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Eigenvalue and eigenvector derivatives of second-order systems using structure-preserving equivalences

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Abstract

A "structure-preserving equivalence" in the sense intended here is a mapping between the stiffness, damping and mass matrices describing some initial second-order system and the corresponding three matrices of another second-order system having identical spectrum. Most second-order systems can be "diagonalised" through a mapping of this sort. The mapping provides a new approach to the evaluation and the understanding of eigenvalue and eigenvector derivatives. In place of pairs of eigenvalues, we think of real scalar stiffness, damping and mass quantities representing decoupled single-degree-of-freedom systems. In place of pairs of eigenvectors, we think of individual columns of the matrices involved in the mapping.

This approach resolves the completely artificial phenomenon that the eigenvalue and eigenvector derivatives become "undefined" at instants when modification of, say, a damping parameter causes a pair of complex eigenvalues to turn into a pair of real eigenvalues or vice-versa. It also has the advantage of being applicable to cases where any one or more of the system matrices are singular.

1. Introduction and conventional equivalences

We consider systems whose equation of motion takes the form

$$\mathbf{M}\frac{d^2\mathbf{q}}{dt^2} + \mathbf{D}\frac{d\mathbf{q}}{dt} + \mathbf{K}\mathbf{q} = \mathbf{f} \tag{1}$$

In this, $\{\mathbf{K}, \mathbf{D}, \mathbf{M}\}$ are the system stiffness, damping and mass matrices, $\mathbf{f}$ is a vector of (generalised) forces, and $\mathbf{q}$ is a vector of generalised displacements. Both vectors are functions of time, t, and the system has N degrees of freedom if $\mathbf{q}$ and $\mathbf{f}$ each contain N entries.

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Nomenclature

d_i	the ith diagonal element of $\mathbf{D}_D$
$\mathbf{D}$	the damping matrix (a function of σ)
$\mathbf{D}_0$, $\mathbf{D}_1$	coefficients of the zeroth and first terms of the Taylor expansion for $\mathbf{D}(\sigma)$
$\mathbf{e}_i$	column i of the $(N \times N)$ identity matrix
$\mathbf{g}_{Ri}, \mathbf{g}_{Li}$	vectors used in the determination of eigenvector derivatives
k_i	the ith diagonal element of $\mathbf{K}_D$
$\mathbf{K}$	the stiffness matrix (a function of σ)
$\mathbf{K}_0$, $\mathbf{K}_1$	coefficients of the zeroth and first terms of the Taylor expansion for $\mathbf{K}(\sigma)$
m_i	the ith diagonal element of $\mathbf{M}_D$
$\mathbf{M}$	the stiffness matrix (a function of σ)
$\mathbf{M}_0$, $\mathbf{M}_1$	coefficients of the zeroth and first terms of the Taylor expansion for $\mathbf{M}(\sigma)$
$\mathbf{Q}_{Li}$, $\mathbf{Q}_{Ri}$	arbitrary orthogonal matrices (Householder reflections)
$\mathbf{S}_i$	all columns of the $(N \times N)$ identity matrix except column i
$\{\mathbf{w}_{Ri}, \mathbf{x}_{Ri}, \mathbf{y}_{Ri}, \mathbf{z}_{Ri}\}$	the ith columns of matrices of right diagonalising transformation
$\{\mathbf{w}_{Li}, \mathbf{x}_{Li}, \mathbf{y}_{Li}, \mathbf{z}_{Li}\}$	the ith columns of matrices of left diagonalising transformation
$\alpha_L, \alpha_R, \beta_L, \beta_R$	scalars used in the determination of eigenvector derivatives
λ_i	the ith eigenvalue
σ	the independent scalar parameter
ϕ_{Li}, ϕ_{Ri}	the left and right ith eigenvectors

It is well known that the spectrum of some original system, $\{\mathbf{K}_O, \mathbf{D}_O, \mathbf{M}_O\}$, is identical to the spectrum of some new system $\{\mathbf{K}_N, \mathbf{D}_N, \mathbf{M}_N\}$ if there are some invertible matrices, $\{\mathbf{T}_L, \mathbf{T}_R\}$ such that

$$\mathbf{T}_L^T \mathbf{K}_O \mathbf{T}_R = \mathbf{K}_N \tag{2}$$

$$\mathbf{T}_L^T \mathbf{D}_O \mathbf{T}_R = \mathbf{D}_N \tag{3}$$

$$\mathbf{T}_L^T \mathbf{M}_O \mathbf{T}_R = \mathbf{M}_N \tag{4}$$

We refer to such a relationship between two systems as a *conventional* equivalence. It is also well known that given any arbitrary system $\{\mathbf{K}_O, \mathbf{D}_O, \mathbf{M}_O\}$, it is not usually possible to find invertible matrices, $\{\mathbf{T}_L, \mathbf{T}_R\}$ such that $\{\mathbf{K}_N, \mathbf{D}_N, \mathbf{M}_N\}$ are all diagonal. Caughey and O'Kelly [1] expressed one necessary and sufficient criterion as

$$\mathbf{K}_O \mathbf{M}_O^{-1} \mathbf{D}_O = \mathbf{D}_O \mathbf{M}_O^{-1} \mathbf{K}_O \tag{5}$$

provided that the mass matrix, $\mathbf{M}_O$, is invertible. In fact, the relationship can be written in alternative ways using the inverse of $\mathbf{K}_O$ or the inverse of $\mathbf{D}_O$, if either of those is invertible. Obviously, where $\mathbf{D}_0 = \mathbf{0}$, Eq. (5) is satisfied. This paper is concerned with how certain properties of the system $\{\mathbf{K}, \mathbf{D}, \mathbf{M}\}$ evolve as the system itself changes. We will consider that the system is a function of one scalar parameter, σ, according to

$$\mathbf{K}(\sigma) = \mathbf{K}_0 + \sigma \mathbf{K}_1 + \cdots \text{higher order terms} \tag{6}$$

$$\mathbf{D}(\sigma) = \mathbf{D}_0 + \sigma \mathbf{D}_1 + \cdots \text{higher order terms} \tag{7}$$

$$\mathbf{M}(\sigma) = \mathbf{M}_0 + \sigma \mathbf{M}_1 + \cdots \text{higher order terms} \tag{8}$$

Even if systems $\{\mathbf{K}_0, \mathbf{D}_0, \mathbf{M}_0\}$ and $\{\mathbf{K}_1, \mathbf{D}_1, \mathbf{M}_1\}$ each obey Eq. (5) and if the higher order terms of Eqs. (6)–(8) can be neglected, $\{\mathbf{K}(\sigma), \mathbf{D}(\sigma), \mathbf{M}(\sigma)\}$ will not generally satisfy Eq. (5). Note that when we take derivatives with respect to σ, these will always be of interest only at $\sigma = 0$ and thus we will have $\dot{\mathbf{K}} = \mathbf{K}_1$, $\dot{\mathbf{D}} = \mathbf{D}_1$ and $\dot{\mathbf{M}} = \mathbf{M}_1$ as well as $\mathbf{K} = \mathbf{K}_0$, $\mathbf{D} = \mathbf{D}_0$ and $\mathbf{M} = \mathbf{M}_0$ for the instant $\sigma = 0$. In this paper, the dot notation always represents the derivative with respect to the scalar parameter, σ.

Although the primary purpose of the paper relates to damped systems, we nevertheless devote Section 2 of the paper to outlining existing methods for the derivatives of eigenvalues and eigenvectors of undamped systems. This section takes a slightly more general view than other papers have done for the undamped case. Instead of being concerned directly with the eigenvalues of a system, it focuses on a description of the eigenvalue problem in terms of *homogeneous coordinates*. In place of the concept of matrices of left and right eigenvectors, we concentrate on transformation matrices $\{\mathbf{T}_L, \mathbf{T}_R\}$ which diagonalise the system according to Eqs. (2) and (4). Section 2.1 shows very concisely how (in effect) eigenvalue derivatives may be obtained. Instead of referring to the eigenvalues directly, however, we discuss the diagonalised system. Section 2.2 shows concisely how the equivalent of eigenvector derivatives may be obtained. Instead of referring to the eigenvectors explicitly, we discuss instead the *diagonalising transformation*. This generalised view of the eigenvalues and eigenvectors of undamped systems—together with their derivatives—equips the reader very well to understand the methods subsequently proposed for general damped systems.

In Section 3 of this paper, the *structure-preserving equivalences* (SPEs) are described. These transformations are more general than the conventional equivalences of Eqs. (2)–(4) and they allow for most systems to be diagonalised. They were first exposed in Refs. [2,3]—but they were referred to as *structure-preserving transformations* in these papers. Each SPE is characterised by one $(2N \times 2N)$ matrix, $\underline{\mathbf{T}_L}$, acting on the left hand side and one $(2N \times 2N)$ matrix, $\underline{\mathbf{T}_R}$, acting on the right hand side in much the same way that Eqs. (2)–(4) involve the $(N \times N)$ matrices $\{\mathbf{T}_L, \mathbf{T}_R\}$. If the original system matrices are real, then the transformation matrices and the diagonalised system matrices are also real. Sections 4 and 5 then present the main results of the paper. Section 4 extends the logic of Section 2.1 to finding the derivatives of the diagonalised system in the context of damped systems. Section 5 extends the logic of Section 2.2 to finding the derivatives of the *diagonalising transformation* in the context of damped systems.

The subsequent section contains four examples. In the first example, there is a pair of repeated real roots and the derivatives of the eigenvalues and eigenvectors are undefined. This example is generated by adjusting a damping parameter such that at one specific value (corresponding to $\sigma = 0$), higher values result in two distinct real roots and lower values result in a complex conjugate pair of eigenvalues. In the second example, the mass matrix is singular at $\sigma = 0$. The conventional concepts of eigenvalue derivatives encounter difficulty in such cases as the rate of change of infinity is difficult to comprehend. The third example is a case of an undamped system having non-symmetric mass and stiffness matrices where some of the eigenvalues of the system are complex. This example shows that whilst eigenvalue and eigenvector derivatives can still be "made to work", the approach to these derivatives through the SPEs is far more elegant and involves only real-valued quantities. The fourth and final example addresses a system of recognisable structure having both singular mass matrix and a pair of identical real roots.

In this paper, matrices are signified by emboldened uppercase roman characters, vectors are denoted by emboldened lowercase roman characters, scalars are denoted using italicised lowercase roman characters where they refer to the diagonal elements of matrices and in other cases they are denoted by italicised Greek symbols.

Underlined quantities are quantities of "double-dimension". Thus $\underline{\mathbf{M}}$ is a matrix of dimension $(2N \times 2N)$ whereas $\mathbf{M}$ is a matrix of size $(N \times N)$, $\underline{\mathbf{w}_{Ri}}$ is a matrix of dimension $(2N \times 2)$ whereas $\mathbf{w}_{Ri}$ is a column-vector of size N and $\underline{k_i}$ is a (2×2) matrix whilst k_i is a scalar. Subscripts O and N indicate that the quantities subscripted belong either to the *original* or *new* system after a discrete transformation has taken place. Subscripts L and R distinguish between left eigenvectors (or left transformation) and right eigenvectors (or right transformation).

2. Eigenvalue and eigenvector derivatives for undamped systems

There is already a substantial literature on eigenvalue and eigenvector derivatives for undamped systems. Fox and Kapoor [4] provided a method applicable to symmetric undamped systems. These expressions have been expanded by numerous authors, e.g. Refs. [5–10] to determine eigenvalue and eigenvector derivatives for more general non-symmetric undamped systems. Nelson [11] simplified the procedure for calculating eigenvector derivatives of undamped systems so that only the eigenvalue and eigenvector under consideration are required.

This section provides an approach based on the above existing methods but taking a slightly more general view of the concept of eigenvalues based on the *homogeneous coordinates* approach of Ref. [12]. The purpose of this section is to provide a framework for undamped systems which can be extended naturally to damped systems. Aside from this purpose, the only added value in this section is that it caters naturally for the case where the mass matrix may be singular whereas the aforementioned methods do not. We consider non-symmetric systems in this section and it is possible that such systems can produce complex eigenvalues and associated complex eigenvectors. The treatment of this section can be applied to such cases but there may be issues of numerical stability in such cases. The intention for the methods of this section is that they would only ever be applied to the case of real eigenvalues. Where an undamped system has complex eigenvalues, the natural course is to treat it as we propose that all damped systems should be treated (Sections 4 and 5).

Consider the undamped system whose stiffness matrix is $\mathbf{K}$ and whose mass matrix is $\mathbf{M}$. The eigenvalues of this system represent squares of natural frequencies in rad/s. The following general definitions will be used

$$(\mathbf{K}m_i - \mathbf{M}k_i)\boldsymbol{\phi}_{Ri} = 0 \tag{9}$$

$$\boldsymbol{\phi}_{Li}^T(\mathbf{K}m_i - \mathbf{M}k_i) = 0 \tag{10}$$

where $\{m_i, k_i\}$ together represent the eigenvalue $\lambda := (k_i/m_i)$ and $\{\boldsymbol{\phi}_{Li}, \boldsymbol{\phi}_{Ri}\}$ represent the corresponding left and right eigenvectors, respectively. For consistency with later sections, we place all emphasis on the pair $\{m_i, k_i\}$ and no emphasis on the eigenvalue itself. In most instances, we could insist that $m_i = 1$. This corresponds to the familiar "mass-normalisation". However a more generally-applicable constraint is that

$$k_i^2 + m_i^2 = 1 \quad \text{for all } i \tag{11}$$

The definitions of eigenvalues comprised by Eqs. (9) and (10) is referred to as a *homogeneous coordinates* definition [12] and the same arbitrary normalisation of each pair $\{m_i, k_i\}$ is used there. The normalisation given by Eq. (11) admits the possibility that either $m_i = 0$ or $k_i = 0$ and it is possible to express $\{m_i, k_i\}$, respectively, as the cosine and sine of a single scalar angle but there is no particular value in pursuing this expression here.

The following orthogonality relationships are easily proven

$$\boldsymbol{\phi}_{Li}^T\mathbf{K}\boldsymbol{\phi}_{Rj} = 0 \quad \forall\, m_ik_j \neq k_im_j \tag{12}$$

$$\boldsymbol{\phi}_{Li}^T\mathbf{M}\boldsymbol{\phi}_{Rj} = 0 \quad \forall\, m_ik_j \neq k_im_j \tag{13}$$

In this section, we will consider that the eigenvalues are all distinct. In the later sections dealing with damped systems, this condition will be relaxed to the milder condition that no pairs of eigenvalues are repeated. Collecting the left and right eigenvectors in the same order produces

$$\boldsymbol{\Phi}_L = [\boldsymbol{\phi}_{L1} \;\; \boldsymbol{\phi}_{L2} \;\; \ldots \;\; \boldsymbol{\phi}_{LN}] \tag{14}$$

$$\boldsymbol{\Phi}_R = [\boldsymbol{\phi}_{R1} \;\; \boldsymbol{\phi}_{R2} \;\; \ldots \;\; \boldsymbol{\phi}_{RN}] \tag{15}$$

and both matrices, $\{\boldsymbol{\Phi}_L, \boldsymbol{\Phi}_R\}$, will be invertible since the eigenvalues are distinct. Then diagonal matrices, $\{\mathbf{K}_D, \mathbf{M}_D\}$ are related to the original stiffness and mass matrices $\{\mathbf{K}_0, \mathbf{M}_0\}$ through the conventional equivalences

$$\mathbf{K}_D = \boldsymbol{\Phi}_L^T\mathbf{K}\boldsymbol{\Phi}_R \tag{16}$$

$$\mathbf{M}_D = \boldsymbol{\Phi}_L^T\mathbf{M}\boldsymbol{\Phi}_R \tag{17}$$

Note that the normalisation of Eq. (11) controls the scaling of $(\boldsymbol{\Phi}_L^T\mathbf{K}\boldsymbol{\Phi}_R)$ and $(\boldsymbol{\Phi}_L^T\mathbf{M}\boldsymbol{\Phi}_R)$. The ith diagonal entries of $\{\mathbf{K}_D, \mathbf{M}_D\}$ are $\{k_i, m_i\}$, respectively. We see that the eigenvectors define a *diagonalising transformation* which maps the original system matrices $\{\mathbf{K}, \mathbf{M}\}$ onto the diagonal matrices $\{\mathbf{K}_D, \mathbf{M}_D\}$.

2.1. Rates of change of the diagonalised system

Differentiate Eqs. (16) and (17) with respect to some scalar parameter, σ. The dot notation employed here and henceforth indicates a derivative with respect to σ.

$$\dot{\mathbf{K}}_D = \dot{\boldsymbol{\Phi}}_L^T\mathbf{K}\boldsymbol{\Phi}_R + \boldsymbol{\Phi}_L^T\dot{\mathbf{K}}\boldsymbol{\Phi}_R + \boldsymbol{\Phi}_L^T\mathbf{K}\dot{\boldsymbol{\Phi}}_R \tag{18}$$

$$\dot{\mathbf{M}}_D = \dot{\boldsymbol{\Phi}}_L^T\mathbf{M}\boldsymbol{\Phi}_R + \boldsymbol{\Phi}_L^T\dot{\mathbf{M}}\boldsymbol{\Phi}_R + \boldsymbol{\Phi}_L^T\mathbf{M}\dot{\boldsymbol{\Phi}}_R \tag{19}$$

The derivatives of each pair, $\{k_i, m_i\}$, must be determined in isolation. Throughout this paper, we use the notation, $\mathbf{e}_i$, to denote the ith column of the $(N \times N)$ identity matrix and we will denote by $\mathbf{S}_i$ the matrix containing the remaining (N–1) columns. Pre-multiplying Eqs. (18) and (19) by $\mathbf{e}_i^T$ and post-multiplying them by $\mathbf{e}_i$ yields

$$\dot{k}_i = \dot{\boldsymbol{\phi}}_{Li}^T \mathbf{K} \boldsymbol{\phi}_{Ri} + \boldsymbol{\phi}_{Li}^T \dot{\mathbf{K}} \boldsymbol{\phi}_{Ri} + \boldsymbol{\phi}_{Li}^T \mathbf{K} \dot{\boldsymbol{\phi}}_{Ri} \tag{20}$$

$$\dot{m}_i = \dot{\boldsymbol{\phi}}_{Li}^T \mathbf{M} \boldsymbol{\phi}_{Ri} + \boldsymbol{\phi}_{Li}^T \dot{\mathbf{M}} \boldsymbol{\phi}_{Ri} + \boldsymbol{\phi}_{Li}^T \mathbf{M} \dot{\boldsymbol{\phi}}_{Ri} \tag{21}$$

Multiplying Eq. (20) by m_i, multiplying Eq. (21) by k_i and subtracting the latter result from the former gives

$$m_i \dot{k}_i - k_i \dot{m}_i = \boldsymbol{\phi}_{Li}^T (m_i \mathbf{K}_1 - k_i \mathbf{M}_1) \boldsymbol{\phi}_{Ri} \tag{22}$$

Eqs. (12) and (13) were invoked to cancel terms from Eq. (22) and Eqs. (6) and (8) were also applied to replace $\{\dot{\mathbf{K}}, \dot{\mathbf{M}}\}$ by $\{\mathbf{K}_1,\ \mathbf{M}_1\}$, respectively. Evidently, Eq. (22) is insufficient to determine $\{\dot{k}_i, \dot{m}_i\}$ uniquely. The normalisation information of Eq. (11) provides the necessary second equation

$$k_i \dot{k}_i + m_i \dot{m}_i = 0 \tag{23}$$

Combining Eqs. (22) and (23) yields

$$\begin{bmatrix} m_i & -k_i \\ k_i & m_i \end{bmatrix} \begin{bmatrix} \dot{k}_i \\ \dot{m}_i \end{bmatrix} = \begin{bmatrix} \diamond \\ 0 \end{bmatrix} \tag{24}$$

where the diamond symbol, $\diamond$, is used here (and later) to represent a known scalar quantity. In the present case, it happens to be $(\boldsymbol{\phi}_{Li}^T (m_i \mathbf{K}_1 - k_i \mathbf{M}_1) \boldsymbol{\phi}_{Ri})$. Note that the (2×2) matrix of Eq. (24) is an orthogonal matrix.

2.2. Rates of change of the diagonalising transformation

Applying Eqs. (9) and (10) for all of the eigenvectors simultaneously results in

$$(\mathbf{K} \boldsymbol{\Phi}_R \mathbf{M}_D - \mathbf{M} \boldsymbol{\Phi}_R \mathbf{K}_D) = 0 \tag{25}$$

$$(\mathbf{M}_D \boldsymbol{\Phi}_L^T \mathbf{K} - \mathbf{K}_D \boldsymbol{\Phi}_L^T \mathbf{M}) = 0 \tag{26}$$

Differentiating Eq. (25) with respect to σ yields

$$\mathbf{K} \dot{\boldsymbol{\Phi}}_R \mathbf{M}_D - \mathbf{M} \dot{\boldsymbol{\Phi}}_R \mathbf{K}_D = (\mathbf{M} \boldsymbol{\Phi}_R \dot{\mathbf{K}}_D + \mathbf{M}_1 \boldsymbol{\Phi}_R \mathbf{K}_D) - (\mathbf{K}_1 \boldsymbol{\Phi}_R \mathbf{M}_D + \mathbf{K} \boldsymbol{\Phi}_R \dot{\mathbf{M}}_D) \tag{27}$$

The quantities on the right hand side of Eq. (27) are known. Note that because $\{\mathbf{K}_D,\ \mathbf{M}_D\}$ are both diagonal, column i of the left hand side involves only the eigenvector, $\boldsymbol{\phi}_{Ri}$. Thus for the general ith right eigenvector,

$$(m_i \mathbf{K} - k_i \mathbf{M}) \dot{\boldsymbol{\phi}}_{Ri} = [(\mathbf{M} \dot{k}_i + \mathbf{M}_1 k_i) - (\mathbf{K}_1 m_i + \mathbf{K} \dot{m}_i)] \boldsymbol{\phi}_{Ri} \tag{28}$$

Similarly, by differentiating Eq. (26) and taking row i we can obtain

$$\dot{\boldsymbol{\phi}}_{Li}^T (m_i \mathbf{K} - k_i \mathbf{M}) = \boldsymbol{\phi}_{Li}^T [(\mathbf{M} \dot{k}_i + \mathbf{M}_1 k_i) - (\mathbf{K}_1 m_i + \mathbf{K} \dot{m}_i)] \tag{29}$$

Now, $(m_i \mathbf{K} - k_i \mathbf{M})$ has one zero singular value (recall our assumption of no repeated eigenvalues) and hence solution of Eqs. (28) and (29) is not simply a matter of finding the inverse of this matrix. In fact, these equations each reveal the fundamental truth that it is only possible to know $(N-1)$ independent facts about the rate of change of any one eigenvector (either left or right). The reason for the remaining unknown is that even when the system is not changing, any multiple of $\boldsymbol{\phi}_{Ri}$ can be added onto $\boldsymbol{\phi}_{Ri}$ itself without compromising its legitimacy as a right eigenvector. The same is true for $\boldsymbol{\phi}_{Li}$. Nelson [11] showed that by writing the eigenvector derivative as a linear combination of the eigenvectors, it becomes clear what can be known about the eigenvector derivative and what cannot. We adopt a more direct (and more efficient) approach here.

Begin by calculating orthogonal matrices, $\{\mathbf{Q}_{Li},\ \mathbf{Q}_{Ri}\}$ such that

$$\mathbf{Q}_{Li} \boldsymbol{\phi}_{Li} = \mathbf{e}_i \alpha_{Li} \tag{30}$$

$$\mathbf{Q}_{Ri} \boldsymbol{\phi}_{Ri} = \mathbf{e}_i \alpha_{Ri} \tag{31}$$

where $\{\alpha_{Li}, \alpha_{Ri}\}$ are arbitrary real scalars. Such matrices $\{\mathbf{Q}_{Li}, \mathbf{Q}_{Ri}\}$ are easily achieved as Householder reflections [13] which—in addition to being orthogonal—have the very attractive properties of being symmetric and low-rank modifications of the identity. Recalling that $\mathbf{S}_i$ represents the $(N \times N)$ identity matrix from which the ith column has been removed, it is clear that $(\mathbf{S}_i^T \mathbf{Q}_{Li} \boldsymbol{\phi}_{Li}) = 0 = (\mathbf{S}_i^T \mathbf{Q}_{Ri} \boldsymbol{\phi}_{Ri})$. Now, write the two desired vector derivatives as

$$\dot{\boldsymbol{\phi}}_{Li} = (\mathbf{Q}_{Li}^T \mathbf{S}_i)\mathbf{g}_{Li} + \beta_{Li}\boldsymbol{\phi}_{Li} \tag{32}$$

$$\dot{\boldsymbol{\phi}}_{Ri} = (\mathbf{Q}_{Ri}^T \mathbf{S}_i)\mathbf{g}_{Ri} + \beta_{Ri}\boldsymbol{\phi}_{Ri} \tag{33}$$

The vectors $\{\mathbf{g}_{Li}, \mathbf{g}_{Ri}\}$ each have (N–1) entries and these can be computed directly from Eqs. (28) and (29), respectively. The transpose symbols in Eqs. (32) and (33) are not necessary if $\{\mathbf{Q}_{Li}, \mathbf{Q}_{Ri}\}$ have been calculated as Householder reflections but we allow that $\{\mathbf{Q}_{Li}, \mathbf{Q}_{Ri}\}$ are not necessarily Householder reflections. Many other options are also available.

To obtain $\mathbf{g}_{Ri}$, substitute for $\dot{\boldsymbol{\phi}}_{Ri}$ in Eq. (28). The term involving the unknown scalar, β_{Ri}, vanishes naturally and the result is a set of N consistent equations in (N–1) unknowns. The most stable solution of these equations is achieved through use of the left pseudo-inverse. To obtain $\mathbf{g}_{Li}$, substitute for $\dot{\boldsymbol{\phi}}_{Li}$ in Eq. (29). The term involving the unknown scalar, β_{Li}, vanishes naturally and the result is a set of N consistent equations in (N–1) unknowns. The most stable solution of these equations is achieved through use of the right pseudo-inverse.

Having evaluated vectors $\{\mathbf{g}_{Li}, \mathbf{g}_{Ri}\}$, it remains only to set values for the unknown scalars $\{\beta_{Li}, \beta_{Ri}\}$. The normalisation of Eq. (11) provides one equation governing these since $\boldsymbol{\phi}_{Li}^T \mathbf{K} \boldsymbol{\phi}_{Ri} = k_i$ and $\boldsymbol{\phi}_{Li}^T \mathbf{M} \boldsymbol{\phi}_{Ri} = m_i$. Thus

$$k_i(\dot{\boldsymbol{\phi}}_{Li}^T \mathbf{K}\boldsymbol{\phi}_{Ri} + \boldsymbol{\phi}_{Li}^T \mathbf{K}_1 \boldsymbol{\phi}_{Ri} + \boldsymbol{\phi}_{Li}^T \mathbf{K} \dot{\boldsymbol{\phi}}_{Ri}) + m_i(\dot{\boldsymbol{\phi}}_{Li}^T \mathbf{M}\boldsymbol{\phi}_{Ri} + \boldsymbol{\phi}_{Li}^T \mathbf{M}_1 \boldsymbol{\phi}_{Ri} + \boldsymbol{\phi}_{Li}^T \mathbf{M} \dot{\boldsymbol{\phi}}_{Ri}) = 0 \tag{34}$$

Substituting for $\{\dot{\boldsymbol{\phi}}_{Li}, \dot{\boldsymbol{\phi}}_{Ri}\}$ using Eqs. (32) and (33) transforms (34) into a linear equation in $\{\beta_{Li}, \beta_{Ri}\}$. Obviously, one linear equation is not sufficient to determine two unknowns and a further arbitrary decision must be made. For symmetric undamped systems, the left and right eigenvectors can be forced to be identical and in this case, $\beta_{Li} = \beta_{Ri}$. In more general cases, a reasonable strategy is to maintain

$$\boldsymbol{\phi}_{Li}^T \boldsymbol{\phi}_{Li} = \boldsymbol{\phi}_{Ri}^T \boldsymbol{\phi}_{Ri} \quad \text{for all } i \tag{35}$$

and this clearly leads to a second linear equation in $\{\beta_{Li}, \beta_{Ri}\}$. The problem of determining the eigenvector derivatives is now solved. Moreover, notice that in order to determine the derivatives of the i^{th} left and right eigenvectors, it is not necessary to know any other eigenvectors.

3. Structure-preserving equivalences

Garvey et al. [2,3] presented a more general approach to coordinate transformations for second-order systems (extended to higher order systems in Refs. [14,15]). These more general coordinate transformations are referred to here as SPEs and they can be understood as left and right transformation matrices which preserve the structure of *Lancaster augmented matrices* (*LAMs*).

Two systems having matrices $\{\mathbf{K}_O, \mathbf{D}_O, \mathbf{M}_O\}$ and $\{\mathbf{K}_N, \mathbf{D}_N, \mathbf{M}_N\}$, respectively, are related by a SPE if for some $(N \times N)$ matrices $\{\mathbf{W}_L, \mathbf{X}_L, \mathbf{Y}_L, \mathbf{Z}_L\}$ and $\{\mathbf{W}_R, \mathbf{X}_R, \mathbf{Y}_R, \mathbf{Z}_R\}$, the following identities hold

$$\begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \mathbf{K}_O \\ \mathbf{K}_O & \mathbf{D}_O \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{K}_N \\ \mathbf{K}_N & \mathbf{D}_N \end{bmatrix} \tag{36}$$

$$\begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} \mathbf{K}_O & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_O \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} = \begin{bmatrix} \mathbf{K}_N & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_N \end{bmatrix} \tag{37}$$

$$\begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} -\mathbf{D}_O & -\mathbf{M}_O \\ -\mathbf{M}_O & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} = \begin{bmatrix} -\mathbf{D}_N & -\mathbf{M}_N \\ -\mathbf{M}_N & \mathbf{0} \end{bmatrix} \tag{38}$$

and if the inverses of the $(2N \times 2N)$ transformation matrices exist. The matrices on the right hand sides of Eqs. (36)–(38) are the *LAMs* of the new system $\{\mathbf{K}_N, \mathbf{D}_N, \mathbf{M}_N\}$ and it is clear that the corresponding *LAMs* for the original system $\{\mathbf{K}_O, \mathbf{D}_O, \mathbf{M}_O\}$ appear on the left hand side. These equations can be written more compactly as

$$\underline{\mathbf{T}}_L^T \underline{\mathbf{M}_O} \underline{\mathbf{T}_R} = \underline{\mathbf{M}_N} \tag{39}$$

$$\underline{\mathbf{T}}_L^T \underline{\mathbf{D}_O} \underline{\mathbf{T}_R} = \underline{\mathbf{D}_N} \tag{40}$$

$$\underline{\mathbf{T}}_L^T \underline{\mathbf{K}_O} \underline{\mathbf{T}_R} = \underline{\mathbf{K}_N} \tag{41}$$

Here the underlining is used to indicate quantities of "double-dimension" i.e. matrices of dimension $(2N \times 2N)$. The notation for the LAMs is chosen deliberately. Note that $\mathbf{M}_O$ does not appear in $\underline{\mathbf{M}_O}$ but $\mathbf{K}_O$ and $\mathbf{D}_O$ do. Similarly $\mathbf{D}_O$ does not appear in $\underline{\mathbf{D}_O}$ but $\mathbf{K}_O$ and $\mathbf{M}_O$ do. Finally, note that $\mathbf{K}_O$ does not appear in $\underline{\mathbf{K}_O}$ but $\mathbf{D}_O$ and $\mathbf{M}_O$. That the two systems $\{\mathbf{K}_O, \mathbf{D}_O, \mathbf{M}_O\}$ and $\{\mathbf{K}_N, \mathbf{D}_N, \mathbf{M}_N\}$ have the same spectrum is clear, since the eigenvalues of the system $\{\mathbf{K}_O, \mathbf{D}_O, \mathbf{M}_O\}$ are most usually calculated as the roots of

$$\det(\underline{\mathbf{D}_O} - \lambda \underline{\mathbf{K}_O}) = 0 \tag{42}$$

If these roots are all distinct, then it is always possible to find matrices $\{\underline{\mathbf{U}_L}, \underline{\mathbf{U}_R}\}$ such that $(\underline{\mathbf{U}}_L^T \underline{\mathbf{D}_O} \underline{\mathbf{U}_R})$ and $(\underline{\mathbf{U}}_L^T \underline{\mathbf{K}_O} \underline{\mathbf{U}_R})$ are diagonal and in all such cases, it is always possible to find some SPEs for the system such that $\{\mathbf{K}_N, \mathbf{D}_N, \mathbf{M}_N\}$ of Eqs. (35)–(37) are all diagonal. An algorithmic approach to determining diagonalising $\{\underline{\mathbf{T}_L}, \underline{\mathbf{T}_R}\}$ from $\{\underline{\mathbf{U}_L}, \underline{\mathbf{U}_R}\}$ is outlined in Refs. [3,14]. In Ref. [15], a process is described by which the matrices $\{\underline{\mathbf{T}_L}, \underline{\mathbf{T}_R}\}$ are developed from numerical solution of the differential equations outlined in Ref. [16]. A major motivation for the present paper is that even when it is not possible to find matrices $\{\underline{\mathbf{U}_L}, \underline{\mathbf{U}_R}\}$ such that $(\underline{\mathbf{U}}_L^T \underline{\mathbf{D}_O} \underline{\mathbf{U}_R})$ and $(\underline{\mathbf{U}}_L^T \underline{\mathbf{K}_O} \underline{\mathbf{U}_R})$ are diagonal, it may still be possible to determine diagonalising $\{\underline{\mathbf{T}_L}, \underline{\mathbf{T}_R}\}$. This happens particularly where there are pairs of repeated real roots.

Define the diagonalising transformation for the general system $\{\mathbf{K}, \mathbf{D}, \mathbf{M}\}$ as

$$\underline{\mathbf{T}}_L^T \underline{\mathbf{M}} \underline{\mathbf{T}_R} = \underline{\mathbf{M}_D} \tag{43}$$

$$\underline{\mathbf{T}}_L^T \underline{\mathbf{D}} \underline{\mathbf{T}_R} = \underline{\mathbf{D}_D} \tag{44}$$

$$\underline{\mathbf{T}}_L^T \underline{\mathbf{K}} \underline{\mathbf{T}_R} = \underline{\mathbf{K}_D} \tag{45}$$

where $\{\underline{\mathbf{K}_D}, \underline{\mathbf{D}_D}, \underline{\mathbf{M}_D}\}$ represent the LAMs for the system whose coefficient matrices are the diagonal matrices $\{\mathbf{K}_D, \mathbf{D}_D, \mathbf{M}_D\}$. We can now develop a new *homogeneous coordinates* definition for the eigenvalues and eigenvectors of a second-order system. We begin this development with the simple observation that

$$\left(m_i \begin{bmatrix} 0 & k_i \\ k_i & d_i \end{bmatrix} + d_i \begin{bmatrix} k_i & 0 \\ 0 & -m_i \end{bmatrix} + k_i \begin{bmatrix} -d_i & -m_i \\ -m_i & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{46}$$

It follows immediately that if $\{k_i, d_i, m_i\}$ are, respectively, the ith diagonal entries of the diagonal matrices are $\{\mathbf{K}_D, \mathbf{D}_D, \mathbf{M}_D\}$, then $(m_i \underline{\mathbf{M}_D} + d_i \underline{\mathbf{D}_D} + k_i \underline{\mathbf{K}_D})$ must have (at least) two zero singular values. It then follows that if Eqs. (43)–(45) apply and if $\{\underline{\mathbf{T}_L}, \underline{\mathbf{T}_R}\}$ are both invertible, then $(m_i \underline{\mathbf{M}} + d_i \underline{\mathbf{D}} + k_i \underline{\mathbf{K}})$ must also have (at least) two zero singular values. Vectors from within the row-kernel of $(m_i \underline{\mathbf{M}} + d_i \underline{\mathbf{D}} + k_i \underline{\mathbf{K}})$ form columns i and $(N+i)$ of $\underline{\mathbf{T}_R}$ and from within the column-kernel of $(m_i \underline{\mathbf{M}} + d_i \underline{\mathbf{D}} + k_i \underline{\mathbf{K}})$ form columns i and $(N+i)$ of $\underline{\mathbf{T}_L}$.

4. Derivatives of the diagonalised system

The derivatives of the eigenvalues and eigenvectors of damped second-order systems provide additional challenges but there is obviously strong motivation for studying these. Cardani and Mantegazza [17] considered damping in context of flutter problems and noted that the eigenvalues, eigenvectors and their derivatives become complex in general. Adhikari [18] derived exact expressions for the derivatives of complex eigenvalues and eigenvectors for systems having non-proportional viscous damping—avoiding the use of a state space representation of the equation of motion. Friswell and Adhikari [19] developed Nelson's method for symmetric non-proportionally damped systems with complex modes and Adhikari and Friswell [20] developed expressions for the first and second derivatives of complex eigensolutions of general asymmetric nonconservative systems.

In this section, we concentrate on the derivatives of three diagonal matrices, $\{\mathbf{K}_D, \mathbf{D}_D, \mathbf{M}_D\}$. In effect, we are finding the derivatives of the eigenvalues of the general second-order system but this approach does not suffer from the eigenvalue derivatives becoming undefined in the presence of a single pair of identical real roots and it does not have any restriction to non-infinite eigenvalues.

The following serve as a general definition for a pair of eigenvalues of a second-order system and the associated pairs of left and right eigenvectors.

$$(k_i\,\underline{\mathbf{K}} + d_i\,\underline{\mathbf{D}} + m_i\,\underline{\mathbf{M}})\,\underline{\mathbf{w}_{Ri}} = 0 \tag{47}$$

$$\underline{\mathbf{w}_{Li}}^T(k_i\,\underline{\mathbf{K}} + d_i\,\underline{\mathbf{D}} + m_i\,\underline{\mathbf{M}}) = 0 \tag{48}$$

where $\{\underline{\mathbf{w}_{Li}}, \underline{\mathbf{w}_{Ri}}\}$ are each $(2N \times 2)$ matrices whose partitions will be denoted as follows

$$\underline{\mathbf{w}_{Li}} = \begin{bmatrix} \mathbf{w}_{Li} & \mathbf{x}_{Li} \\ \mathbf{y}_{Li} & \mathbf{z}_{Li} \end{bmatrix} \tag{49}$$

$$\underline{\mathbf{w}_{Ri}} = \begin{bmatrix} \mathbf{w}_{Ri} & \mathbf{x}_{Ri} \\ \mathbf{y}_{Ri} & \mathbf{z}_{Ri} \end{bmatrix} \tag{50}$$

These partitions are related to the full matrices of the diagonalising transformation through

$$\mathbf{W}_L = [\mathbf{w}_{L1}\ \ \mathbf{w}_{L2}\ \ \ldots\ \ \mathbf{w}_{LN}] \tag{51}$$

$$\mathbf{W}_R = [\mathbf{w}_{R1}\ \ \mathbf{w}_{R2}\ \ \ldots\ \ \mathbf{w}_{RN}] \tag{52}$$

$$\mathbf{X}_L = [\mathbf{x}_{L1}\ \ \mathbf{x}_{L2}\ \ \ldots\ \ \mathbf{x}_{LN}] \tag{53}$$

$$\mathbf{X}_R = [\mathbf{x}_{R1}\ \ \mathbf{x}_{R2}\ \ \ldots\ \ \mathbf{x}_{RN}] \tag{54}$$

$$\mathbf{Y}_L = [\mathbf{y}_{L1}\ \ \mathbf{y}_{L2}\ \ \ldots \mathbf{y}_{LN}] \tag{55}$$

$$\mathbf{Y}_R = [\mathbf{y}_{R1}\ \ \mathbf{w}_{R2}\ \ \ldots\ \ \mathbf{w}_{RN}] \tag{56}$$

$$\mathbf{Z}_L = [\mathbf{z}_{L1}\ \ \mathbf{z}_{L2}\ \ \ldots\ \ \mathbf{z}_{LN}] \tag{57}$$

$$\mathbf{Z}_R = [\mathbf{z}_{R1}\ \ \mathbf{z}_{R2}\ \ \ldots\ \ \mathbf{z}_{RN}] \tag{58}$$

and the matrices $\{\mathbf{W}_L, \mathbf{X}_L, \mathbf{Y}_L, \mathbf{Z}_L, \mathbf{W}_R, \mathbf{X}_R, \mathbf{Y}_R, \mathbf{Z}_R\}$ collectively form $\{\underline{\mathbf{T}_L}, \underline{\mathbf{T}_R}\}$ as Eqs. (36)–(41) indicate. Define $\underline{\mathbf{e}_i}$ as

$$\underline{\mathbf{e}_i} = \begin{bmatrix} \mathbf{e}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{e}_i \end{bmatrix} \tag{59}$$

and it is then evident that

$$\underline{\mathbf{T}_L}\,\underline{\mathbf{e}_i} = \underline{\mathbf{w}_{Li}} \tag{60}$$

$$\underline{\mathbf{T}_R}\,\underline{\mathbf{e}_i} = \underline{\mathbf{w}_{Ri}} \tag{61}$$

Differentiate Eqs. (43)–(45) to obtain:

$$\underline{\dot{\mathbf{T}}_L^T}\,\underline{\mathbf{M}}\,\underline{\mathbf{T}_R} + \underline{\mathbf{T}_L^T}\,\underline{\dot{\mathbf{M}}}\,\underline{\mathbf{T}_R} + \underline{\mathbf{T}_L^T}\,\underline{\mathbf{M}}\,\underline{\dot{\mathbf{T}}_R} = \underline{\dot{\mathbf{M}}_D} \tag{62}$$

$$\underline{\dot{\mathbf{T}}_L^T}\,\underline{\mathbf{D}}\,\underline{\mathbf{T}_R} + \underline{\mathbf{T}_L^T}\,\underline{\dot{\mathbf{D}}}\,\underline{\mathbf{T}_R} + \underline{\mathbf{T}_L^T}\,\underline{\mathbf{D}}\,\underline{\dot{\mathbf{T}}_R} = \underline{\dot{\mathbf{D}}_D} \tag{63}$$

$$\underline{\dot{\mathbf{T}}_L^T\,\mathbf{K}\,\mathbf{T}_R} + \underline{\mathbf{T}_L^T\,\dot{\mathbf{K}}\,\mathbf{T}_R} + \underline{\mathbf{T}_L^T\,\mathbf{K}\,\dot{\mathbf{T}}_R} = \underline{\dot{\mathbf{K}}_D} \tag{64}$$

Following a close parallel to the logic used in Section 2.1, for some given i, multiply Eq. (62) by m_i, multiply Eq. (63) by d_i and multiply Eq. (64) by k_i. Adding the three resulting equations and pre- and post-multiplying by $\underline{\mathbf{e}_i}^T$ and $\underline{\mathbf{e}_i}$, respectively, yields

$$\underline{\mathbf{w}_{Li}}^T(k_i\,\dot{\mathbf{K}} + d_i\,\dot{\mathbf{D}} + m_i\,\dot{\mathbf{M}})\,\underline{\mathbf{w}_{Ri}} = (k_i\,\underline{\dot{k}_i} + d_i\,\underline{\dot{d}_i} + m_i\,\underline{\dot{m}_i}) \tag{65}$$

where the definitions of $\{\underline{k_i}, \underline{d_i}, \underline{m_i}\}$ are (obviously)

$$\underline{k_i} = \begin{bmatrix} -d_i & -m_i \\ -m_i & 0 \end{bmatrix}, \quad \underline{d_i} = \begin{bmatrix} k_i & 0 \\ 0 & -m_i \end{bmatrix}, \quad \underline{m_i} = \begin{bmatrix} 0 & k_i \\ k_i & d_i \end{bmatrix} \tag{66}$$

In determining (65), Eqs. (47), (48), (60) and (61) were applied. Now, since all quantities on the LHS of (65) are known along with $\{k_i, d_i, m_i\}$, it is clear that there are four scalar equations in (65) involving the three unknowns $\{\dot{k}_i, \dot{d}_i, \dot{m}_i\}$. It is immediately obvious, by symmetry that, at most, only three scalar equations are independent. In fact, these three resulting equations can be rearranged to have the form:

$$\begin{bmatrix} d_i & -k_i & 0 \\ m_i & 0 & -k_i \\ 0 & m_i & -d_i \end{bmatrix} \begin{bmatrix} \dot{k}_i \\ \dot{d}_i \\ \dot{m}_i \end{bmatrix} = \begin{bmatrix} \diamond \\ \diamond \\ \diamond \end{bmatrix} \tag{67}$$

where, once again, the diamond symbol, $\diamond$, is being used to indicate a known scalar quantity. The (3×3) matrix on the left hand side of (67) always has one zero singular value. The fact that $\{\underline{\dot{k}_i}, \underline{\dot{d}_i}, \underline{\dot{m}_i}\}$ are not uniquely defined is consistent with the finding from the undamped case that the scaling of the diagonal entries themselves is not unique. We choose to set the scaling

$$k_i^2 + d_i^2 + m_i^2 = 1 \quad \text{for all } i \tag{68}$$

Then Eq. (67) becomes

$$\begin{bmatrix} d_i & -k_i & 0 \\ m_i & 0 & -k_i \\ 0 & m_i & -d_i \\ k_i & d_i & m_i \end{bmatrix} \begin{bmatrix} \dot{k}_i \\ \dot{d}_i \\ \dot{m}_i \end{bmatrix} = \begin{bmatrix} \diamond \\ \diamond \\ \diamond \\ 0 \end{bmatrix} \tag{69}$$

Rather than eliminate any one row from Eq. (67) when forming Eq. (69), it recommended to leave four scalar equations in Eq. (69) and to solve this using the pseudo-inverse. The columns of the (4×3) matrix in Eq. (69) are mutually orthogonal and thus very stable solutions can be found for the three rates of change.

Eigenvalues and their derivatives can be calculated from $\{m_i, d_i, k_i\}$ and their derivatives $\{\dot{m}_i, \dot{d}_i, \dot{k}_i\}$ using the equation $k_i + d_i\lambda + m_i\lambda^2 = 0$. Then, a pair of eigenvalues is represented by

$$\lambda = \left(-d_i \pm \sqrt{d_i^2 - 4k_i m_i}\right) \Big/ 2m_i$$

Differentiating the equation $k_i + d_i\lambda + m_i\lambda^2 = 0$ with respect to σ gives an expression for the eigenvalue derivatives $\dot{\lambda} = -(\dot{k}_i + \dot{d}_i\lambda + \dot{m}_i\lambda^2)/(d_i + 2m_i\lambda)$.

5. Derivatives of the diagonalising transformation

In this, we follow a close parallel to the development of Section 2.2. We begin with Eqs. (47) and (48) as definitions of the left and right "eigenvectors"—in the sense that they contain the eigenvector information associated with the pair of eigenvalues which are roots of $(m_i\lambda^2 + d_i\lambda + k_i) = 0$. As was the case with Section 2.2, we shall see that no other eigenvector information is required to find the required derivatives—$\{\underline{\dot{\mathbf{w}}_{Li}}, \underline{\dot{\mathbf{w}}_{Ri}}\}$.

Differentiating Eqs. (47) and (48) produces

$$(k_i\,\underline{\mathbf{K}}+d_i\,\underline{\mathbf{D}}+m_i\,\underline{\mathbf{M}})\,\underline{\dot{\mathbf{w}}_{Ri}}=-\frac{d}{d\sigma}[(k_i\,\underline{\mathbf{K}}+d_i\,\underline{\mathbf{D}}+m_i\,\underline{\mathbf{M}})]\,\underline{\mathbf{w}_{Ri}} \tag{70}$$

$$\underline{\dot{\mathbf{w}}_{Li}}^T(k_i\,\underline{\mathbf{K}}+d_i\,\underline{\mathbf{D}}+m_i\,\underline{\mathbf{M}})=-\underline{\mathbf{w}_{Li}}^T\frac{d}{d\sigma}[(k_i\,\underline{\mathbf{K}}+d_i\,\underline{\mathbf{D}}+m_i\,\underline{\mathbf{M}})] \tag{71}$$

The right hand sides of the above equations are known. We now follow a procedure almost identical to that of Section 2.2 where the derivatives of the eigenvectors of an undamped system were derived. First, recall the definition of $\underline{\mathbf{e}_i}$ from Eq. (58). In the same way that $\{\mathbf{e}_i, \mathbf{S}_i\}$ together span N-space, we define new $(2N\times(2N\text{–}2))$ matrix, $\underline{\mathbf{S}_i}$, such that $\{\underline{\mathbf{e}_i},\underline{\mathbf{S}_i}\}$ together span $2N$-space. A logical format for $\underline{\mathbf{S}_i}$ is this

$$\underline{\mathbf{S}_i}=\begin{bmatrix}\mathbf{S}_i & \mathbf{0}\\ \mathbf{0} & \mathbf{S}_i\end{bmatrix} \tag{72}$$

Let $\{\underline{\mathbf{Q}_{Li}},\underline{\mathbf{Q}_{Ri}}\}$ represent two orthogonal matrices satisfying

$$\underline{\mathbf{Q}_{Li}}\,\underline{\mathbf{w}_{Li}}=\underline{\mathbf{e}_i}\,\underline{\alpha_{Li}} \tag{73}$$

$$\underline{\mathbf{Q}_{Ri}}\,\underline{\mathbf{w}_{Ri}}=\underline{\mathbf{e}_i}\,\underline{\alpha_{Ri}} \tag{74}$$

where $\{\underline{\alpha_{Li}},\underline{\alpha_{Ri}}\}$ are any two arbitrary (2×2) matrices. Then $(\underline{\mathbf{S}_i^T}\,\underline{\mathbf{Q}_{Li}}\,\underline{\mathbf{w}_{Li}})=\mathbf{0}=(\underline{\mathbf{S}_i^T}\,\underline{\mathbf{Q}_{Ri}}\,\underline{\mathbf{w}_{Ri}})$. Now, write the two desired derivatives, $\{\underline{\dot{\mathbf{w}}_{Li}},\underline{\dot{\mathbf{w}}_{Ri}}\}$, as

$$\underline{\dot{\mathbf{w}}_{Li}}=(\underline{\mathbf{Q}_{Li}^T}\,\underline{\mathbf{S}_i})\,\underline{\mathbf{g}_{Li}}+\underline{\mathbf{w}_{Li}}\,\underline{\beta_{Li}} \tag{75}$$

$$\underline{\dot{\mathbf{w}}_{Ri}}=(\underline{\mathbf{Q}_{Ri}^T}\,\underline{\mathbf{S}_i})\,\underline{\mathbf{g}_{Ri}}+\underline{\mathbf{w}_{Ri}}\,\underline{\beta_{Ri}} \tag{76}$$

where $\{\underline{\beta_{Li}},\underline{\beta_{Ri}}\}$ are (2×2) matrices and where $\{\underline{\mathbf{g}_{Li}},\underline{\mathbf{g}_{Ri}}\}$ are matrices of dimension $(2(N-1)\times 2)$ which can be determined uniquely by substituting for $\{\underline{\dot{\mathbf{w}}_{Li}},\underline{\dot{\mathbf{w}}_{Ri}}\}$ in Eqs. (70) and (71), respectively, using Eqs. (75) and (76). Terms involving the unknown ((2 × 2) matrix) quantities, $\{\underline{\beta_{Li}},\underline{\beta_{Ri}}\}$, vanish naturally and the result in each case is an overdetermined but consistent set of equations which can be solved directly and stably using a pseudo-inverse.

It remains only to state how the two (2 × 2) matrices $\{\underline{\beta_{Li}},\underline{\beta_{Ri}}\}$ should be determined. Much of the requisite information is present through differentiating the following three equations

$$\underline{\mathbf{w}_{Li}^T}\,\underline{\mathbf{K}}\,\underline{\mathbf{w}_{Ri}}=\underline{k_i} \tag{77}$$

$$\underline{\mathbf{w}_{Li}^T}\,\underline{\mathbf{D}}\,\underline{\mathbf{w}_{Ri}}=\underline{d_i} \tag{78}$$

$$\underline{\mathbf{w}_{Li}^T}\,\underline{\mathbf{M}}\,\underline{\mathbf{w}_{Ri}}=\underline{m_i} \tag{79}$$

These equations arise by post-multiplying each of Eqs. (43)–(45) by $\underline{\mathbf{e}_i}$ and pre-multiplying each one by its transpose. The derivatives of the right hand sides of Eqs. (77)–(79) are known (see Section 4) as are the derivatives of the LAMs. When Eqs. (75) and (76) are used to substitute for $\{\underline{\dot{\mathbf{w}}_{Li}},\underline{\dot{\mathbf{w}}_{Ri}}\}$, Eqs. (77)–(79) yield 12 equations in only 8 unknowns (the 8 scalar entries of $\{\underline{\beta_{Li}},\underline{\beta_{Ri}}\}$). In fact, only 6 of these equations are independent. When the system matrices are symmetric, forcing $\underline{\beta_{Li}}=\underline{\beta_{Ri}}$ is always possible and this is sufficient to determine $\{\underline{\beta_{Li}},\underline{\beta_{Ri}}\}$ uniquely. When the system matrices are not symmetric, some arbitrary choice must be made in the determination of these quantities. A reasonable general approach (which is consistent with the symmetric case) is to impose the constraint:

$$\mathrm{diag}(\underline{\beta_{Li}^T}\,\underline{\beta_{Li}})=\mathrm{diag}(\underline{\beta_{Ri}^T}\,\underline{\beta_{Ri}}) \tag{80}$$

6. Examples

Four examples are presented here. For three of the cases, the systems are symmetric and have dimension (3 × 3). In the other case, the system is undamped and non-symmetric with dimension (2 × 2). All systems are described in terms of the scalar parameter, σ, through Eqs. (6)–(8) where $\{\mathbf{M}_0, \mathbf{D}_0, \mathbf{K}_0\}$ and $\{\mathbf{M}_1, \mathbf{D}_1, \mathbf{K}_1\}$ are given explicitly.

6.1. A pair of identical real roots

In this case, there is a pair of identical real roots. Here, the point $\sigma = 0$ coincides exactly with a point on the root-locus plot where two repeated real roots are just about to turn into two complex conjugate roots or vice-versa. Established methods indicate (correctly) that eigenvalue derivatives are undefined in such cases. However, the coefficients of the corresponding quadratic polynomial vary smoothly and are well-behaved.

$$\mathbf{K}_0 = \begin{bmatrix} 800 & -300 & 0 \\ -300 & 900 & 50 \\ 0 & 50 & 1200 \end{bmatrix}, \quad \mathbf{M}_0 = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 5 & 1 \\ 0 & 1 & 9 \end{bmatrix}, \quad \mathbf{K}_1 = \mathbf{M}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{D}_0 = \begin{bmatrix} 26 & 10 & 24 \\ 10 & 18 & 15 \\ 24 & 15 & 40 \end{bmatrix} + \sigma_{\text{ref}}\mathbf{D}_1, \quad \mathbf{D}_1 = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}, \quad \sigma_{\text{ref}} = 13.4120370573992091$$

The "eigenvector scaling" is selected such that $(m_i^2 + d_i^2 + k_i^2) \equiv 1$ for all i.

Table 1 summarises the values $\{m_i, d_i, k_i\}$ for each of the three pairs of modes as well as $\{\dot{m}_i, \dot{d}_i, \dot{k}_i\}$.

Table 1
The diagonalised system and its rates of change.

	Mode pair 1	Mode pair 2	Mode pair 3
m_i	4.9413E−3	6.5974E−3	6.7012E−3
d_i	1.5678E−2	1.9336E−2	1.6263E−1
k_i	9.9986E−1	9.9979E−1	9.8666E−1
$\dot{m}_i$	6.278E−6	−6.254E−6	−9.165E−6
$\dot{d}_i$	0.3514E−3	0.6544E−3	0.6149E−2
$\dot{k}_i$	−5.54E−6	−1.261E−5	−0.1013E−2

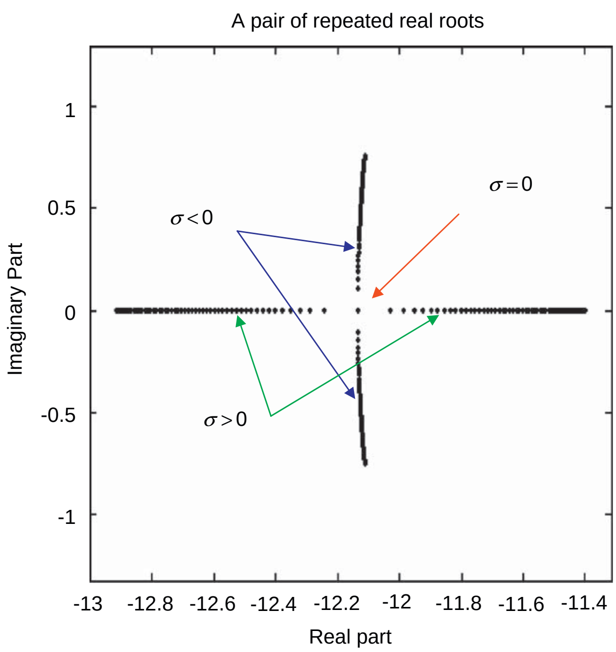

Fig. 1. Root locus for one pair of roots.

Pairs of eigenvalues can be calculated from $\{m_i, d_i, k_i\}$ using $k_i + d_i\lambda + m_i\lambda^2 = 0$. The following eigenvalues are found:

$$\lambda = \begin{bmatrix} -1.5862 \pm 14.1363i \\ -1.4654 \pm 12.2227i \\ -12.1341 \pm 0i \end{bmatrix}$$

Fig. 1 shows the root locus for the pair of repeated real roots for positive and negative values of σ close to $\sigma = 0$.

6.2. Singular mass matrix

In this case, the mass matrix, M_0, is singular. This is deliberately selected to be a problematic case for the conventional eigenvalue solutions because one eigenvalue is the point at infinity.

$$\mathbf{M}_0 = \begin{bmatrix} 4 & 1 & -1 \\ 1 & 2 & -2 \\ -1 & -2 & 2 \end{bmatrix}, \quad \mathbf{M}_1 = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 6 & 0 \\ 0 & 0 & 5 \end{bmatrix},$$

$$\mathbf{K}_0 = \begin{bmatrix} 800 & -300 & 0 \\ -300 & 900 & 50 \\ 0 & 50 & 1200 \end{bmatrix}, \quad \mathbf{K}_1 = \begin{bmatrix} 29 & 26 & -14 \\ 26 & 29 & -24 \\ -14 & -24 & 36 \end{bmatrix}$$

$$\mathbf{D}_0 = \begin{bmatrix} 26 & 10 & 24 \\ 10 & 18 & 15 \\ 24 & 15 & 40 \end{bmatrix}, \quad \mathbf{D}_1 = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

The eigenvalues for the system are shown below

$$\lambda = \begin{bmatrix} -1.6401 \pm 11.1827i \\ -18.5730 \\ -5.4110 \pm 22.0597i \\ \text{Inf} \end{bmatrix}$$

Since the mass matrix is singular, Eqs. (36)–(38) cannot be used easily to determine the diagonalised system. We take two linear combinations of the *LAMs*.

$$\underline{\mathbf{F}} = a\,\underline{\mathbf{M}_0} + b\,\underline{\mathbf{D}_0} + c\,\underline{\mathbf{K}_0}$$

$$\underline{\mathbf{G}} = d\,\underline{\mathbf{M}_0} + e\,\underline{\mathbf{D}_0} + f\,\underline{\mathbf{K}_0}$$

where $\{a, b, c, d, e, f\}$ are selected scalars and $\underline{\mathbf{G}}$ is an invertible matrix. The matrix of eigenvectors, $\mathbf{X}$, diagonalises all three LAMs in the sense that $X^T\,\underline{M_0}\,X$, $X^T\,\underline{D_0}\,X$ and $X^T\,\underline{K_0}\,X$ are all diagonal. From this point, it is straightforward to determine the left and right parts of the structure-preserving diagonalising equivalence. These are identical and they both are given by

$$\underline{T} = 10^{-3}\begin{bmatrix} -28.2204 & -21.7725 & 13.8422 & 0.5128 & -0.0109 & -0.0000 \\ -22.4223 & 19.6328 & 20.6002 & 0.2288 & 0.2550 & -0.0000 \\ 16.9498 & -6.5772 & 26.0390 & 0.7345 & 0.4515 & 0.0000 \\ -65.5050 & 5.6367 & -257.0914 & -29.9025 & -21.6542 & 0.0000 \\ -29.2315 & -131.5579 & 76.4689 & -23.1729 & 16.8731 & 24.7175 \\ -93.8257 & -232.9398 & -24.5455 & 14.5405 & -11.4635 & 24.7175 \end{bmatrix}$$

The diagonalised system and its derivatives are calculated from Eq. (69).

Table 2
Diagonalised system and its rates of change.

	Mode pair 1	Mode pair 2	Mode pair 3
m_i	7.8254E−3	1.9378E−3	1.0692E−20
d_i	2.5669E−2	2.0972E−2	5.3763E−2
k_i	9.9964E−1	9.9977E−1	9.9855E−1
$\dot{m}_i$	0.4139E−2	0.3712E−2	0.672E−2
$\dot{d}_i$	−0.2159E−2	−0.144E−1	0.1921E−1
$\dot{k}_i$	0.2304E−4	0.2949E−3	−0.1034E−2

Table 3
Diagonalised system and its rates of change.

	Mode pair 1	Mode pair 2
m_i	0.2174	0.2174
d_i	0.6981	−0.6981
k_i	0.6822	0.6822
$\dot{m}_i$	−0.0055	−0.0055
$\dot{d}_i$	0.0414	−0.0414
$\dot{k}_i$	−0.0406	−0.0406

Table 2 summarises the values of $\{m_i, d_i, k_i\}$ for each of the three pairs of modes as well as $\{\dot{m}_i, \dot{d}_i, \dot{k}_i\}$.

The rate of change of the diagonalising transformation $\underline{\dot{\mathbf{T}}}$ is computed by substituting Eqs. (75) and (76) into Eqs. (70) and (71). Terms involving the unknown scalars, $\{\underline{\beta_{Li}}, \underline{\beta_{Ri}}\}$, vanish naturally because of Eqs. (47) and (48). The derivatives of the Eqs. (77)–(79) are sufficient to determine $\{\underline{\beta_{Li}}, \underline{\beta_{Ri}}\}$ uniquely. We find

$$\underline{\dot{\mathbf{T}}} = 10^{-3}\begin{bmatrix} 6.3979 & -6.1790 & 16.5810 & 0.3525 & 0.3174 & 1.7303 \\ -4.3913 & 8.4139 & -13.8721 & 0.0949 & 0.2566 & -0.5147 \\ 1.4526 & 22.3316 & 8.5647 & 0.2828 & 0.8830 & 0.1652 \\ -10.3857 & -174.5571 & -1.1403 & 6.2727 & -9.9218 & 11.5605 \\ 3.3407 & 119.5562 & 6.7647 & -4.2423 & 12.8174 & -12.0328 \\ 13.4884 & -9.4664 & 227.8947 & 2.0016 & 25.4895 & 20.3615 \end{bmatrix}$$

6.3. Undamped non-symmetric system

This example is an undamped system having non-symmetric (2×2) mass and stiffness matrices. It is selected to be an interesting case for conventional eigenvalue solutions because all of the eigenvalues of the system are complex.

$$K_0 = \begin{bmatrix} 11 & -16 \\ 8 & -2 \end{bmatrix}, \quad M_0 = \begin{bmatrix} 4 & 3 \\ -5 & 3 \end{bmatrix}, \quad K_1 = \begin{bmatrix} -4 & 0 \\ -5 & 4 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 4 & 0 \\ -1 & -4 \end{bmatrix}$$

The eigenvalues comprise two conjugate pairs—one pair being the negative of the other.

$$\lambda = \pm(1.6058 \pm 0.7484i)$$

The decoupled single-degree-of-freedom system $\{m_i, d_i, k_i\}$ and their derivatives are calculated.

Table 3 below summarises the values $\{m_i, d_i, k_i\}$ for each of the two pairs of eigenvalues as well as $\{\dot{m}_i, \dot{d}_i, \dot{k}_i\}$.

The right and left parts of the diagonalising transformations $\{\underline{T_R}, \underline{T_L}\}$ are given by

$$\underline{T_R} = \begin{bmatrix} -0.0481 & -0.1409 & -0.0601 & 0.0612 \\ -0.2683 & -0.0420 & -0.1478 & -0.0391 \\ 0.1886 & -0.1922 & 0.1449 & 0.0558 \\ 0.4640 & 0.1226 & 0.2065 & -0.1675 \end{bmatrix}$$

$$\underline{T_L} = \begin{bmatrix} 0.1118 & 0.1873 & 0.0149 & -0.1240 \\ 0.2426 & -0.2061 & 0.1707 & 0.0444 \\ -0.0468 & 0.3892 & 0.0639 & -0.2109 \\ -0.5357 & -0.1392 & -0.3055 & -0.0636 \end{bmatrix}$$

The rates of change of the right and left diagonalising transformations $\{\underline{\dot{T}_R}, \underline{\dot{T}_L}\}$ are found to be

$$\underline{\dot{T}_R} = \begin{bmatrix} 0.0378 & 0.0268 & 0.0394 & -0.0063 \\ -0.1374 & 0.0333 & -0.0971 & -0.0300 \\ -0.1300 & 0.0264 & -0.0723 & 0.0232 \\ 0.2891 & 0.0901 & 0.2149 & -0.0738 \end{bmatrix}$$

$$\underline{\dot{T}_L} = \begin{bmatrix} -0.0647 & -0.0938 & -0.0020 & 0.0518 \\ 0.3657 & -0.0133 & 0.1673 & -0.0178 \\ 0.0080 & -0.1758 & -0.0622 & 0.0387 \\ -0.5068 & 0.0605 & -0.2181 & -0.0583 \end{bmatrix}$$

6.4. Physical system

In this example we try to give a practical application of the proposed approach. Fig. 2 shows a system with 3 degrees of freedom whose system matrices are

$$\mathbf{M}_0 = \begin{bmatrix} \theta_{m1} & 0 & 0 \\ 0 & \theta_{m2} & 0 \\ 0 & 0 & \theta_{m3} \end{bmatrix}, \quad \mathbf{K}_0 = \begin{bmatrix} \theta_{k1} & -\theta_{k1} & 0 \\ -\theta_{k1} & (\theta_{k1}+\theta_{k2}) & -\theta_{k2} \\ 0 & -\theta_{k2} & (\theta_{k2}+\theta_{k3}) \end{bmatrix}, \quad \mathbf{D}_0 = \begin{bmatrix} \theta_{d1} & -\theta_{d1} & 0 \\ -\theta_{d1} & (\theta_{d1}+\theta_{d2}) & -\theta_{d2} \\ 0 & -\theta_{d2} & (\theta_{d2}+\theta_{d3}) \end{bmatrix}$$

We will investigate the case where $\theta_{m1} = 2$, $\theta_{m2} = 1$, $\theta_{m3} = 0$, $\theta_{k1} = 3.0E4$, $\theta_{k2} = 2.0E4$, $\theta_{k3} = 1.0E4$, $\theta_{d2} = \theta_{d3} = 100$ and $\theta_{d1} = 278.9616508149336$. The value for d_1 has been chosen so as to cause a pair of identical real roots. The rate of change of the system parameters are $\dot{\theta}_{m1} = 3$, $\dot{\theta}_{m2} = 1$, $\dot{\theta}_{m3} = 5$, $\dot{\theta}_{k1} = 1600$, $\dot{\theta}_{k2} = 1400, \dot{\theta}_{k3} = 2000$ and $\dot{\theta}_{d1} = 6$, $\dot{\theta}_{d2} = 7$, $\dot{\theta}_{d3} = 2$.

The eigenvalues of the system comprise one pair of complex conjugate, one pair of identical real roots, one other finite real root and one infinite eigenvalue. The inverses of the system eigenvalues are

$$\lambda^{-1} = \begin{bmatrix} -0.004179171 \pm 0.021767786i \\ -0.004298863 \pm 0.00000i \\ -0.007342651 \\ -0.000000000 \end{bmatrix}$$

The diagonalised system and its derivatives are calculated from Eq. (69).

Table 4 summarises the values of $\{m_i, d_i, k_i\}$ for each of the three pairs of modes as well as $\{\dot{m}_i, \dot{d}_i, \dot{k}_i\}$.

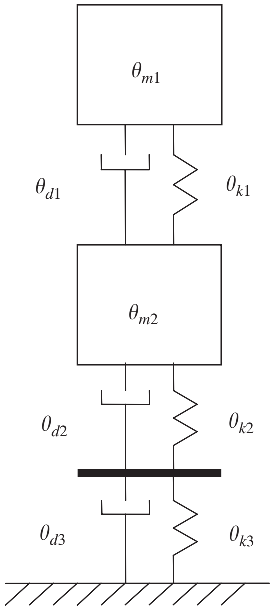

Fig. 2. Physical system; mass, stiffness and damping.

Table 4
Diagonalised system and its rates of change.

	Mode pair 1	Mode pair 2	Mode pair 3
m_i	4.9128E−4	1.8481E−5	0
d_i	8.3581E−3	8.5976E−3	7.3425E−3
k_i	9.9996E−1	9.9993E−1	9.9997E−1
$\dot{m}_i$	0.1134E−2	0.1278E−3	0.1836E−3
$\dot{d}_i$	−0.6694E−2	0.1207E−1	−0.1407E−1
$\dot{k}_i$	0.5539E−6	−0.1038E−3	0.1033E−3

7. Conclusions

Previous papers have presented computation methods for the derivatives of eigenvalues and eigenvectors for general non-symmetric and damped vibrating systems. Separate papers have also defined structure-preserving equivalences and shown that almost all second-order systems can be transformed to diagonal form using real-valued coordinate transformations. The triples of diagonal entries of the diagonalised system contain the eigenvalues. Pairs of columns of the left and right transformation matrices contain the left and right eigenvector information. The present paper has shown how the existing methods for eigenvalue and eigenvalue derivatives for undamped systems can be extended to the concept of structure-preserving equivalences to yield general methods for calculating the derivatives of both the diagonalised system and the diagonalising transformations.

The new construction for these derivatives has several advantages over the conventional approaches to eigenvalue and eigenvector derivatives. Firstly, cases where the existence of a pair of identical real roots causes the derivatives of two eigenvalues and their corresponding eigenvectors to become undefined present no such problem in this case. Secondly, cases of infinite eigenvalues (corresponding to some zero singular values in the mass matrix) produce no difficulty whatsoever. Thirdly, some other cases where the Jordan form for the system is non-diagonal can have well defined derivatives for their diagonalised systems and diagonalising transformations.

The method proposed has practical applications-beyond simply resolving the quandary that eigenvalue derivatives become undefined in the presence of a pair of identical real roots and the difficulty with expressing derivatives of infinite eigenvalues. The additional advantages are associated with the transformation of measured data into ‘‘modal form’’. A pair of complex modes, as it is normally understood, actually represents a real two-dimensional subspace within state-space. The diagonalising transformations discussed in this paper are direct representations of these two-dimensional subspaces and they can be extracted directly from physical measurements of displacements and velocities. By contrast, extraction of pairs of complex modes from sets of physical measurements inherently requires use of the corresponding complex eigenvalues.

As a by-product of the development of these new formulae for the derivatives, new *homogeneous coordinates* expressions for pairs of roots of a quadratic eigenvalue problem have been presented (Eqs. (47) and (48)). These expressions potentially have substantial value in their own right and they form obvious prototypes for related expressions for groups of l eigenvalues of general matrix polynomials of order l.

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