstration of the phenomenon. Then, in his subsequent, less known paper Stephenson [2] showed that an inverted double and even triple pendulum can be stabilized in the same way.

Among other works on stabilization of statically unstable systems we should mention the contributions by Kapitza [3,4], Chelomei [5,6], Bogoliubov and Mitropol'sky [7], Acheson [8], Acheson and Mullin [9], Chelomei [10], Blekhman [11], Champneys and Fraser [12], Thomsen [13], Seyranian and Seyranian [14], Yabuno and Tsumoto [15] and many others. All these works deal with high-frequency stabilization problems. However, in the recent paper by Seyranian and Seyranian [16] it is shown that a statically unstable elastic beam, compressed by an axial periodic force, can be stabilized in its horizontal position by the excitation frequencies of the order of the main frequency of transverse vibrations of the beam.

Formally, the theory of high-frequency excitation is based on the assumption that the excitation frequency $\Omega$ is much higher than all eigenfrequencies of the system. Thus, $1/\Omega$ is considered as a small parameter. In our

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paper, we do not impose restrictions on the excitation frequency. It is quite natural and necessary, for example, for stabilization problems of infinite degree-of-freedom systems, having arbitrarily high eigenfrequencies corresponding to higher modes.

In this paper we raise a new question: Can a statically unstable system be stabilized by medium- or low-frequency excitation? In our study, we focus on systems, which are weakly unstable. Such systems depend on a parameter $p$, taken close to a critical stability value $p_0$. In other words, we study stabilization by periodic excitation for arbitrary $\Omega$ assuming that $\Delta p = p - p_0$ is a small parameter (on the contrary to the high-frequency approach when $\Delta p$ is arbitrary and $1/\Omega$ is small). Excitation amplitude $\delta$ is also assumed to be small.

Our approach is based on the analysis of bifurcations of multiple multipliers of a periodic system [17] with respect to small parameters $\Delta p$ and $\delta$ (a double multiplier with a Jordan block that appears at $p_0$ and $\delta = 0$). This approach is used in Section 2 to derive a stabilization condition for a general linear finite-degrees-of-freedom vibrational system with arbitrary excitation frequency $\Omega$. In Section 3 we rewrite this condition in terms of eigenfrequencies and modes of the undisturbed conservative system, determining explicitly the stabilizing or destabilizing effect of each mode. The effect of dissipative forces is discussed in Section 4. We show that this effect is typically very small. In Section 5, from the obtained formulae we derive approximation for high-frequency excitation. In Section 6 we present simple examples of systems with one and two degrees of freedom. In the second example it is shown that, away from parametric resonance regions, stabilization by parametric excitation is possible for the whole range of excitation frequencies: low, medium and high compared to the second eigenfrequency of the system. Some complicated derivations are collected in Appendices A and B.

## 2. Influence of periodic excitation on the stability bound

Consider a linear vibrational system of the form

$$\mathbf{M}\ddot{\mathbf{q}} + (\mathbf{C}(p) + \delta\mathbf{B}(t))\mathbf{q} = 0, \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{B}$ are real symmetric $n \times n$ matrices, $p$ and $\delta$ are real constant parameters, and the dots denote derivatives with respect to time $t$. The matrix $\mathbf{M}$ is positive definite, and the matrix $\mathbf{B}(t) = \mathbf{B}(t + T)$ is time-periodic with period $T = 2\pi/\Omega$ and frequency $\Omega$. Taking $\delta = 0$, we obtain the autonomous conservative system

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\mathbf{q} = 0. \tag{2}$$

The trivial solution $\mathbf{q} \equiv 0$ is stable if the matrix $\mathbf{C}(p)$ is positive definite. In this case the eigenfrequencies $0 < \omega_1 \leqslant \cdots \leqslant \omega_n$ and corresponding eigenmodes $\mathbf{w}_k$ for system (2) satisfy the following equations and orthonormality conditions

$$\mathbf{C}\mathbf{w}_k = \omega_k^2\mathbf{M}\mathbf{w}_k, \quad \mathbf{w}_k^{\mathrm{T}}\mathbf{M}\mathbf{w}_{k'} = \delta_k^{k'}, \tag{3}$$

where $\delta_k^{k'}$ is the Kronecker delta.

Eqs. (1)–(3) remain unchanged under the transformation (changing the time scale)

$$t = T\tilde{t}, \quad \mathbf{C} = \tilde{\mathbf{C}}/T^2, \quad \mathbf{B}(t) = \tilde{\mathbf{B}}(\tilde{t})/T^2, \quad \omega_k = \tilde{\omega}_k/T, \tag{4}$$

where the matrix $\tilde{\mathbf{B}}(\tilde{t}) = \tilde{\mathbf{B}}(\tilde{t} + \tilde{T})$ has period $\tilde{T} = 1$. So, below we assume that $T = 1$, omitting the tildes.

Let $p = p_0$ be the critical value, such that system (2) is stable at $p < p_0$ and unstable at $p > p_0$. Then the matrix $\mathbf{C}_0 = \mathbf{C}(p_0)$ is singular, i.e., has zero eigenvalue. We consider a generic case, when there is a single critical mode described by the eigenvector $\mathbf{w}_1$:

$$\mathbf{C}_0\mathbf{w}_1 = 0, \tag{5}$$

i.e., $\omega_1 = 0$ and $0 < \omega_2 \leqslant \cdots \leqslant \omega_n$. Since the system is stable for $p < p_0$, the matrix $\mathbf{C}(p) \approx \mathbf{C}_0 + (\mathrm{d}\mathbf{C}/\mathrm{d}p)\Delta p$ is positive definite for small negative $\Delta p = p - p_0$. In particular, $\mathbf{w}_1^{\mathrm{T}}\mathbf{C}\mathbf{w}_1 > 0$, which via Eq. (5) leads to the condition

$$\mathbf{w}_1^{\mathrm{T}}\mathbf{C}_1\mathbf{w}_1 < 0, \quad \mathbf{C}_1 = \mathrm{d}\mathbf{C}/\mathrm{d}p\big|_{p=p_0}. \tag{6}$$

System (1) can be written as

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} \tag{7}$$

with

$$\mathbf{x} = \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{pmatrix}, \quad \mathbf{A}(t) = \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{M}^{-1}(\mathbf{C}(p) + \delta\mathbf{B}(t)) & 0 \end{pmatrix}, \tag{8}$$

where $\mathbf{I}$ and 0 are the identity and zero $n \times n$ matrices, respectively. The matrix $\mathbf{A}(t) = \mathbf{A}(t+1)$ is periodic and depends on constant parameters $p$ and $\delta$. At $\delta = p - p_0 = 0$, the matrix

$$\mathbf{A}_0 = \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{C}_0 & 0 \end{pmatrix} \tag{9}$$

is time independent. It has a double zero eigenvalue with second-order Jordan block structure:

$$\mathbf{A}_0\mathbf{u}_0 = 0, \quad \mathbf{A}_0\mathbf{u}_1 = \mathbf{u}_0 \tag{10}$$

with the eigenvector and associated vector (generalized eigenvectors)

$$\mathbf{u}_0 = \begin{pmatrix} \mathbf{w}_1 \\ 0 \end{pmatrix}, \quad \mathbf{u}_1 = \begin{pmatrix} 0 \\ \mathbf{w}_1 \end{pmatrix}. \tag{11}$$

We also define the left eigenvectors

$$\mathbf{v}_0 = \begin{pmatrix} 0 \\ \mathbf{M}\mathbf{w}_1 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} \mathbf{M}\mathbf{w}_1 \\ 0 \end{pmatrix} \tag{12}$$

that satisfy the equations and orthonormality conditions

$$\mathbf{v}_0^{\mathrm{T}}\mathbf{A}_0 = 0, \quad \mathbf{v}_1^{\mathrm{T}}\mathbf{A}_0 = \mathbf{v}_0^{\mathrm{T}}, \quad \mathbf{v}_0^{\mathrm{T}}\mathbf{u}_0 = \mathbf{v}_1^{\mathrm{T}}\mathbf{u}_1 = 0, \quad \mathbf{v}_1^{\mathrm{T}}\mathbf{u}_0 = \mathbf{v}_0^{\mathrm{T}}\mathbf{u}_1 = 1. \tag{13}$$

Let us introduce a fundamental matrix $\mathbf{X}(t)$ for system (7) satisfying the equation with the initial condition

$$\dot{\mathbf{X}} = \mathbf{A}(t)\mathbf{X}, \quad \mathbf{X}(0) = \mathbf{I}. \tag{14}$$

This matrix gives a solution $\mathbf{x}(t) = \mathbf{X}(t)\mathbf{x}_0$ of system (1) with an arbitrary initial condition $\mathbf{x}(0) = \mathbf{x}_0$. The Floquet (monodromy) matrix is defined as

$$\mathbf{F} = \mathbf{X}(1). \tag{15}$$

The eigenvalues $\rho$ of the Floquet matrix are called multipliers. Multipliers of system (1) possess the symmetry: if $\rho$ is the multiplier, then $1/\rho$ is also a multiplier [18]. The trivial solution $\mathbf{q} \equiv 0$ is stable if and only if all the multipliers lie on the unit circle $|\rho| = 1$ and do not form Jordan blocks.

For $\delta = 0$ and $p = p_0$, system (14) becomes $\dot{\mathbf{X}} = \mathbf{A}_0\mathbf{X}$. Since $\mathbf{A}_0$ is time independent, we have

$$\mathbf{X}_0(t) = \exp(t\mathbf{A}_0), \quad \mathbf{F}_0 = \exp(\mathbf{A}_0). \tag{16}$$

It is easy to see that

$$\mathbf{X}_0\mathbf{u}_0 = \mathbf{u}_0, \quad \mathbf{X}_0\mathbf{u}_1 = \mathbf{u}_1 + t\mathbf{u}_0, \quad \mathbf{v}_0^{\mathrm{T}}\mathbf{X}_0 = \mathbf{v}_0^{\mathrm{T}}, \quad \mathbf{v}_1^{\mathrm{T}}\mathbf{X}_0 = \mathbf{v}_1^{\mathrm{T}} + t\mathbf{v}_0^{\mathrm{T}}, \tag{17}$$

and similar expressions for $\mathbf{F}_0$ with $t = 1$. This means that $\mathbf{F}_0$ has a double multiplier $\rho_0 = 1$ with the right and left Jordan chains $\mathbf{u}_0$, $\mathbf{u}_1$ and $\mathbf{v}_0$, $\mathbf{v}_1$.

The other multipliers $\rho = \exp(\pm \mathrm{i}\omega_k)$, $k = 2, \ldots, n$, are assumed to be simple and complex (the system is not at the resonance). This means that

$$\omega_k \pm \omega_{k'} \neq 2\pi j \tag{18}$$

for any positive integers $k$, $k'$, $j$. For arbitrary period $T = 2\pi/\Omega$, this condition takes the form $\Omega \neq (\omega_k \pm \omega_{k'})/j$. Stability analysis in the resonance cases $\omega_k \pm \omega_{k'} \approx 2\pi j$ with $k, k' > 1$ can be carried out using methods of parametric resonance theory, see e.g., Ref. [17]. The resonances with $k' = 1$, so that

$\omega_k \approx 2\pi j$, are degenerate: at the resonance point, the multiplicity of the multiplier $\rho = 1$ increases to 4. These cases require special study and are not considered in this paper. Below in this section we assume that the system is not close to resonances.

For small perturbations of parameters $p$ and $\delta$, simple multipliers $\rho = \exp(\pm i\omega_k)$, $k = 2, \ldots, n$, remain on the unit circle $|\rho| = 1$. Hence, for stability, we have to study only the perturbation of the double multiplier $\rho = 1$.

Perturbation of this double multiplier is described by the asymptotic formula [17, pp. 37, 38]

$$\rho = 1 \pm \sqrt{g_p \Delta p + g_\delta \delta}, \quad g_\alpha = \mathbf{v}_0^{\mathrm{T}} \frac{\partial \mathbf{F}}{\partial \alpha} \mathbf{u}_0, \ \ \alpha \in \{p, \delta\}, \tag{19}$$

where derivatives are taken at $p = p_0$, $\delta = 0$. Using Eq. (19) in the equality $|\rho| = 1$ gives the first-order stability condition as

$$g_p \Delta p + g_\delta \delta < 0. \tag{20}$$

For derivatives of the Floquet matrix one has the formula [17, p. 280]

$$\frac{\partial \mathbf{F}}{\partial \alpha} = \mathbf{F}_0 \int_0^1 \mathbf{H}_\alpha(t)\, \mathrm{d}t, \quad \mathbf{H}_\alpha(t) = \mathbf{X}_0^{-1}(t) \frac{\partial \mathbf{A}}{\partial \alpha} \mathbf{X}_0(t). \tag{21}$$

Using Eqs. (8), (11), (12), (17), (21) in Eq. (19), we find

$$g_p = -\mathbf{w}_1^{\mathrm{T}} \mathbf{C}_1 \mathbf{w}_1, \quad g_\delta = -\mathbf{w}_1^{\mathrm{T}} \overline{\mathbf{B}} \mathbf{w}_1, \quad \overline{\mathbf{B}} = \int_0^1 \mathbf{B}(t)\, \mathrm{d}t. \tag{22}$$

Thus, condition (20) takes the form

$$\mathbf{w}_1^{\mathrm{T}} \mathbf{C}_1 \mathbf{w}_1 (p - p_0) + \mathbf{w}_1^{\mathrm{T}} \overline{\mathbf{B}} \mathbf{w}_1 \delta > 0. \tag{23}$$

Recall that $\mathbf{w}_1^{\mathrm{T}} \mathbf{C}_1 \mathbf{w}_1 < 0$.

Below let us consider the case $\overline{\mathbf{B}} = 0$. Then the first approximation (23) yields $p < p_0$. Thus, the stabilization effect is described by the second-order approximation. The general form of the stabilization condition becomes

$$p < p_0 + a\delta^2/2 + o(\delta^2) \tag{24}$$

with an unknown coefficient $a$. The critical value of $p$ (stability boundary) is

$$p_{cr} = p_0 + a\delta^2/2 + o(\delta^2). \tag{25}$$

Consider a perturbation along the stability boundary $p = p_0 + a\delta^2/2 + o(\delta^2)$. Since $\Delta p \sim \delta^2$ and $g_\delta = 0$, the square root term in Eq. (19) is proportional to $\delta$ and, hence, exceeds the order of the expression $o(\delta^{1/2})$. In this degenerate case, the asymptotic expression for $\rho$ starts with the first power of $\delta$ as [17]

$$\rho = 1 + \mu\delta + o(\delta), \tag{26}$$

where two different values of $\mu$ are determined from the quadratic equation

$$\mu^2 + \alpha_1 \mu + \alpha_2 = 0. \tag{27}$$

The coefficients $\alpha_1$ and $\alpha_2$ are [17, p. 39]

$$\alpha_1 = -\mathbf{v}_0^{\mathrm{T}} \frac{\partial \mathbf{F}}{\partial \delta} \mathbf{u}_1 - \mathbf{v}_1^{\mathrm{T}} \frac{\partial \mathbf{F}}{\partial \delta} \mathbf{u}_0,$$

$$\alpha_2 = \mathbf{v}_0^{\mathrm{T}} \left( \frac{\partial \mathbf{F}}{\partial \delta} \mathbf{G}^{-1} \frac{\partial \mathbf{F}}{\partial \delta} - \frac{1}{2} \frac{\partial^2 \mathbf{F}}{\partial \delta^2} \right) \mathbf{u}_0 - \frac{1}{2} \mathbf{v}_0^{\mathrm{T}} \frac{\partial \mathbf{F}}{\partial p} \mathbf{u}_0 \frac{\mathrm{d}^2 p}{\mathrm{d}\delta^2},$$

$$\mathbf{G} = \mathbf{F}_0 - \mathbf{I} + \mathbf{u}_1 \mathbf{v}_1^{\mathrm{T}}. \tag{28}$$

Note that we took the term $\mathbf{u}_1 \mathbf{v}_1^{\mathrm{T}}$ in the matrix $\mathbf{G}$ instead of $\mathbf{v}_0 \mathbf{v}_1^{\mathrm{T}}$ suggested in Ref. [17]; in fact, one can show that using any diadic product $\mathbf{z}\mathbf{v}_1^{\mathrm{T}}$ with the vector $\mathbf{z}$, not proportional to $\mathbf{u}_0$, gives the same value of $\alpha_2$.

Using Eqs. (8), (11), (12), (17), (21) in Eq. (28), we find $\alpha_1 = \mathbf{w}_0^T \overline{\mathbf{B}} \mathbf{w}_0 = 0$ due to $\overline{\mathbf{B}} = 0$. Then $\mu = \pm\sqrt{-\alpha_2}$. The stability condition $|\rho| = 1$ with the expansion (26) yields $\alpha_2 > 0$; equation $\alpha_2 = 0$ is the critical (stability boundary) condition. The last term for $\alpha_2$ in Eq. (28) is found in Eqs. (19), (22) with $\mathrm{d}^2 p/\mathrm{d}\delta^2 = a$ according to Eq. (25). Using Eq. (28) in the condition $\alpha_2 = 0$ yields the unknown constant

$$a = 2\mathbf{v}_0^T \left( \frac{1}{2} \frac{\partial^2 \mathbf{F}}{\partial \delta^2} - \frac{\partial \mathbf{F}}{\partial \delta} \mathbf{G}^{-1} \frac{\partial \mathbf{F}}{\partial \delta} \right) \mathbf{u}_0 \Big/ (\mathbf{w}_1^T \mathbf{C}_1 \mathbf{w}_1). \tag{29}$$

The second derivative of the Floquet matrix can be found as [17, p. 281]

$$\frac{\partial^2 \mathbf{F}}{\partial \delta^2} = 2\mathbf{F}_0 \int_0^1 \int_0^t \mathbf{H}_\delta(t) \mathbf{H}_\delta(\tau) \, \mathrm{d}\tau \, \mathrm{d}t \tag{30}$$

with $\mathbf{H}_\delta(t)$ defined in Eq. (21).

Expression (25) with the coefficient $a$ given by Eqs. (29), (21), (30) provides the critical load of the system under parametric excitation. These expressions give $a$ explicitly in the form of integrals in terms of the critical mode $\mathbf{w}_1$ and the matrices $\mathbf{M}$, $\mathbf{C}_0$, $\mathbf{B}(t)$ according to Eqs. (8), (9), (16). Stabilization of a statically unstable system with $p > p_0$ is possible only if $a > 0$. Then, according to Eq. (24), the system is stabilized by the excitation with amplitudes $\delta > \sqrt{2\Delta p / a}$.

## 3. Modal expansion of the critical load

Let us express the coefficient $a$ in terms of the eigenfrequencies $\omega_k$ and eigenmodes $\mathbf{w}_k$ of the conservative system (2), (3). In addition to $\lambda_0 = 0$, the matrix $\mathbf{A}_0$ possesses the eigenvalues $\lambda_{k\sigma} = \sigma \mathrm{i} \omega_k$ with $k = 2, \ldots, n$ and $\sigma = \pm 1$. The corresponding right and left eigenvectors are

$$\mathbf{u}_{k\sigma} = \begin{pmatrix} \mathbf{w}_k \\ \sigma \mathrm{i} \omega_k \mathbf{w}_k \end{pmatrix}, \quad \mathbf{v}_{k\sigma} = \begin{pmatrix} \mathbf{M}\mathbf{w}_k/2 \\ \mathbf{M}\mathbf{w}_k/(2\sigma \mathrm{i} \omega_k) \end{pmatrix}. \tag{31}$$

These vectors satisfy the equations and orthonormality conditions

$$\mathbf{A}_0 \mathbf{u}_{k\sigma} = \sigma \mathrm{i} \omega_k \mathbf{u}_{k\sigma}, \quad \mathbf{v}_{k\sigma}^T \mathbf{A}_0 = \sigma \mathrm{i} \omega_k \mathbf{v}_{k\sigma}^T,$$

$$\mathbf{v}_{k\sigma}^T \mathbf{u}_{k'\sigma'} = \delta_k^{k'} \delta_\sigma^{\sigma'}, \quad \mathbf{v}_0^T \mathbf{u}_{k\sigma} = \mathbf{v}_1^T \mathbf{u}_{k\sigma} = \mathbf{v}_{k\sigma}^T \mathbf{u}_0 = \mathbf{v}_{k\sigma}^T \mathbf{u}_1 = 0, \quad k, k' = 2, \ldots, n; \ \sigma, \sigma' = \pm 1. \tag{32}$$

Expression (29) can be written in terms of eigenmodes and eigenfrequencies as (the lengthy derivation of this formula is given in Appendix A)

$$a = \frac{4}{\mathbf{w}_1^T \mathbf{C}_1 \mathbf{w}_1} \int_0^1 \int_0^t \left[ B_1(t) B_1(\tau)(t-1)\tau + \sum_{k=2}^n \mathrm{Im} \frac{B_k(t) B_k(\tau) \mathrm{e}^{\mathrm{i}\omega_k (t-\tau)}}{\omega_k (1 - \mathrm{e}^{\mathrm{i}\omega_k})} \right] \mathrm{d}\tau \, \mathrm{d}t, \tag{33}$$

where the real scalar quantities $B_k(t)$ describe the modes interaction through the excitation term

$$B_k(t) = \mathbf{w}_1^T \mathbf{B}(t) \mathbf{w}_k. \tag{34}$$

For systems (1) with arbitrary period $T$, backward substitution of Eq. (4) in Eq. (33) yields

$$a = \frac{4}{\mathbf{w}_1^T \mathbf{C}_1 \mathbf{w}_1} \int_0^T \int_0^t \left[ \frac{B_1(t) B_1(\tau)(t-T)\tau}{T^2} + \sum_{k=2}^n \mathrm{Im} \frac{B_k(t) B_k(\tau) \mathrm{e}^{\mathrm{i}\omega_k (t-\tau)}}{\omega_k T (1 - \mathrm{e}^{\mathrm{i}\omega_k T})} \right] \mathrm{d}\tau \, \mathrm{d}t. \tag{35}$$

Expression (35) determines the change of the critical load (25) in terms of the frequencies and modes of the initial non-excited system.

In the particular case of harmonic excitation $\mathbf{B}(t) = \mathbf{B}_0 \cos \Omega t$, integration in (35) yields

$$a = -\frac{1}{\mathbf{w}_1^T \mathbf{C}_1 \mathbf{w}_1} \sum_{k=1}^n \frac{(\mathbf{w}_1^T \mathbf{B}_0 \mathbf{w}_k)^2}{\Omega^2 - \omega_k^2}. \tag{36}$$

According to Eq. (6), $\mathbf{w}_1^T\mathbf{C}_1\mathbf{w}_1 < 0$. Thus, in the case of harmonic excitation, the eigenmodes with frequencies $\omega_k < \Omega$ supply positive terms in the coefficient $a$ (stabilizing effect), while the eigenmodes with frequencies $\omega_k > \Omega$ give negative terms in $a$ (destabilizing effect). The term corresponding to the critical mode $\omega_1 = 0$ is always positive (stabilizing).

## 4. Effect of dissipative forces

Let us consider system (1) taking into account small dissipation:

$$\mathbf{M}\ddot{\mathbf{q}} + \gamma\mathbf{D}\dot{\mathbf{q}} + (\mathbf{C}(p) + \delta\mathbf{B}(t))\mathbf{q} = 0, \tag{37}$$

where $\mathbf{D}$ is a real symmetric positive definite $n \times n$ matrix, $\gamma > 0$ is a small dissipation parameter. As in the previous section, we take the period $T = 1$.

The critical parameter $p_{cr}(\delta, \gamma)$ can be expanded in the power series in both $\delta$ and $\gamma$:

$$p_{cr} = p_0 + (b_1\gamma + b_2\gamma^2 + \cdots) + (c_1\gamma + c_2\gamma^2 + \cdots)\delta + a\delta^2/2 + \cdots. \tag{38}$$

In this section, we study the structure of terms containing $\gamma$ in this expansion.

In the absence of parametric excitation $\delta = 0$, dissipative forces cannot stabilize or destabilize a linear autonomous conservative system [19]. Hence, $p_{cr}(0, \gamma) \equiv p_0$, i.e., none of the terms $\gamma^k$, $k = 1, 2, \ldots$ is present in expansion (38). Remark that this statement can also be confirmed by the perturbation technique used in Section 2.

Assume that the parameter $\gamma > 0$ is fixed. System (37) is transformed to the first-order equation (7) with the matrix

$$\mathbf{A}(t) = \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{M}^{-1}(\mathbf{C} + \delta\mathbf{B}(t)) & -\gamma\mathbf{M}^{-1}\mathbf{D} \end{pmatrix}. \tag{39}$$

At $\delta = p - p_0 = 0$, this matrix is time independent:

$$\mathbf{A}_0 = \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{C}_0 & -\gamma\mathbf{M}^{-1}\mathbf{D} \end{pmatrix}. \tag{40}$$

It has the simple zero eigenvalue with the eigenvectors

$$\mathbf{u}_0 = \begin{pmatrix} \mathbf{w}_1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_0 = \frac{1}{\gamma\mathbf{w}_1^T\mathbf{D}\mathbf{w}_1}\begin{pmatrix} \gamma\mathbf{D}\mathbf{w}_1 \\ \mathbf{M}\mathbf{w}_1 \end{pmatrix}, \tag{41}$$

satisfying the normalization condition $\mathbf{v}_0^T\mathbf{u}_0 = 1$.

Let us introduce the matrices (14)–(16) with the new $\mathbf{A}$ and $\mathbf{A}_0$. Then the eigenvectors (41) satisfy the equations

$$\mathbf{F}_0\mathbf{u}_0 = \mathbf{X}_0\mathbf{u}_0 = \mathbf{u}_0, \quad \mathbf{v}_0^T\mathbf{X}_0 = \mathbf{v}_0^T\mathbf{F}_0 = \mathbf{v}_0^T, \tag{42}$$

similar to Eq. (17). This implies that $\rho = 1$ is the simple multiplier of the Floquet matrix $\mathbf{F}_0$. Derivatives of this multiplier with respect to $p$ and $\delta$ are given by [17, p. 290]

$$\frac{\partial\rho}{\partial\alpha} = \rho\int_0^1 \mathbf{v}_0^T\mathbf{X}_0^{-1}(t)\frac{\partial\mathbf{A}}{\partial\alpha}\mathbf{X}_0(t)\mathbf{u}_0\,dt, \quad \alpha \in \{p, \delta\}. \tag{43}$$

Using Eqs. (39), (41), (42) in Eq. (43), we obtain the derivatives of the multiplier at the point $p - p_0 = \delta = 0$ as

$$\frac{\partial\rho}{\partial p} = -\frac{\mathbf{w}_1^T\mathbf{C}_1\mathbf{w}_1}{\gamma\mathbf{w}_1^T\mathbf{D}\mathbf{w}_1} > 0, \quad \frac{\partial\rho}{\partial\delta} = -\frac{\mathbf{w}_1^T\overline{\mathbf{B}}\mathbf{w}_1}{\gamma\mathbf{w}_1^T\mathbf{D}\mathbf{w}_1} = 0. \tag{44}$$

Here we used Eq. (6) and the assumption $\overline{\mathbf{B}} = 0$.

System (37) is asymptotically stable if all the multipliers lie inside the unit circle $|\rho| < 1$. First consider the autonomous system (37) with $p = p_0$ and $\delta = 0$. This system has a bounded solution $\mathbf{q}(t) = \mathbf{w}_1$. The other $2n - 1$ linear independent solutions decay in time exponentially due to dissipation. This means that all the multipliers lie inside the unit circle, except for a simple multiplier $\rho = 1$. Under a change of $p$ and $\delta$, the multiplier $\rho = 1$ moves along the real axis. The stability condition is $\rho < 1$. Using Eq. (44), we write this condition as $p - p_0 + o(\Delta p, \delta) < 0$. Hence, the critical value $p_{\text{cr}} = p_0 + o(\delta)$. Since this condition is obtained for arbitrary $\gamma$, none of the terms $\gamma^k \delta$, $k = 1, 2, \ldots$ is present in expansion (38).

We conclude that the small dissipation changes the coefficient in relation (25):

$$p_{\text{cr}} = p_0 + a(\gamma)\delta^2/2 + o(\delta^2), \tag{45}$$

where $a(0)$ is given by Eq. (35). We see that the effect of dissipation on the critical parameter is usually very small. If the excitation matrix is time-reversible $\mathbf{B}(t) = \mathbf{B}(t_0 - t)$ for some $t_0$, then system (37) is invariant under the transformation $t \to t_0 - t$ and $\gamma \to -\gamma$. Hence, odd powers of $\gamma$, that change sign under this transformation, cannot appear in the expansion of $p_{\text{cr}}(\delta, \gamma)$. In this case, the first correcting term due to dissipation is of order $\gamma^2\delta^2$.

## 5. High-frequency excitation

Let us consider the case when the frequency of parametric excitation $\Omega$ is much higher than all natural frequencies of the system $\omega_2, \ldots, \omega_n$. In this case $\omega_k T \ll 1$, so we can expand the exponents in Eq. (35) with respect to $\omega_k T$ and $\omega_k(t - \tau)$ as

$$\operatorname{Im} \frac{e^{i\omega_k(t-\tau)}}{\omega_k T(1 - e^{i\omega_k T})} = \frac{1}{\omega_k^2 T^2} + \frac{t - \tau}{2T} - \frac{(t - \tau)^2}{2T^2} - \frac{1}{12} + O(T^2). \tag{46}$$

Then, using relation (88) from Appendix B with $s = 0, 1, 2$, we obtain

$$\int_0^T \int_0^t \operatorname{Im} \frac{B_k(t)B_k(\tau)e^{i\omega_k(t-\tau)}}{\omega_k T(1 - e^{i\omega_k T})} \, d\tau \, dt = \frac{1}{T^2} \int_0^T \int_0^t B_k(t)B_k(\tau)(t - T)\tau \, d\tau \, dt + O(T^4). \tag{47}$$

Using Eq. (47) in Eq. (35) yields

$$a = \frac{4}{T^2 \mathbf{w}_1^{\mathrm{T}} \mathbf{C}_1 \mathbf{w}_1} \sum_{k=1}^{n} \int_0^T \int_0^t B_k(t)B_k(\tau)(t - T)\tau \, d\tau \, dt + O(T^4). \tag{48}$$

This expression gives the high excitation frequency asymptotics of Eq. (35). Since the value of the integral in Eq. (48) is of order $T^4$, we have $a \sim T^2 \sim \Omega^{-2}$.

We take the matrix $\mathbf{B}(t)$ in the form of Fourier series

$$\mathbf{B}(t) = \sum_{m=1}^{\infty} \mathbf{B}'_m \cos(m\Omega t) + \mathbf{B}''_m \sin(m\Omega t), \quad \Omega = \frac{2\pi}{T} \tag{49}$$

(there is no constant term since $\overline{\mathbf{B}} = 0$). Then the integration in Eq. (48) can be done (see Appendix B) giving

$$a = -\frac{1}{\mathbf{w}_1^{\mathrm{T}} \mathbf{C}_1 \mathbf{w}_1} \sum_{k=1}^{n} \sum_{m=1}^{\infty} \frac{(\mathbf{w}_1^{\mathrm{T}} \mathbf{B}'_m \mathbf{w}_k)^2 + (\mathbf{w}_1^{\mathrm{T}} \mathbf{B}''_m \mathbf{w}_k)^2}{(m\Omega)^2} + O(\Omega^{-4}). \tag{50}$$

Using expansion (81) from Appendix A, the summation with respect to $k$ in Eq. (50) is carried out giving

$$a = -\frac{1}{\mathbf{w}_1^{\mathrm{T}} \mathbf{C}_1 \mathbf{w}_1} \sum_{m=1}^{\infty} \frac{\mathbf{w}_1^{\mathrm{T}} (\mathbf{B}'_m \mathbf{M}^{-1} \mathbf{B}'_m + \mathbf{B}''_m \mathbf{M}^{-1} \mathbf{B}''_m) \mathbf{w}_1}{(m\Omega)^2} + O(\Omega^{-4}). \tag{51}$$

This formula yields the coefficient $a$ for high excitation frequency $\Omega$ in terms of Fourier coefficients of the excitation matrix $\mathbf{B}(t)$. In the particular case $\mathbf{B}(t) = \mathbf{B}_0 \cos \Omega t$, expression (51) becomes

$$a = -\frac{\mathbf{w}_1^{\mathrm{T}} \mathbf{B}_0 \mathbf{M}^{-1} \mathbf{B}_0 \mathbf{w}_1}{\mathbf{w}_1^{\mathrm{T}} \mathbf{C}_1 \mathbf{w}_1} \Omega^{-2} + O(\Omega^{-4}). \tag{52}$$

Another form of this expression follows from Eq. (50) as

$$a = -\frac{\Omega^{-2}}{\mathbf{w}_1^{\mathrm{T}}\mathbf{C}_1\mathbf{w}_1}\sum_{k=1}^{n}(\mathbf{w}_1^{\mathrm{T}}\mathbf{B}_0\mathbf{w}_k)^2 + O(\Omega^{-4}), \tag{53}$$

which agrees with Eq. (36) for high excitation frequencies $\Omega$.

Note that expression (51), which we obtained as the asymptotics of Eq. (35) for high excitation frequencies $\Omega$, can also be derived by means of averaging method. Indeed, for large $\Omega$, the term $\delta(\mathbf{B}'_m\cos(m\Omega t) + \mathbf{B}''_m\sin(m\Omega t))\mathbf{q}$ in the right-hand side of Eq. (1) can be substituted by the time-independent effective stiffness term $\delta^2\mathbf{B}_m^{\mathrm{eff}}\mathbf{q}$ with [20]

$$\mathbf{B}_m^{\mathrm{eff}} = \frac{\mathbf{B}'_m\mathbf{M}^{-1}\mathbf{B}'_m + \mathbf{B}''_m\mathbf{M}^{-1}\mathbf{B}''_m}{2(m\Omega)^2}. \tag{54}$$

Then by using methods of perturbation theory for autonomous systems [17], we obtain the critical load (25) with

$$a = -2\sum_{m=1}^{\infty}\frac{\mathbf{w}_1^{\mathrm{T}}\mathbf{B}_m^{\mathrm{eff}}\mathbf{w}_1}{\mathbf{w}_1^{\mathrm{T}}\mathbf{C}_1\mathbf{w}_1}, \tag{55}$$

which coincides with Eq. (51).

## 6. Examples

In this section we consider examples for systems with one and two degrees of freedom. For a system with one degree of freedom Eq. (1) is the Hill equation

$$\ddot{q} + (-p + \delta b(t))q = 0, \tag{56}$$

where $b(t)$ is $T$-periodic function with zero mean value $\int_0^T b(t)\,\mathrm{d}t = 0$. Then, from relations (24) and (35) we obtain the stabilization region in the first approximation as

$$p < \frac{a\delta^2}{2}, \quad a = \frac{4}{T}\int_0^T b(t)\int_0^t b(\tau)\tau\,\mathrm{d}\tau\,\mathrm{d}t - \frac{2}{T^2}\left(\int_0^T b(t)t\,\mathrm{d}t\right)^2. \tag{57}$$

For $T = 2\pi$ this formula agrees with that of derived in Ref. [14]. If $b(t) = \cos t$, we obtain $a = 1$, which is well known, see e.g., Refs. [13,21].

As a second example, consider an inverted double pendulum consisting of two point masses $m_1$ and $m_2$ connected by rigid massless rods of equal length $l$ with two elastic joints of equal stiffness $c$ in the gravitational field, see Fig. 1. The kinetic and potential energy for this system have the form

$$K = \tfrac{1}{2}(m_1 + m_2)l^2\dot{\theta}_1^2 + \tfrac{1}{2}m_2 l^2\dot{\theta}_2^2 + m_2 l^2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2),$$

$$V = \frac{c\theta_1^2}{2} + \frac{c(\theta_2 - \theta_1)^2}{2} + (m_1 + m_2)gl\cos\theta_1 + m_2 gl\cos\theta_2, \tag{58}$$

where $g$ is the acceleration of gravity. With the Lagrange function $L = K - V$ we derive equations of motion of the system linearized near the vertical position $\theta_1 = \theta_2 = 0$ as

$$(m_1 + m_2)l^2\ddot{\theta}_1 + m_2 l^2\ddot{\theta}_2 + (2c - (m_1 + m_2)gl)\theta_1 - c\theta_2 = 0,$$

$$m_2 l^2(\ddot{\theta}_1 + \ddot{\theta}_2) - c\theta_1 + (c - m_2 gl)\theta_2 = 0. \tag{59}$$

Let us consider a periodic excitation of the support $z = a\cos\Omega t$. Then, according to the d'Alembert principle, in the equations of motion (59) we must substitute $g$ by $g + \ddot{z}$.

For convenience we introduce non-dimensional time and parameters

$$\tilde{t} = \Omega^* t, \quad \delta = \frac{a}{l}, \quad p = -\frac{c}{m_1 g l}, \quad \tilde{\Omega} = \frac{\Omega}{\Omega^*}, \quad \eta = \frac{m_2}{m_1}, \tag{60}$$

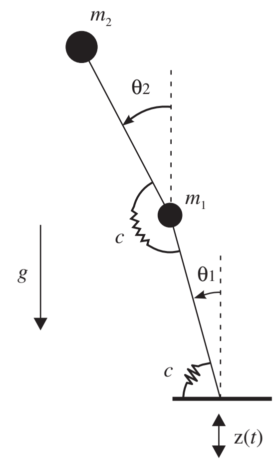


Fig. 1. Inverted double pendulum subjected to periodic excitation of the support.

where $\Omega^* = \sqrt{g/l}$. In non-dimensional variables the linearized equations of motion of the system take the form of Eq. (1) with $\mathbf{B}(t) = \mathbf{B}_0 \cos \Omega t$ and the matrices

$$\mathbf{M} = \begin{pmatrix} \eta + 1 & \eta \\ \eta & \eta \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} -2p - \eta - 1 & p \\ p & -p - \eta \end{pmatrix}, \quad \mathbf{B}_0 = \Omega^2 \begin{pmatrix} 1 + \eta & 0 \\ 0 & \eta \end{pmatrix}. \tag{61}$$

Here and below we omit tildes. The negative parameter $p$ is introduced in order to match the theoretical part of the paper, where the unstable system corresponds to $p > p_0$.

Taking into account viscous friction forces in the hinges determined by the dissipative function $F = \gamma \dot{\theta}_1^2/2 + \gamma(\dot{\theta}_2 - \dot{\theta}_1)^2/2$ yields the system in the form (37) with the dimensionless dissipation parameter $\tilde{\gamma} = \gamma/(\Omega^* m_1 l^2)$ and the matrix

$$\mathbf{D} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}. \tag{62}$$

For the case of no excitation $\delta = 0$, system (59) is stable if and only if the matrix $\mathbf{C}$ is positive definite. Simple analysis shows that the matrix $\mathbf{C}$ is positive definite when the parameter $p$ satisfies the inequality

$$p < p_0, \quad p_0 = -\frac{3\eta + 1 + \sqrt{5\eta^2 + 2\eta + 1}}{2}. \tag{63}$$

At $p = p_0$, we find the eigenfrequencies and the corresponding eigenmodes from Eqs. (3) and (61) as

$$\omega_1 = 0, \quad \mathbf{w}_1 = \alpha_1 \begin{pmatrix} p_0 \\ 2p_0 + \eta + 1 \end{pmatrix}, \tag{64}$$

$$\omega_2 = \sqrt{-\left(5 + \frac{1}{\eta}\right) p_0 - 2(\eta + 1)}, \quad \mathbf{w}_2 = \alpha_2 \begin{pmatrix} \eta + \omega_2^2 \eta + p_0 \\ -\omega_2^2 \eta + p_0 \end{pmatrix}, \tag{65}$$

where the scaling coefficients $\alpha_1$ and $\alpha_2$ are obtained from orthonormality conditions (3).

The matrix $\mathbf{C}_1$ according to Eqs. (6) and (61) is

$$\mathbf{C}_1 = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}. \tag{66}$$

The constant $a$ is given by Eq. (36) as

$$a = a_1\Omega^2 + \frac{a_2\Omega^4}{\Omega^2 - \omega_2^2}, \quad a_k = -\frac{(\mathbf{w}_1^T\mathbf{B}_0\mathbf{w}_k)^2}{\Omega^4\mathbf{w}_1^T\mathbf{C}_1\mathbf{w}_1}. \tag{67}$$

According to Eq. (61), the coefficients $a_1$ and $a_2$ depend only on the mass-ratio parameter $\eta$. This dependence is shown in Fig. 2; both $a_1$ and $a_2$ are positive, and $a_2$ vanishes at $\eta = 0$ and 1. The stability condition is given by inequality (24), which can be written using Eq. (67) as

$$\left(a_1 + \frac{a_2\Omega^2}{\Omega^2 - \omega_2^2}\right)\frac{\delta^2\Omega^2}{2} > \Delta p, \quad \Delta p = p - p_0. \tag{68}$$

The second term in the parentheses describes the effect of the second mode $\omega_2$. It is unimportant for $\eta \lesssim 2$ when $a_2 \ll a_1$, and plays an important role for larger $\eta$.

Inequality (68) is a condition for stabilization of the first mode, which is unstable without excitation ($\delta = 0$). Thus, Eq. (68) is a necessary condition giving a lower bound on the stabilizing amplitude $\delta$. The instability can also be associated with the second mode due to resonances, giving upper bounds on the stabilizing excitation amplitude.

When $\Omega \approx 2\omega_2/j$ with integer $j$, the system is subjected to parametric resonance. It is known that the secondary resonances corresponding to $j > 1$ are effectively suppressed by introduction of the damping term $\gamma\mathbf{D}$ in Eq. (37). The primary resonance $\Omega \approx 2\omega_2$ represents the most important instability region. In the first approximation it is given by the inequality [17, p. 350]

$$\zeta^2\gamma^2 + (\Omega - 2\omega_2)^2 < \xi\delta^2, \tag{69}$$

where

$$\zeta = \mathbf{w}_2^T\mathbf{D}\mathbf{w}_2, \quad \xi = \frac{(\mathbf{w}_2^T\mathbf{B}_0\mathbf{w}_2)^2}{4\omega_2^2}. \tag{70}$$

As we mentioned in Section 2, the values $\Omega \approx \omega_2/j$ correspond to a specific type of degenerate resonance associated with the multiplier $\rho = 1$ of multiplicity 4, which requires special study.

Let us consider two specific values of $\eta$. For $\eta = 1$, we have $p_0 = -3.414$, $\omega_2 = 4.040$ and the coefficients $a_1 = 2$, $a_2 = 0$. Thus, stabilization condition (68) becomes very simple

$$\delta^2\Omega^2 > \Delta p. \tag{71}$$

This is a degenerate case since the effect of the second mode disappears, and the high-frequency stabilization condition is valid in the whole range of excitation frequencies $\Omega$.

Fig. 3 shows the stability diagram computed numerically for $\eta = 1$, $\Delta p = 0.5$ and the damping coefficient $\gamma = 0.01$ using the Floquet method (calculating system multipliers $\rho$ and checking the asymptotic stability condition $|\rho| < 1$). The stability region is shown grey. The analytical lower stability bound (71) is shown by the

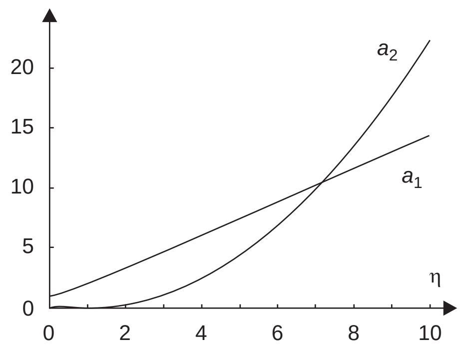


Fig. 2. Coefficients $a_1$ and $a_2$ in formula (67) depending on $\eta$.

solid line which matches perfectly with the numerical stability boundary. In Fig. 3 the resonances at $\Omega \approx \omega_2$ and $2\omega_2$ are seen. The primary resonance region (69) is indicated by a bold V-shaped line; of course, this approximation is not good for large amplitudes $\delta$.

Now let $\eta = 10$. Then $p_0 = -26.91$, $\omega_2 = 10.74$ and condition (68) becomes

$$\left(14.34 + \frac{22.30\Omega^2}{\Omega^2 - 10.74^2}\right)\frac{\delta^2\Omega^2}{2} > \Delta p. \tag{72}$$

For high excitation frequency $\Omega \gg \omega_2$ it reduces to

$$18.32\delta^2\Omega^2 > \Delta p. \tag{73}$$

Fig. 4 shows the stability diagram computed numerically for $\eta = 10$, $\Delta p = 0.1$ and $\gamma = 0.01$ using the Floquet method. The analytical lower stability bound (72) is shown by solid lines. This theoretical bound is in a very good agreement with numerical stability region. The high-frequency asymptotic (73) is presented by the dashed line. For frequencies $\Omega \gtrsim 20$, the boundaries given by expressions (72) and (73) are very close.

Upper stability bounds correspond to resonances, when the second mode becomes unstable. The resonances at $\Omega \approx 2\omega_2/k$ with $k = 1, 2$ and 4 can be seen in Fig. 4. The primary resonance region (69) is indicated by a bold V-shaped line. The upper bounds depend weakly on a small parameter $\Delta p$. Fig. 4 corresponds to a rather small value $\Delta p = 0.1$. With the increase of $\Delta p$, the lower stability bound gets higher, and the middle stabilization region disappears at about $\Delta p \sim 10$, while the low-frequency region is shifted to higher amplitudes $\delta$ and gets thinner due to resonances. This is demonstrated in Fig. 5 corresponding to $\Delta p = 2$.

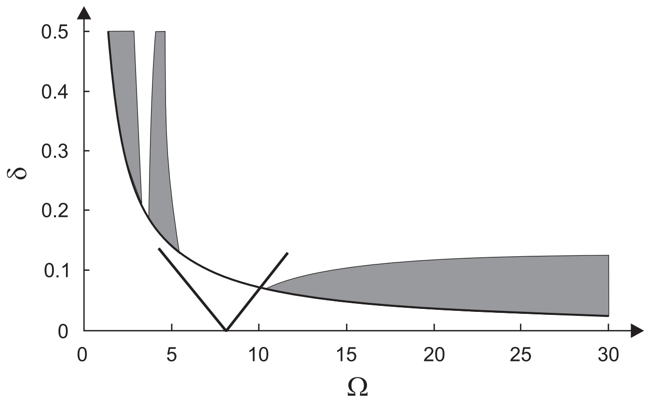


Fig. 3. Stability diagram for $\eta = 1$ and $\Delta p = 0.5$. Stability region is shaded. Solid line shows the theoretical lower stability bound. V-line indicates the primary resonance zone approximation.

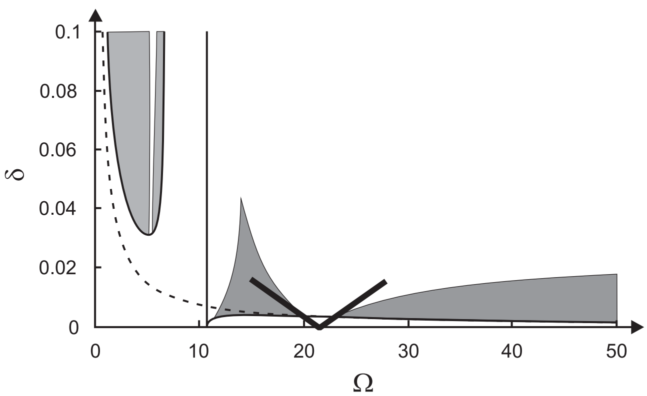


Fig. 4. Stability diagram for $\eta = 10$ and $\Delta p = 0.1$.

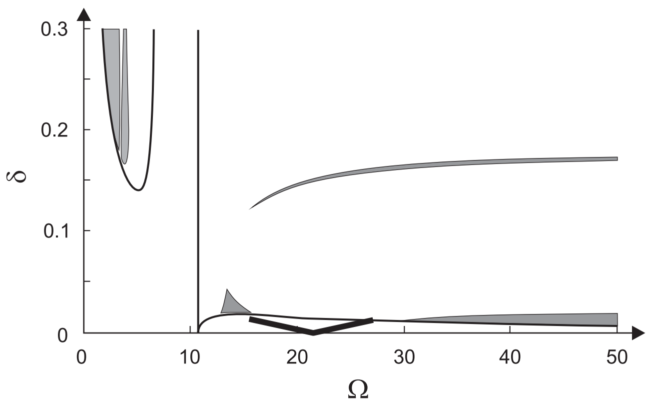


Fig. 5. Stability diagram for $\eta = 10$ and $\Delta p = 2$.

We conclude that, away from parametric resonance regions, stabilization by periodic excitation is possible for the whole range of excitation frequencies: low, medium and high compared with the second eigenfrequency of the system.

## 7. Conclusion

In this paper we considered the problem of stabilization of a weakly unstable system by parametric excitation of arbitrary frequency. A general stabilization condition is obtained. This condition contains a coefficient $a$, which is given explicitly by Eq. (29) in the form of integrals with the critical mode and the system matrices. Another form of the coefficient $a$ with the modal expansion is given in Eq. (35). Both forms of this coefficient are useful for applications. In the case of harmonic excitation a simple formula (36) is derived showing the influence of eigenfrequencies and modes on the stabilization bound. The case of high-frequency excitation follows as a limit from general formulas.

The obtained stabilization condition is valid for non-resonant excitation frequencies. It is shown that, in addition to usual parametric resonances, there are special cases associated with the multiplier $\rho = 1$ of multiplicity 4. The analysis of this type of resonance requires separate study. It seems that oscillations observed at about the second eigenfrequency in Ref. [15] (Figs. 3 and 7) correspond to this special resonance.

The obtained results can be extended to the case of infinite degrees-of-freedom systems. In this case, the system matrices must be substituted by the differential operators, and the product like $\mathbf{v}^T\mathbf{u}$ must be understood as a scalar product in the corresponding functional space.

The considered simple example with two degrees of freedom reveals an interesting phenomenon: the statically unstable system can be stabilized by low-, medium- and high-frequency excitation compared with the second eigenfrequency of the system. This analytical result agrees very well with the numerical stability analysis. Experimental verification of the recognized phenomenon appears to be a new challenge.

## Acknowledgements

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## Appendix A

Let us derive formula (33) for the coefficient $a$ in relation (29). First, we prove the following expansions:

$$\mathbf{I} = \sum \mathbf{u}_{k\sigma}\mathbf{v}_{k\sigma}^T + \mathbf{u}_0\mathbf{v}_1^T + \mathbf{u}_1\mathbf{v}_0^T,$$

$$\mathbf{A}_0 = \sum \sigma \mathrm{i}\omega_k \mathbf{u}_{k\sigma}\mathbf{v}_{k\sigma}^T + \mathbf{u}_0\mathbf{v}_0^T,$$

$$\mathbf{X}_0 = \sum e^{\sigma i\omega_k t}\mathbf{u}_{k\sigma}\mathbf{v}_{k\sigma}^T + \mathbf{u}_0\mathbf{v}_1^T + \mathbf{u}_1\mathbf{v}_0^T + t\mathbf{u}_0\mathbf{v}_0^T,$$

$$\mathbf{X}_0^{-1} = \sum e^{-\sigma i\omega_k t}\mathbf{u}_{k\sigma}\mathbf{v}_{k\sigma}^T + \mathbf{u}_0\mathbf{v}_1^T + \mathbf{u}_1\mathbf{v}_0^T - t\mathbf{u}_0\mathbf{v}_0^T,$$

$$\mathbf{F}_0 = \sum e^{\sigma i\omega_k}\mathbf{u}_{k\sigma}\mathbf{v}_{k\sigma}^T + \mathbf{u}_0\mathbf{v}_1^T + \mathbf{u}_1\mathbf{v}_0^T + \mathbf{u}_0\mathbf{v}_0^T,$$

$$\mathbf{G}^{-1} = \sum (e^{\sigma i\omega_k} - 1)^{-1}\mathbf{u}_{k\sigma}\mathbf{v}_{k\sigma}^T + \mathbf{u}_0\mathbf{v}_0^T + \mathbf{u}_1\mathbf{v}_1^T, \quad (74)$$

where $\sum$ denotes the summation of the first term with respect to $k = 2, \ldots, n$ and $\sigma = \pm 1$; the diadic products like $\mathbf{u}_0\mathbf{v}_1^T$ represent $2n \times 2n$ matrices.

Expressions for $\mathbf{I}$ and $\mathbf{A}_0$ are checked by multiplying both sides by the vectors $\mathbf{u}_0$, $\mathbf{u}_1$, $\mathbf{u}_{k\sigma}$ and using Eqs. (10), (13), (32); these vectors form the basis of $2n$-dimensional space $\mathbf{x}$. Expansion for the matrix $\mathbf{X}_0 = \exp(\mathbf{A}_0 t)$ is found using the Taylor series of exponent

$$\exp(\mathbf{A}_0 t) = \mathbf{I} + \mathbf{A}_0 t + \frac{1}{2!}\mathbf{A}_0^2 t^2 + \frac{1}{3!}\mathbf{A}_0^3 t^3 + \cdots. \quad (75)$$

With Eqs. (10) and (32), it is easy to show that for $s > 1$

$$\mathbf{A}_0^s\mathbf{u}_0 = \mathbf{A}_0^s\mathbf{u}_1 = 0, \quad \mathbf{A}_0^s\mathbf{u}_{k\sigma} = (\sigma i\omega_k)^s\mathbf{u}_{k\sigma}. \quad (76)$$

Hence,

$$\mathbf{A}_0^s = \sum (\sigma i\omega_k)^s\mathbf{u}_{k\sigma}\mathbf{v}_{k\sigma}^T, \quad s > 1. \quad (77)$$

This expansion can be checked by multiplying $\mathbf{A}_0^s$ by the basis vectors $\mathbf{u}_0$, $\mathbf{u}_1$, $\mathbf{u}_{k\sigma}$ and using Eq. (76) with orthonormality conditions (13), (32). Substitution of the expansions for $\mathbf{I}$, $\mathbf{A}_0$, $\mathbf{A}_0^s$ from Eqs. (74), (77) into Eq. (75) yields the expansion for $\mathbf{X}_0$ in Eq. (74). Since $\mathbf{X}_0^{-1} = \exp(-\mathbf{A}_0 t)$ and $\mathbf{F}_0 = \exp(\mathbf{A}_0)$, the corresponding expansions in Eq. (74) are obtained from the expansion for $\mathbf{X}_0$ by changing $t \to -t$ and $t \to 1$, respectively.

Using expansions (74) for $\mathbf{F}_0$ and $\mathbf{I}$, we obtain

$$\mathbf{G} = \mathbf{F}_0 - \mathbf{I} + \mathbf{u}_1\mathbf{v}_1^T = (e^{\sigma i\omega_k} - 1)\mathbf{u}_{k\sigma}\mathbf{v}_{k\sigma}^T + \mathbf{u}_0\mathbf{v}_0^T + \mathbf{u}_1\mathbf{v}_1^T. \quad (78)$$

This matrix satisfies the equations

$$\mathbf{G}\mathbf{u}_0 = \mathbf{u}_1, \quad \mathbf{G}\mathbf{u}_1 = \mathbf{u}_0, \quad \mathbf{G}\mathbf{u}_{k\sigma} = (e^{\sigma i\omega_k} - 1)\mathbf{u}_{k\sigma}. \quad (79)$$

Multiplying both sides of these equations by $\mathbf{G}^{-1}$, one obtains

$$\mathbf{G}^{-1}\mathbf{u}_0 = \mathbf{u}_1, \quad \mathbf{G}^{-1}\mathbf{u}_1 = \mathbf{u}_0, \quad \mathbf{G}^{-1}\mathbf{u}_{k\sigma} = (e^{\sigma i\omega_k} - 1)^{-1}\mathbf{u}_{k\sigma}. \quad (80)$$

Now the last expansion in Eq. (74) can be checked by multiplying $\mathbf{G}^{-1}$ by the basis vectors $\mathbf{u}_0$, $\mathbf{u}_1$, $\mathbf{u}_{k\sigma}$ and using Eq. (80) with orthonormality conditions (13), (32).

Note also the following modal expansion for the mass matrix:

$$\mathbf{M}^{-1} = \sum_{k=1}^{n} \mathbf{w}_k\mathbf{w}_k^T. \quad (81)$$

It can be checked by multiplying both sides by the vectors $\mathbf{M}\mathbf{w}_{k'}$, $k' = 1, \ldots, n$ (these vectors form a basis in the $n$-dimensional space), and using the orthonormality conditions (3).

Introducing the new vectors $\mathbf{u}(t)$, $\mathbf{v}(t)$ and using Eqs. (8), (11), (12), (17) with similar equations for $\mathbf{F}_0 = \mathbf{X}_0(1)$, we obtain

$$\mathbf{u}(t) := \frac{\partial \mathbf{A}}{\partial \delta}\mathbf{X}_0(t)\mathbf{u}_0 = \begin{pmatrix} 0 \\ -\mathbf{M}^{-1}\mathbf{B}(t)\mathbf{w}_1 \end{pmatrix},$$

$$\mathbf{v}^T(t) := \mathbf{v}_0^T\mathbf{F}_0\mathbf{X}_0^{-1}(t)\frac{\partial \mathbf{A}}{\partial \delta} = (-\mathbf{w}_1^T\mathbf{B}(t)\ 0). \quad (82)$$

Via Eqs. (11), (12), (31), we find

$$\mathbf{v}_0^{\mathrm{T}}\mathbf{u}(t) = \mathbf{v}^{\mathrm{T}}(t)\mathbf{u}_0 = -B_1(t), \quad \mathbf{v}_1^{\mathrm{T}}\mathbf{u}(t) = \mathbf{v}^{\mathrm{T}}(t)\mathbf{u}_1 = 0,$$

$$\mathbf{v}^{\mathrm{T}}(t)\mathbf{u}_{k\sigma} = -B_k(t), \quad \mathbf{v}_{k\sigma}^{\mathrm{T}}\mathbf{u}(t) = -\frac{B_k(t)}{2\sigma \mathrm{i}\omega_k} \tag{83}$$

with $B_k(t) = \mathbf{w}_1^{\mathrm{T}}\mathbf{B}(t)\mathbf{w}_k$; here we also used symmetry of the matrix $\mathbf{B}$.

Expression (29) after substitution of Eqs. (21), (30), (82) takes the form

$$a\frac{\mathbf{w}_1^{\mathrm{T}}\mathbf{C}_1\mathbf{w}_1}{2} = \int_0^1\int_0^t \mathbf{v}^{\mathrm{T}}(t)\mathbf{X}_0(t)\mathbf{X}_0^{-1}(\tau)\mathbf{u}(\tau)\,\mathrm{d}\tau\,\mathrm{d}t - \int_0^1\int_0^1 \mathbf{v}^{\mathrm{T}}(t)\mathbf{X}_0(t)\mathbf{G}^{-1}\mathbf{F}_0\mathbf{X}_0^{-1}(\tau)\mathbf{u}(\tau)\,\mathrm{d}\tau\,\mathrm{d}t. \tag{84}$$

Expansions (74) with orthonormality conditions (13), (32) yield

$$\mathbf{X}_0(t)\mathbf{X}_0^{-1}(\tau) = \sum \mathrm{e}^{\sigma \mathrm{i}\omega_k(t-\tau)}\mathbf{u}_{k\sigma}\mathbf{v}_{k\sigma}^{\mathrm{T}} + (t-\tau)\mathbf{u}_0\mathbf{v}_0^{\mathrm{T}} + \cdots,$$

$$\mathbf{X}_0(t)\mathbf{G}^{-1}\mathbf{F}_0\mathbf{X}_0^{-1}(\tau) = \sum \frac{\mathrm{e}^{\sigma \mathrm{i}\omega_k(1+t-\tau)}}{\mathrm{e}^{\sigma \mathrm{i}\omega_k}-1}\mathbf{u}_{k\sigma}\mathbf{v}_{k\sigma}^{\mathrm{T}} + (1+t-t\tau)\mathbf{u}_0\mathbf{v}_0^{\mathrm{T}} + \cdots, \tag{85}$$

skipping the terms with $\mathbf{u}_1$ and $\mathbf{v}_1^{\mathrm{T}}$ (denoted by dots), which vanish after multiplication by $\mathbf{v}^{\mathrm{T}}(t)$ and $\mathbf{u}(t)$ in Eq. (84) due to Eq. (83). Substituting Eq. (85) into Eq. (84) and using Eq. (83), we obtain

$$\begin{aligned} a\frac{\mathbf{w}_1^{\mathrm{T}}\mathbf{C}_1\mathbf{w}_1}{2} = &\int_0^1\int_0^t \left( B_1(t)B_1(\tau)(t-\tau) + \sum_{k=2}^{n}\left[\frac{B_k(t)B_k(\tau)\mathrm{e}^{\mathrm{i}\omega_k(t-\tau)}}{2\mathrm{i}\omega_k} + \mathrm{c.c.}\right]\right)\mathrm{d}\tau\,\mathrm{d}t \\ &+ \int_0^1\int_0^1 \left( B_1(t)B_1(\tau)t\tau + \sum_{k=2}^{n}\left[\frac{B_k(t)B_k(\tau)\mathrm{e}^{\mathrm{i}\omega_k(1+t-\tau)}}{2\mathrm{i}\omega_k(1-\mathrm{e}^{\mathrm{i}\omega_k})} + \mathrm{c.c.}\right]\right)\mathrm{d}\tau\,\mathrm{d}t, \end{aligned} \tag{86}$$

where, in the second integral, we used $\int_0^1 B_1(\tau)\,\mathrm{d}\tau = 0$ following from the condition $\overline{\mathbf{B}} = 0$. The first terms in square brackets correspond to $\sigma = +1$, and c.c. denotes the complex conjugate terms corresponding to $\sigma = -1$.

Now recall the general formula of calculus

$$\int_0^1\int_0^1 f(t,\tau)\,\mathrm{d}\tau\,\mathrm{d}t = \int_0^1\int_0^t (f(t,\tau)+f(\tau,t))\,\mathrm{d}\tau\,\mathrm{d}t, \tag{87}$$

and note that the relation

$$\int_0^1\int_0^t B_k(t)B_k(\tau)(t^s+\tau^s)\,\mathrm{d}\tau\,\mathrm{d}t = 0 \tag{88}$$

is valid for $k = 1,\ldots,n$ and any $s \geqslant 0$ (it follows from Eq. (87) with $f(t,\tau) = B_k(t)B_k(\tau)t^s$ and $\overline{\mathbf{B}} = 0$).

The $B_1(t)B_1(\tau)$ terms in Eq. (86) after transformations using Eq. (87) and then Eq. (88) with $s = 1$ reduce to

$$\int_0^1\int_0^t B_1(t)B_1(\tau)(2t\tau + t - \tau)\,\mathrm{d}\tau\,\mathrm{d}t = 2\int_0^1\int_0^t B_1(t)B_1(\tau)(t-1)\tau\,\mathrm{d}\tau\,\mathrm{d}t. \tag{89}$$

Similarly, using Eq. (87) in $B_k(t)B_k(\tau)$ terms, we get

$$\int_0^1\int_0^t \frac{B_k(t)B_k(\tau)}{2\mathrm{i}\omega_k}\left(\frac{\mathrm{e}^{\mathrm{i}\omega_k(t-\tau)}}{1-\mathrm{e}^{\mathrm{i}\omega_k}} + \frac{\mathrm{e}^{\mathrm{i}\omega_k(1-t+\tau)}}{1-\mathrm{e}^{\mathrm{i}\omega_k}}\right)\mathrm{d}\tau\,\mathrm{d}t + \mathrm{c.c.} \tag{90}$$

Interchanging the second term in the parentheses by its complex conjugate counterpart $\mathrm{e}^{\mathrm{i}\omega_k(-1+t-\tau)}/(\mathrm{e}^{-\mathrm{i}\omega_k}-1)$ in c.c., we obtain

$$\int_0^1\int_0^t \frac{B_k(t)B_k(\tau)\mathrm{e}^{\mathrm{i}\omega_k(t-\tau)}}{\mathrm{i}\omega_k(1-\mathrm{e}^{\mathrm{i}\omega_k})}\,\mathrm{d}\tau\,\mathrm{d}t + \mathrm{c.c.} = 2\int_0^1\int_0^t \mathrm{Im}\frac{B_k(t)B_k(\tau)\mathrm{e}^{\mathrm{i}\omega_k(t-\tau)}}{\omega_k(1-\mathrm{e}^{\mathrm{i}\omega_k})}\,\mathrm{d}\tau\,\mathrm{d}t. \tag{91}$$

Using Eqs. (89) and (91) in Eq. (86), we get expression (33).

## Appendix B

Here we derive formula (50). Let us introduce the function

$$F_k(t) = \sum_{m=1}^{\infty} \frac{\mathbf{w}_1^{\mathrm{T}} \mathbf{B}'_m \mathbf{w}_k \sin(m\Omega t) - \mathbf{w}_1^{\mathrm{T}} \mathbf{B}''_m \mathbf{w}_k \cos(m\Omega t)}{m\Omega}, \tag{92}$$

which satisfies the equations

$$\dot{F}_k = B_k(t), \quad \int_0^T F_k(t)\,\mathrm{d}t = 0, \quad \int_0^T B_k(t)t\,\mathrm{d}t = \int_0^T t\,\mathrm{d}F_k(t) = TF_k(T). \tag{93}$$

In the last equality we used integration by parts and the previous equations. Using Eq. (87) with $f(t,\tau) = B_k(t)B_k(\tau)t\tau$ and Eq. (93), we find

$$\frac{4}{T^2}\int_0^T \int_0^t B_k(t)B_k(\tau)(t-T)\tau\,\mathrm{d}\tau\,\mathrm{d}t = 2F_k^2(T) - \frac{4}{T}\int_0^T \int_0^t B_k(t)B_k(\tau)\tau\,\mathrm{d}\tau\,\mathrm{d}t. \tag{94}$$

Then the integral is transformed using integration by parts and Eq. (93) as

$$\begin{aligned}\frac{4}{T}\int_0^T \int_0^t B_k(t)B_k(\tau)\tau\,\mathrm{d}\tau\,\mathrm{d}t &= \frac{4}{T}\int_0^T \left(\int_0^t B_k(\tau)\tau\,\mathrm{d}\tau\right)\mathrm{d}F_k(t) \\ &= 4F_k^2(T) - \frac{4}{T}\int_0^T F_k(t)B_k(t)t\,\mathrm{d}t = 4F_k^2(T) - \frac{2}{T}\int_0^T t\,\mathrm{d}F_k^2(t) \\ &= 2F_k^2(T) + \frac{2}{T}\int_0^T F_k^2(t)\,\mathrm{d}t.\end{aligned} \tag{95}$$

Using Eqs. (94), (95) in Eq. (48), we obtain

$$a = -\frac{2}{T\mathbf{w}_1^{\mathrm{T}}\mathbf{C}_1\mathbf{w}_1}\sum_{k=1}^{n}\int_0^T F_k^2(t)\,\mathrm{d}t + O(T^4). \tag{96}$$

Expression (50) follows from Eq. (96) after substitution of Eq. (92).

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