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Sky-hook control of nonlinear quarter car model traversing rough road matching performance of LQR control

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Abstract

The control of the stationary random response of a two degree of freedom (dof) quarter car vehicle model with nonlinear passive elements traversing a homogenous rough road with sky-hook damper control strategy is considered. The sky-hook damper control strategy is realized through a feedback control scheme. The parameters of the sky-hook damper are optimally determined by equating the control force of the feedback system to that obtained by linear quadratic regulator (LQR) control in a mean square equivalence sense. The nonlinear suspension is of hysteretic type and modeled by the Bouc–Wen model. The equivalent linearization method is used to linearize the system and the stochastic optimal control LQR theory is applied to the equivalent linear system. Results show the enhanced performance of feedback control based on sky-hook to levels of performance of LQR control which are also verified by Monte Carlo simulation.

1. Introduction

In active vehicle suspension systems the external road excitation is countered with the generation of a control force depending on the vehicle response through an actuator driven by an external energy source. However, difficulties in control hardware implementation, high cost and relatively less robustness restrict the use of active suspensions. The semi-active suspension system first proposed by Karnopp et al. [1] is an alternative to the active suspension system and can combine the advantages of the active suspension system to an extent while providing more robustness to the suspension performance. Usually, in a semi-active suspension system, the actuator is replaced by a rapidly adjusted damper which acts in parallel with a spring. Such semi-active suspensions have been used to control the vibration response of vehicles traversing rough roads [2–7]. More recently, electrorheological (ER) and magnetorheological (MR) fluids which have the property of changing their viscosity by orders of magnitude in quick time on application of electric and magnetic fields, respectively, have been used as variable fluid damper devices in vehicle suspensions [8–11]. Choi and Han [12] designed a semi-active ER seat suspension system for commercial vehicle using sliding

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mode control strategy. Sun et al. [11] proposed a novel feed forward fuzzy control method, based on the identification of the signals main frequency to control a two stage vibration isolation system with ER damper.

Semi-active suspensions of the sky-hook damper type have been extensively researched in the literature. The idea of the sky-hook control was introduced by Crosby and Karnopp [13] and Karnopp and Crosby [14] in the context of vehicle suspensions. The concept follows an ideal configuration of a passive damper connected between the sprung mass and a fictional fixed point in inertial space with beneficial effects of decreased resonant transmissibility and improved high frequency isolation with increase in the sky-hook damping ratio, unlike in the case of a conventional passive suspension where an increase in the passive damping while improving the resonant transmissibility reduces the high frequency isolation performance. Thus in the sky-hook damper configuration, the damper is partitioned such that the damper force is a function of the absolute velocity of the sprung mass and since it is not connected to the unsprung mass, the sky-hook damper provides damping without transmitting the unsprung mass vibrations to the body. Since the sky-hook concept cannot be realized in a moving vehicle, Karnopp at al. [1] have tried to realize the same by means of a variable rate damper which applies the same force to the sprung mass as the sky-hook damper, except that when energy is required to be added to the system, rather than dissipated, the damping is set to almost zero. The necessary control force is obtained by the feed back of absolute velocity of the mass to a controller where the control force corresponding to the sky-hook damper is calculated and applied to the mass. Liu et al. [15] studied the performance of semi-active damper using five control applications based on, on–off sky-hook, continuous sky-hook, on–off balance, continuous and adaptive damping control. Roebuck et al. [16] developed a vehicle model with semi-active tri-axle air suspension with nonlinear damping element to improve the performance of the vehicle using modified sky-hook damping and optimal full state feedback control strategies and compared these performances with the performance of vehicles with passive suspension. Choi et al. [17] used the sky-hook control strategy to achieve the desired damping force in an ER damper by controlling the electric field appropriately. Holdmann and Holle [6] modeled the sky-hook damper control scheme and analyzed the performance of a delivery truck model using the multi-body simulation tool SIMPACK. It was shown that at higher frequencies the sky-hook control system did not have much influence over the performance. Yao et al. [8] showed that a semi-active MR suspension based on sky-hook control improved performance of a quarter car model with respect to sprung mass acceleration, suspension travel and tire deflection.

Sammier et al. [18] proposed a continuous feedback control strategy to approximately realize the sky-hook damper control. Ahmadian et al. [19] suggested two types of sky-hook control strategies, the sky-hook function which is an analytical continuous function used to avoid the damping force discontinuity and no-jerk sky-hook to reduce the dynamic jerk. Valasek and Sveda [20] introduced off-road vehicle suspensions based on the concept of extended ground-hook control strategy which adapt to the local terrain conditions. Li and Nagai [21] presented two nonlinear control strategies for applying the sky-hook control laws in railway secondary active suspensions to improve the ride quality of railway vehicles. Gopalarao and Narayanan [22] considered the vibration control of a quarter-car vehicle model to random road excitation with the sky-hook damper control realized by an approximate continuous feedback control strategy. The sky-hook damper parameters based on feedback control are optimally determined by equating in a mean square sense the semi-active control force to the control force of a fully active linear quadratic regulator (LQR) control thereby trying to match the performance of the semi-active control to that of a fully active control.

Generally in the vibration analysis of vehicle systems, the suspension elements are modeled as linear springs and dampers to reduce the complexity of analysis. In reality, the deformation characteristics of springs exhibit nonlinear behavior especially of hysteresis type. Similarly, damping characteristics of suspension and the stiffness characteristics of the tyres also exhibit nonlinear behavior. Hence for a realistic vehicle dynamic model it is necessary to consider the nonlinearities in the passive suspension systems.

The response of vehicle models with nonlinear passive suspension elements can be obtained using either simulation or analytical techniques. The simulation technique like the Monte Carlo simulation requires more computational effort due to the necessity of simulating time series of excitation processes and applying numerical methods to integrate the equations of motion of the system to compute the response statistics. Yadav and Nigam [23] have used the simulation technique to study the response behavior of nonlinear vehicle models. Analytical methods are preferred wherever possible, due to the large computational time involved in simulation techniques. The analytical methods include the perturbation, equivalent linearization and various closure techniques. The equivalent linearization technique is one of the most widely used techniques in the analysis of nonlinear systems subjected to random excitation [24,25]. Solutions obtained by the equivalent linearization technique have been shown to compare favorably with Monte Carlo simulation results. Caughey [26] was the first one to introduce the equivalent linearization technique for single degree of freedom (dof) nonlinear random vibration problems which was extended to multi-dof systems by Iwan and Yang [27], Atalik and Utku [28] and Spanos [29]. The advantages of the method and various developments are given in the book by Roberts and Spanos [30].

In vehicular vibration systems, nonlinearities in the suspension have been taken into consideration in the vibration analysis by Kirk and Perry [31] and Kirk [32] in the context of landing gear response to runway unevenness and by Yadav and Nigam [23], Harrison and Hammond [33], Narayanan [34] and Elmadany [35] in the context of response of road vehicles to random road undulations and by Narayanan and Raju [36] and Narayanan and Senthil [37] in the context of active control of nonstationary response of road vehicles.

Most of the works in the literature concerning sky-hook control suspensions have dealt with vehicle models consisting of linear passive suspension elements. The problem of response of vehicle models with nonlinear passive elements with sky-hook control has not been considered in the stochastic framework so far. In this paper, we consider sky-hook control concepts in the nonlinear passive suspension elements in the context of the vibration control of a two dof quarter car vehicle model.

Specifically, the control of the stationary random response of a two dof quarter car vehicle model with nonlinear passive elements traversing a homogenous rough road with sky-hook damper control strategy is considered. The nonlinear passive suspension spring is assumed to be of hysteretic nature which is modeled by the Bouc–Wen model [38–40] in which the restoring force of the spring is assumed to be a combination of a pre-yielding component and a hysteretic component. The sky-hook damper control is assumed to be approximately realized by a continuous feedback control scheme [18]. The performance of the semi-active control scheme is sought to be enhanced to the levels of a fully active control scheme like the LQR by a mean square equivalence of the control forces in both cases as considered by Gopalarao and Narayanan [22] in the case of a linear vehicle model.

The sky-hook damper parameters based on the feedback control scheme are optimally determined from a combination of values by minimizing the rms difference with respect to the overall vehicle performance between the LQR control and the feedback control. The nonlinear system is linearized using the equivalent linearization method and the LQR control and the feedback control are applied to the equivalent linear system and the response and the control of the equivalent linear system are obtained by an iterative procedure. The results of the equivalent linearization are verified by Monte Carlo simulation.

The random road profile is modeled as the response of a first-order linear shaping filter to white noise excitation. The average behavior of the system is described by the zero-lag covariance matrix of the response state vector. In the LQR control the vehicle suspension is realized by minimizing a performance index which is a weighted integral of the mean square acceleration, suspension stroke, road holding and control force to improve the overall vehicle performance.

This paper is organized in the following manner. Section 2 gives the equations of motion of the vehicle with the nonlinear suspension, the LQR control scheme and the equivalent linearization relations for the Bouc–Wen model. Section 3 gives the sky-hook damper model based on feedback control and the method of estimating the optimum sky-hook damper parameters. Section 4 gives the results including Monte Carlo simulation and Section 5 gives the important conclusions of the paper.

2. Mathematical modeling

2.1. Vehicle model with LQR control

The quarter car vehicle model with nonlinear hysteretic suspension and with LQR control is shown in Fig. 1. The hysteretic nature of the nonlinear spring is modeled by the Bouc–Wen hysteretic model [38–40]. In the Bouc–Wen model the restoring force of the spring is assumed to be a combination of a preyielding component and a hysteretic component. The preyielding component of the restoring force is proportional to the relative displacement between the sprung mass and the unsprung mass with the proportionality constant

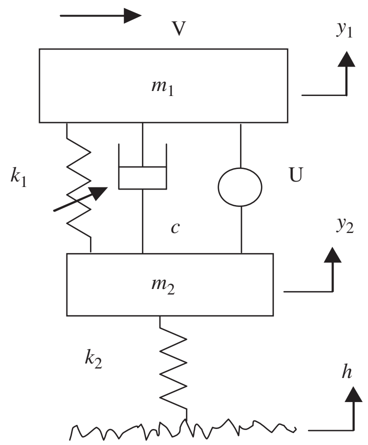

Fig. 1. Quarter car model with LQR control.

equal to $\alpha_1 k_1$. The restoring force corresponding to the hysteretic component is given by $(1-\alpha_1)k_1 z_q$, where z_q is the hysteretic displacement. For $\alpha_1 = 1$, the restoring force corresponds to a linear system. Hence, the equations of motion of the quarter car vehicle model given in Fig. 1 are given by

$$m_1\ddot{y}_1 + \alpha_1 k_1(y_1 - y_2) + (1-\alpha_1)k_1 z_q + c(\dot{y}_1 - \dot{y}_2) - U = 0 \tag{1}$$

$$m_2\ddot{y}_2 + c(\dot{y}_2 - \dot{y}_1) + \alpha_1 k_1(y_2 - y_1) - (1-\alpha_1)k_1 z_q + k_2(y_2 - h) + U = 0 \tag{2}$$

where m_1 and m_2 are sprung and unsprung masses, respectively, c is the damping coefficient of the linear viscous damper connecting the sprung and unsprung masses. The suspension spring is of hysteresis type given by the Bouc–Wen model, whose restoring force consists of a hysteretic component with parameters k_1 and α_1 and k_2 is the tyre stiffnesses. U is the control force and h is the random road input. The hysteretic displacement z_q is governed by the following equations as per the Bouc–Wen model [38–40]:

$$\dot{z}_q = -\gamma_q|\dot{y}_1 - \dot{y}_2|z_q|z_q|^{n-1} - \beta_q|z_q|^n(\dot{y}_1 - \dot{y}_2) + A_q(\dot{y}_1 - \dot{y}_2) \tag{3}$$

where γ_q, β_q and A_q are the parameters of the hysteretic suspension. These parameters control the shape of the hysteresis loop. The parameter n determines the smoothness of the force–displacement curve. In this paper the hysteretic parameters are chosen as given in Refs. [36,37], which are $A_q = 1.5$, $\gamma_q = 0.5$, $\beta_q = 0.5$, $\alpha_1 = 0.2$. For these parameters the force-deflection characteristics are given in Fig. 2.

2.2. Equivalent linear system

The equations of motion of a multi-dof nonlinear system (Eqs. (1)–(3)) can be written in matrix form as

$$M\ddot{X} + C\dot{X} + KX + g(X, \dot{X}) = f(t) \tag{4}$$

where X is the state vector which includes z_q, the hysteretic displacement given by Eq. (3), $g(X, \dot{X})$ is the vector containing the nonlinear terms and $f(t)$ is the excitation vector. The system of equations (1) and (2) can be converted to an equivalent linear system of equations represented by

$$M\ddot{X} + C^*\dot{X} + K^*X = f(t) \tag{5}$$

where $C^* = C + C'$ and $K^* = K + K'$ where C and K are the linear part of damping and stiffness matrices, respectively, and C' and K' are obtained by minimizing the mean square equation error $E[e^{\mathrm{T}}e]$, where e is defined by

$$e = g(X, \dot{X}) - C'\dot{X} - K'X \tag{6}$$

The excitation vector $f(t)$ is assumed to be a zero mean Gaussian random vector. Since the system of equation (5) is linear, the response of the equivalent linear system can also be assumed to be Gaussian as an

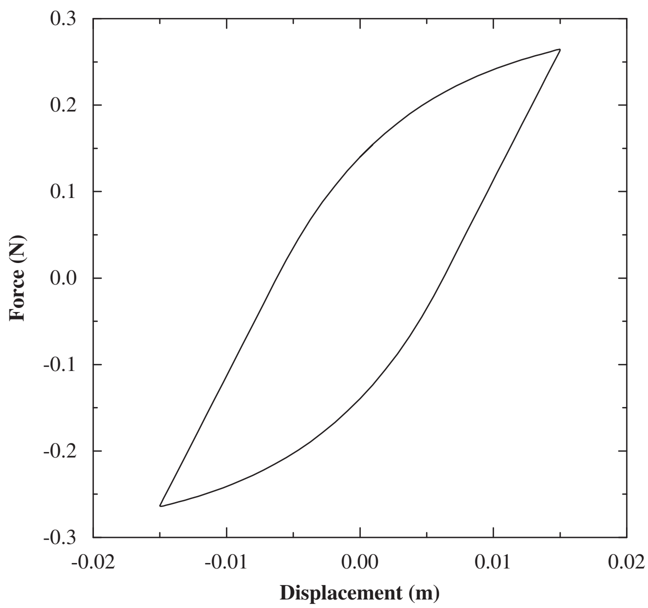

Fig. 2. Force deformation characteristics of a hysteretic system for $A_q = 1.5$, $\gamma_q = 0.5$, $\beta_q = 0.5$, $\alpha_1 = 0.2$.

approximation. Under this condition, using the results from Refs. [27,28], the elements of the matrices C' and K' are given by

$$C'_{ij} = E\left[\frac{\partial g_i}{\partial \dot{X}_j}\right] \tag{7}$$

$$K'_{ij} = E\left[\frac{\partial g_i}{\partial X_j}\right] \tag{8}$$

where g_i is the ith element of the nonlinear vector g, X_j is the jth response and $\dot{X}_j$ is the derivative of $X_j(t)$ with respect to time.

Using the above results, Eq. (3) can be represented in equivalent linear form as [40]

$$\dot{z}_q = C_h(\dot{y}_2 - \dot{y}_1) + K_h z_q \tag{9}$$

where C_h and K_h are the equivalent damping coefficient and stiffness, which are obtained by minimizing the mean square error between Eqs. (3) and (9). These values are given by Hurdato and Barbat [41]

$$C_h = -\gamma_q g_{1f} - \beta_q g_{2f} + A_h \tag{10}$$

$$K_h = -\gamma_q g_{3f} - \beta_q g_{4f} \tag{11}$$

$$g_{1f} = \sigma_{z_f}^n \Gamma\left(\frac{n+2}{2}\right) 2^{n/2} F_s \tag{12}$$

$$g_{2f} = \frac{\sigma_{z_f}^n}{\sqrt{\pi}} \Gamma\left(\frac{n+1}{2}\right) 2^{n/2} \tag{13}$$

$$g_{3f} = \frac{n\sigma_{(\dot{y}_2-\dot{y}_1)}\sigma_{z_f}^{n-1}}{\pi} \Gamma\left(\frac{n+1}{2}\right) 2^{n/2}\left[2(1-\rho^2_{(\dot{y}_2-\dot{y}_1)z_f})^{(n+1)/2} + \rho_{(\dot{y}_2-\dot{y}_1)z_f} F_s\right] \tag{14}$$

$$g_{4f} = \frac{n\rho_{(\dot{y}_2-\dot{y}_1)z_f}\sigma_{(\dot{y}_2-\dot{y}_1)}\sigma_{z_f}^{n-1}}{\sqrt{\pi}} \Gamma\left(\frac{n+1}{2}\right) 2^{n/2} \tag{15}$$

where

$$F_s = 2\int_{l}^{\pi/2} \sin^n\theta \, \mathrm{d}\theta$$

$$l = \tan^{-1}\left(\frac{\sqrt{1-\rho^2_{(\dot{y}_2-\dot{y}_1)z_f}}}{\rho_{(\dot{y}_2-\dot{y}_1)z_f}}\right)$$

$$\rho_{(\dot{y}_2-\dot{y}_1)z_f} = \frac{E[(\dot{y}_2-\dot{y}_1)z_f]}{\sigma_{(\dot{y}_2-\dot{y}_1)}\sigma_{z_f}}$$

where $\sigma_{(\dot{y}_2-\dot{y}_1)}$ is the standard deviation of the front suspension relative velocity. σ_{z_f} is the standard deviation of the Bouc–Wen model hysteretic displacement, $\rho_{(\dot{y}_2-\dot{y}_1)z_f}$ is the correlation coefficient between relative velocity and hysteretic displacement and $\Gamma(.)$ is the gamma function.

The power spectral density of the random road roughness is assumed to be in the form

$$S_h(\omega) = \frac{\sigma^2}{\pi}\frac{\alpha_r V}{(\omega^2 + (\alpha_r V)^2} \tag{16}$$

where σ^2 is the variance of the road irregularities, V is the vehicle forward velocity, ω is circular frequency and α_r is a coefficient depending on the type of road surface with $h(t)$ being the output of a linear first-order filter to white noise excitation expressed by

$$\dot{h}(t) + \alpha_r V h(t) = w(t) \tag{17}$$

where $w(t)$ is a zero-mean stationary Gaussian white noise process with covariance function $E[w(t)w^{\mathrm{T}}(t+\tau)] = 2\sigma^2 V\alpha_r\delta(\tau)$ where $\delta(.)$ is the Dirac delta function. Defining the state variables $x_1 = y_1$, $x_2 = \dot{y}_1$, $x_3 = y_2$, $x_4 = \dot{y}_2$, $x_5 = z_q$, $x_6 = h$, Eqs. (1), (2), (9) and (17) can be combined in matrix form as

$$\dot{x} = Fx + GU(t) + Dw(t) \tag{18}$$

where F, G, U, D, x are system matrix, control distribution vector, control vector, excitation distribution vector and state vector, respectively, given by

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{\alpha_1 k_1}{m_1} & -\frac{c}{m_1} & \frac{\alpha_1 k_1}{m_1} & \frac{c}{m_1} & \frac{(\alpha_1-1)k_1}{m_1} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{\alpha_1 k_1}{m_2} & \frac{c}{m_2} & -\frac{\alpha_1 k_1 + k_2}{m_2} & -\frac{c}{m_2} & \frac{(1-\alpha_1)k_1}{m_2} & \frac{k_2}{m_2} \\ 0 & c_h & 0 & -c_h & k_h & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha_r V \end{bmatrix} \tag{19}$$

$$G = \begin{bmatrix} 0 & \frac{1}{m_1} & 0 & \frac{-1}{m_2} & 0 & 0 \end{bmatrix}^{\mathrm{T}}; \quad D = [0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1]^{\mathrm{T}} \tag{20}$$

2.3. Performance criterion

Stochastic optimal control theory is applied to the equivalent linear system to obtain a fully active optimal control strategy. Defining the different performance measures, the sprung mass acceleration, suspension stroke, road holding and the control effort in terms of the mean square values given, respectively, by

$$J_1 = E[\ddot{y}_1^{\,2}] \tag{21}$$

$$J_2 = E[(y_1 - y_2)^2]; \quad J_3 = E[(y_2 - h)^2]; \quad J_4 = E[U^2] \tag{22}$$

the overall performance index J can be written as

$$J = \rho_1 J_1 + \rho_2 J_2 + \rho_3 J_3 + \rho_4 J_4 \tag{23}$$

where ρ_i, $i = 1, \ldots, 4$ are the weighting factors. These weighting factors may be chosen depending on the relative importance given to ride comfort and handling characteristics of the vehicle.

Eq. (23) can be expressed in the standard form as

$$J = E\left[[x^{\mathrm{T}} \;\; U^{\mathrm{T}}]\begin{bmatrix} A & N \\ N^{\mathrm{T}} & B \end{bmatrix}\begin{bmatrix} x \\ U \end{bmatrix}\right] \tag{24}$$

where matrices A and B are symmetric positive semi-definite and positive definite matrices, respectively.

Assuming that all the system states are measurable, the optimal control force minimizing the overall performance index is given by

$$U(t) = -Cx(t) \tag{25}$$

where $C = B^{-1}[N^{\mathrm{T}} + G^{\mathrm{T}}S]$ and S is the positive definite matrix, given by the solution of the following Riccati equation

$$S[F - GB^{-1}N^{\mathrm{T}}] + [F - GB^{-1}N^{\mathrm{T}}]S - SGB^{-1}G^{\mathrm{T}}S + [A - NB^{-1}N^{\mathrm{T}}] = 0 \tag{26}$$

Since, U is a linear function of state vector and since $x(t)$ is assumed to be approximately Gaussian as per the equivalent linearization scheme the control vector U is also approximately Gaussian.

The system response described by the zero-lag covariance matrix $P(t) = E[x(t)x(t)^{\mathrm{T}}]$ is obtained as the solution of the following Lyapunov equation:

$$[F - GC]P(t) + P(t)[F - GC]^{\mathrm{T}} + DQD^{\mathrm{T}} = 0 \tag{27}$$

2.4. Evaluation of performance index terms

The performance index terms J_1, J_2, J_3 and J_4 can be calculated in terms of the elements of the covariance matrix as follows [37]:

$$J_1 = E[\ddot{y}_1^2] = E\left[\sum_{i=1}^{6}(F(2,i) - C(i)/m_1)y_i\right]^2 \tag{28}$$

$$J_2 = P(1,1) - P(1,3) + P(3,3) \tag{29}$$

$$J_3 = P(3,3) - P(3,6) + P(6,6) \tag{30}$$

$$J_4 = \sum_{i=1}^{6}\sum_{i=1}^{6} C(i)C(j)P_{ij} \tag{31}$$

Since the equivalent linear parameters of the system given by Eqs. (10) and (11) are functions of the response statistics the control given by Eq. (25) and the solutions of the Riccati equation (26) and the Lyapunov equation (27) and the response statistics given by Eqs. (28)–(31) have to be obtained in an iterative manner. In the iterative scheme the control gains corresponding to the linear vehicle model are first computed. These control gains are used in obtaining the response statistics of the nonlinear vehicle model. Using the current response statistics of the nonlinear vehicle model the control gains are again computed. This iterative procedure is repeated until the values of the control gain and the response statistics converge to a specified degree of accuracy of the order of 10^{-4}. It has been observed that the convergence is achieved within four to five iterations.

3. Sky-hook control

3.1. Equations of motion with sky-hook damper

The equations of motion of the quarter car model with sky-hook damper as shown in Fig. 3 are given by

$$m_1\ddot{y}_1 + \alpha_1 k_1(y_1 - y_2) + (1-\alpha_1)k_1 z_q + c(\dot{y}_1 - \dot{y}_2) + c_s\dot{y}_1 - \alpha_s c_s \dot{y}_2 = 0 \tag{32}$$

$$m_2\ddot{y}_2 + \alpha_1 k_1(y_2 - y_1) - (1-\alpha_1)k_1 z_q + c(\dot{y}_2 - \dot{y}_1) + \alpha_s c_s(\dot{y}_2 - \dot{y}_1) + k_2(y_2 - h) = 0 \tag{33}$$

where α_s and c_s are parameters of the sky-hook damper. For $\alpha_s = 1$, Eqs. (32) and (33) reduce to that of the equations of motion with passive suspension.

In Eqs. (32) and (33), $c_s\dot{y}_1 - \alpha_s c_s\dot{y}_2$ and $\alpha_s c_s(\dot{y}_2 - \dot{y}_1)$, respectively, are the control force terms which are different. But as per the system equations (1) and (2) with active control the control force term U is the same in both the equations with opposite sign representing the proper feedback. Since in this paper, the performance of the semi-active sky-hook damper is sought to be enhanced to that of a fully active system by a mean square equivalence of the control forces, it is necessary that the control force terms in the sprung mass and unsprung mass equations of motion are the same implying proper feedback. Hence the sky-hook damper is approximated by a feedback control scheme suggested by Sammier et al. [18] with the following control law:

$$U_s = \alpha_s c_s(\dot{y}_2 - \dot{y}_1) + v \tag{34}$$

where $v = -(1-\alpha_s)c_s\dot{y}_2$.

Eqs. (32) and (33) can be written in state space form as after equivalent linearization as

$$\dot{x} = Fx(t) + Dw(t) \tag{35}$$

where $x = [x_1\ \ x_2\ \ x_3\ \ x_4\ \ x_5\ \ x_6\ ]^{\mathrm{T}}$ and $x_1 = y_1,\ \ x_2 = \dot{y}_1,\ \ x_3 = y_2,\ \ x_4 = \dot{y}_2,\ \ x_5 = z_q,\ \ x_6 = h$

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \dfrac{-\alpha_1 k_1}{m_1} & \dfrac{-(c_s + c)}{m_1} & \dfrac{\alpha_1 k_1}{m_1} & \dfrac{\alpha_s c_s + c}{m_1} & \dfrac{(1-\alpha_1)k_1}{m_1} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \dfrac{\alpha_1 k_1}{m_2} & \dfrac{\alpha_s c_s + c}{m_2} & \dfrac{-(\alpha_1 k_1 + k_2)}{m_2} & \dfrac{-(\alpha_s c_s + c)}{m_2} & \dfrac{(1-\alpha_1)k_1}{m_2} & \dfrac{k_2}{m_2} \\ 0 & C_h & 0 & -C_h & K_h & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha_r V \end{bmatrix}; \quad d = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{36}$$

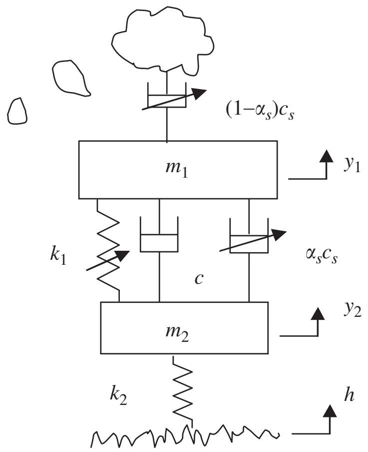

Fig. 3. Quarter car model with sky-hook suspension.

with C_h and K_h as before are given in Eqs. (10) and (11). The system response described by the zero-lag covariance matrix $P(t) = E[x(t)x(t)^{\mathrm{T}}]$ is obtained as the solution of the following Lyapunov equation:

$$\dot{P}(t) = FP(t) + P(t)F^{\mathrm{T}} + DQD^{\mathrm{T}} \tag{37}$$

For obtaining the stationary response we set $\dot{P}(t) = 0$.

As in the case of the LQR control, the response statistics in the case of the vehicle with feedback control based on sky-hook are obtained iteratively as the equivalent linear parameters are functions of the response statistics.

3.2. Optimal sky-hook damper parameters

Given the sky-hook damper parameters α_s and c_s the mean square value of the feedback control force as per Eq. (34) and the response statistics of the vehicle can be obtained for a particular velocity.

Since the aim is to find optimal values for parameters α_s and c_s of the feedback control based on sky-hook damper which will match the performance of the fully active suspension, the following method is adopted. The value of α_s is varied in the admissible range (0–1) and a corresponding value of c_s is obtained for a particular velocity equating the mean square value of the control force corresponding to the feedback control based on sky-hook damper as per Eq. (34) to the mean square value of the control force obtained using the LQR control as per Eq. (25). This is given by

$$c_s = \sqrt{\frac{E[U^2]}{E[(\dot{y}_1 - \alpha_s \dot{y}_2)^2]}} \tag{38}$$

Thus for a given velocity, a set of α_s and c_s values are obtained which ensures the mean square equivalence of the control force corresponding to the feedback control based on sky-hook damper and the LQR control. However, these values may not be optimal, in that the response statistics of the vehicle using the feedback control with these parameters may not be as good as those obtained by the LQR control. Thus there is a need to choose one combination of α_s and c_s which not only is equivalent in terms of the control force generated by the LQR control theory but will also match the vehicle performance of the LQR. This can be effected by minimizing the rms difference between the vehicle performance of the feedback control based on sky-hook damper and LQR control with respect to the overall vehicle response, including sprung mass acceleration, suspension stroke, road holding, etc. Thus the α_s and c_s values obtained correspond to the minimum rms difference between the overall performance of the vehicle given by

$$\min \sqrt{\sum_{i=1}^{3} ((J_i)_{SH} - (J_i)_{LQR})} \tag{39}$$

where the subscripts SH and LQR refer to sky-hook control and LQR control, respectively, and the J_i's are given by Eq. (22). Thus one combination of optimal values of α_s and c_s for a particular velocity of the vehicle is obtained from the point view of equivalent control force and equivalent vehicle performance of the feedback control based on sky-hook with that of LQR control.

The optimal values of α_s and c_s for a particular velocity obtained as described above may not be optimum for different velocities in the normal range of operation of the vehicle. An average of the optimal α_s and c_s values over a velocity range above a specific value is considered to be adequate for the sky-hook damper to match the performance of the active suspension over this velocity range as these values tend to reach almost constant values beyond a velocity which is around 15 m/s.

4. Results and discussion

Typical results of the response statistics of the nonlinear quarter car vehicle model of the semi-active suspensions with optimal sky-hook damper parameters and active suspensions using LQR control are presented here. The parameters of the vehicle model, the road profile and the weighting factors are

$m_1 = 1000\,\text{kg}$, $m_2 = 100\,\text{kg}$, $k_1 = 36\,000\,\text{N/m}$, $k_2 = 360\,000\,\text{N/m}$, $c = 1000\,\text{N s/m}$, $\alpha_r = 0.15\,\text{rad/m}$, $\sigma^2 = 9 * 10^{-6}\,\text{m}^2$, $\rho_1 = 1$, $\rho_2 = 10^4$, $\rho_3 = 10^4$, $\rho_4 = 10^{-6}$. These values are similar to that adopted in Refs. [22,37,42].

The Bouc–Wen model parameters for the nonlinear suspension are assumed to be $A_q = 1.5$, $\gamma_q = 0.5$, $\beta_q = 0.5$, $\alpha_1 = 0.2$.

The optimal values of the sky-hook damper parameters matching the parameters of the LQR control as per the procedure described in Section 3.2 are $\alpha_s = 0.194$, $c_s = 5804.9\,\text{N s/m}$.

The overall performance, rms sprung mass acceleration, suspension stroke and road holding of the nonlinear quarter car vehicle model with LQR control and equivalent semi-active sky-hook damper control are presented in Figs. (4)–(7), respectively. From Figs. (4)–(6) it can be observed that the overall performance of the vehicle, the sprung mass acceleration response and the suspension stroke response for the LQR control and for the feedback control based on sky-hook damper are significantly better than the vehicle performance with the passive suspension. It is further observed that the performance of the vehicle with the optimal feed back control parameters based on the sky-hook damper are almost as good as that of the active suspension with LQR control with respect to these responses.

From Fig. 7 it can be observed that the vehicle performance with respect to road holding using LQR control is much better than that of the passive suspension while the road holding response using the feedback control based on sky-hook damper with optimal parameters performs marginally better than even that of the active suspension with LQR.

It is seen from Fig. 8 that the rms control force for the feedback control based on sky-hook strategy is almost the same as for the LQR control as it should be since the feedback control parameters are obtained through a mean square equivalence of the two control forces. While for lower velocities the rms control force for the sky-hook control is almost the same as the rms control force for LQR control, at higher velocities the rms control force for the feedback control based on sky-hook is marginally higher than that of the LQR control. This is due to the fact that the optimum sky-hook damper parameters are obtained by averaging the optimum values of the parameters obtained for different velocities of the vehicle as explained in Section 3.2.

4.1. Monte Carlo simulation

To verify the response statistics obtained for the nonlinear quarter car vehicle model with LQR control and feedback control based on sky-hook damper control for the random road excitation by the equivalent

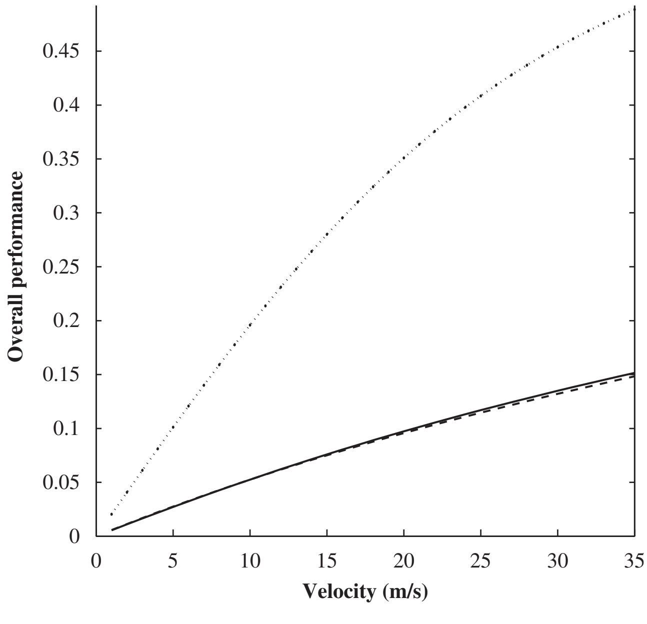

Fig. 4. Overall response for different velocities of the quarter car vehicle model with LQR control. (…) passive, (- - -) LQR, (—) optimal sky-hook.

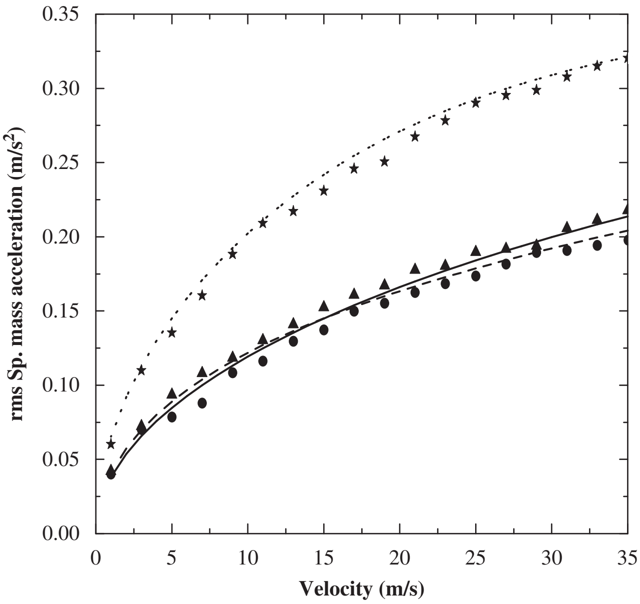

Fig. 5. The rms acceleration response: equivalent linearization results, (—) optimal sky-hook; (- - -) LQR; (…) passive; Monte Carlo simulation results, (▲) optimal sky-hook; (•) LQR; (⋆) passive.

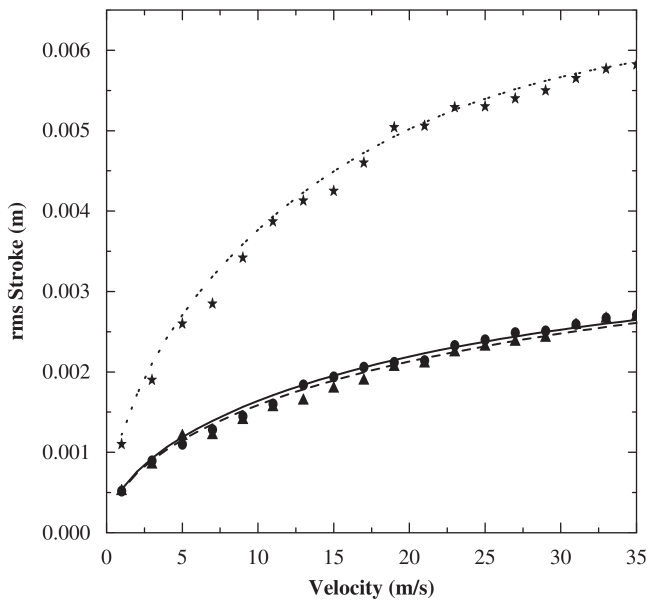

Fig. 6. The rms stroke response with equivalent linearization and Monte Carlo simulation. (Legends as in Fig. 5.)

linearization technique, Monte Carlo simulation studies are carried out. Time histories of the road input $h(t)$ corresponding to the power spectral density function given in Eq. (17) are generated using the method given by Shinozuka and Jan [43]. The time histories are generated by the following series:

$$h(t) = \sqrt{2}\sum_{k=1}^{N}[G_h(\omega_k)\,\Delta\omega]^{1/2}\cos(\omega_k' + \phi_k) \tag{40}$$

where $G_h(\omega_k) = 2S_h(\omega_k)$ is the one sided power spectral density function at frequency ω_k, with $\omega_k = \omega_l + (k - \frac{1}{2})\Delta\omega$, $k = 1, 2, \ldots, N$; $\omega_k' = \omega_k + \delta\omega$, $k = 2, ..., N$ and $\Delta\omega = (\omega_u - \omega_l)/N$ with N being the

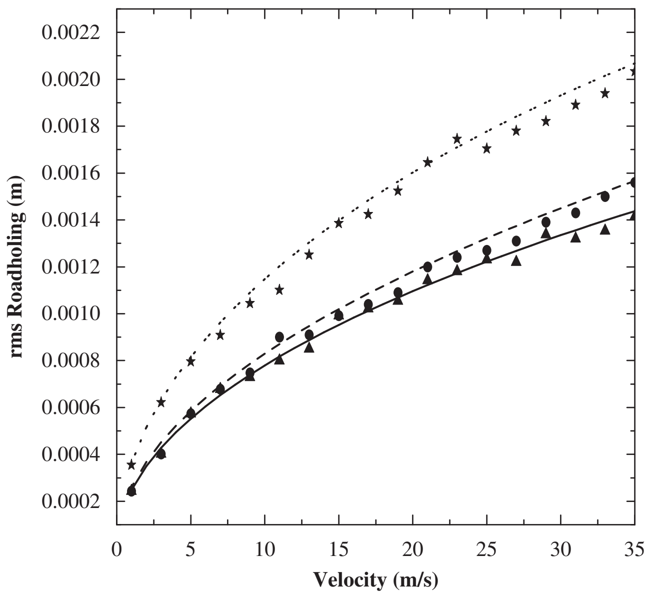

Fig. 7. The rms road holding response with equivalent linearization and Monte Carlo simulation. (Legends as in Fig. 5.)

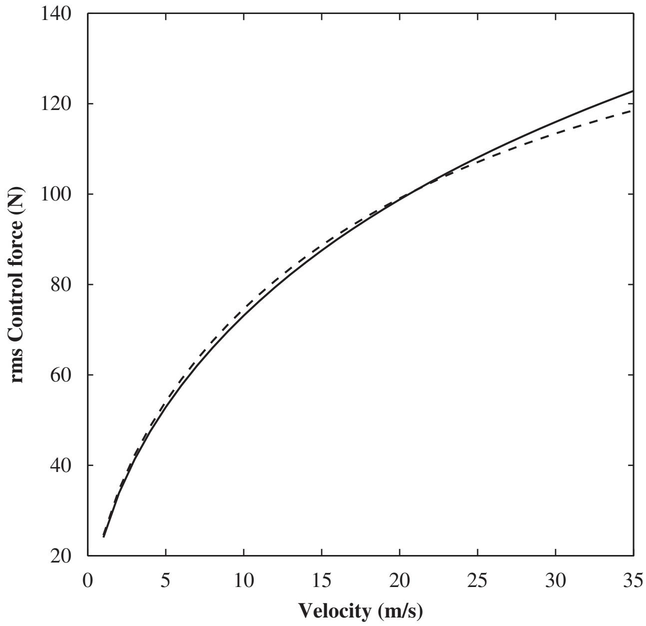

Fig. 8. The rms control force of nonlinear quarter car vehicle model. (Legends as in Fig. 4.)

number of equal intervals in which the frequency interval is divided. $\delta\omega$ is a small random frequency uniformly distributed between $-\Delta\omega'/2$ and $\Delta\omega'/2$ with $\Delta\omega' \ll \Delta\omega$ introduced to avoid the periodicity of the process which is simulated. ϕ_k's are independent random phase angles uniformly distributed in the interval 0 to 2π. ω_l and ω_u are, respectively, the lower and upper cut-off frequencies. The time history $h_f(t)$ is generated with $N = 1000$, $\omega_l = 0$, $\omega_u = 2\pi * 100\,\text{rad/s}$, $\Delta\omega' = 0.05\,\Delta\omega$.

From the generated time history $h(t)$ the power spectral density function of the road input is obtained by using the MATLAB function 'psd' and is compared with the target power spectral density corresponding to Eq. (16) in Fig. 9 showing good agreement between the target psd and the simulated psd. The equations of

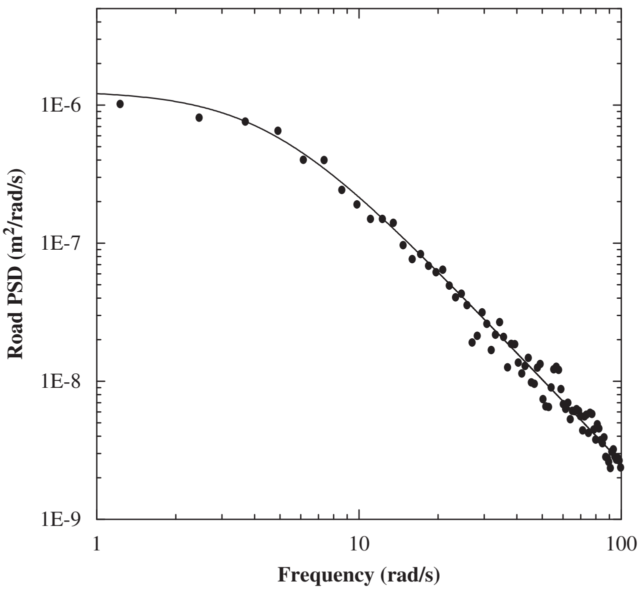

Fig. 9. Simulation '•' and target '–' psd of road profile.

motion (1), (2) and (17) with the nonlinear equations of the Bouc–Wen model equations (3) are numerically integrated with the control input given by Eq. (25) in the case of LQR control and by Eq. (34) in the case of feed back control based on sky-hook control. The control input $U(t)$ is obtained as explained in Sections 2.3 and 3.1 for the equivalent linear model. The response statistics, rms sprung mass acceleration, rms stroke and rms roadholding are obtained from the generated time histories with LQR control and the feedback control based on sky-hook damper with optimum parameters and for the passive system. These are plotted by the symbols '⋆', '•' and '▲', respectively, in Figs. 5–7 along with the results obtained by the equivalent linearization method. The agreement between the Monte Carlo simulated results and the results of the equivalent linearization method are very good validating that the equivalent linearization technique can be used not only for obtaining the response statistics of the nonlinear passive system but can also be effectively used for the control problem so that linear optimal control theories can be applied to the equivalent linear system.

5. Conclusions

In this paper, the control of the stationary response of a quarter car vehicle model with nonlinear stiffness traversing a rough road by use of feedback control based on sky-hook damper strategy is considered. The sky-hook damper control is realized through a feed back control scheme. A new method as explained in Section 3.2 is proposed to choose optimal parameters of the feedback control based on sky-hook damper. Optimal parameters are obtained by matching the control force of the feedback control based on sky-hook damper with that of the fully active suspension using LQR in a mean square equivalence sense. The optimal parameters are also chosen on the basis of reduction in the rms error in comparison with the fully active control effort based on the rms acceleration, rms stroke and rms road holding respectively. The performance of the feedback control based on sky-hook damper suspension is sought to be enhanced to the levels of performance of the fully active suspension with LQR control. The results corroborate the expectation of the enhanced performance of the feedback control based on sky-hook damper with optimal parameters almost to the levels of the fully active suspension.

From the results, it can be concluded that the quarter car vehicle model with the nonlinear suspension of the hysteretic type modeled by the Bouc–Wen model can be linearized using the equivalent linearization method and optimal control strategies such as the LQR and the feedback control based on sky-hook damper control can be successfully applied to the equivalent linear model. It can also be concluded that the performance of the semi-active suspension such as the sky-hook control can be improved to the levels of performance of a fully active suspension by a suitable choice of the semi-active suspension parameters such that the control forces in both the cases are mean square equivalent.

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