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The decoupling of damped linear systems in oscillatory free vibration

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Abstract

The purpose of this paper is to extend classical modal analysis to decouple any viscously damped linear system in oscillatory free vibration. Based upon an exposition of how viscous damping causes phase drifts in the components of a linear system, the concept of non-classically damped modes of vibration is introduced. These damped modes are real and physically excitable. By synchronizing the phase angles in each damped mode, a time-varying transformation is constructed to decouple damped oscillatory free vibration. The decoupling procedure devised herein reduces to classical modal analysis for systems that are undamped or classically damped. This paper constitutes the first part of a solution to the ''classical decoupling problem'' of linear systems.

1. Introduction

It is well known that an undamped linear dynamical system possesses classical normal modes, and that in each mode different parts of the system vibrate in a synchronous manner. The normal modes constitute a modal matrix, which defines a linear coordinate transformation that decouples the undamped system. This process of decoupling the equation of motion of a dynamical system is a time-honored procedure termed modal analysis. Upon decoupling, an undamped linear system can be treated as a series of independent single-degree-of-freedom systems.

In the presence of damping, a linear system cannot be decoupled by modal analysis unless it possesses a full set of classical normal modes, in which case the damped linear system is said to be classically damped. Rayleigh [1] showed that a system is classically damped if its damping matrix is a linear combination of its inertia and stiffness matrices. A damped system possessing such a special property is said to be proportionally damped. Classical and in particular proportional damping is routinely assumed in design and computations.

There is no reason why damping in a linear system should be classical. Practically speaking, classical damping means that energy dissipation is almost uniformly distributed throughout the system. This assumption is violated for systems consisting of two or more parts with significantly different levels of

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damping. Examples of such systems include soil–structure systems [2], base-isolated structures [3–5], and systems in which coupled vibrations of structures and fluids occur. Increasing use of special energy-dissipating devices in control constitutes another important example. In fact, experimental modal testing suggests that no physical system is strictly classically damped [6]. Linear systems are generally non-classically damped.

The purpose of this paper is to develop a method to decouple any viscously damped linear system in oscillatory free vibration. The method devised is an extension of classical modal analysis, which is only applicable to undamped or classically damped systems. Based upon an exposition of the mechanics of viscous damping, a real time-varying transformation is constructed in the configuration space to decouple oscillatory free vibration. The organization of this paper is as follows. In Section 2, the traditional theory of coordinate coupling in linear dynamical systems is concisely surveyed. This survey sets up the terminology and notation used throughout the paper. The problem of decoupling is then formulated in Section 3, for which the quadratic eigenvalue problem is called into play. Assumptions that are not essential but are made for gaining physical insight are discussed. The concept of non-classically damped modes of vibration is introduced in Section 4. An exposition of how an oscillatory system can be decoupled by phase synchronization of its damped modes is given in Section 5. It is also shown that the method of phase synchronization reduces to classical modal analysis for systems that are undamped or classically damped. In Section 6, three examples are given to illustrate the process of decoupling. A summary of findings is provided in Section 7. This paper constitutes the first part of a method to decouple any damped linear dynamical system. Non-oscillatory free vibration and forced vibration will be considered in a future paper.

2. Coordinate coupling in damped linear systems

The equation of motion of an n-degree-of-freedom viscously damped linear system in free vibration can be written in the form

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}, \tag{1}$$

where the generalized coordinate

$$\mathbf{q} = [q_1 \;\; q_2 \;\; \cdots \;\; q_n]^{\mathrm{T}}. \tag{2}$$

For passive systems, the mass matrix $\mathbf{M}$, damping matrix $\mathbf{C}$, and stiffness matrix $\mathbf{K}$ are real, symmetric and positive definite of order n. These characteristics are not arbitrary, but in fact have solid footing in the theory of Lagrangian dynamics. For example, symmetry of $\mathbf{M}$ is based upon the fact that the quadratic form of kinetic energy can always be defined in terms of a symmetric matrix. In addition, any linear system is passive if the rigid-body modes are eliminated, which is not an essential restriction at all. In general, the matrix differential equation (1) is coupled. The ith component equation involves not only q_i and its derivatives but also other coordinates and their derivatives as well. Coupling is not an inherent property of a system but depends on the generalized coordinates used.

2.1. Decoupling by classical modal analysis

Associated with the undamped system is a symmetric eigenvalue problem [7]

$$\mathbf{K}\mathbf{u} = \lambda\mathbf{M}\mathbf{u}. \tag{3}$$

Owing to the positive definiteness of the matrices $\mathbf{M}$ and $\mathbf{K}$, all eigenvalues λ_i are real and positive, and the corresponding natural modes $\mathbf{u}_i$ are real and orthogonal with respect to either $\mathbf{M}$ or $\mathbf{K}$. Define the modal and spectral matrices, respectively, by

$$\mathbf{U} = [\mathbf{u}_1|\mathbf{u}_2|\cdots|\mathbf{u}_n], \tag{4}$$

$$\mathbf{\Omega} = \mathrm{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]. \tag{5}$$

Upon normalization of the natural modes with respect to the mass matrix, the generalized orthogonality of the modes can be expressed in a compact form:

$$\mathbf{U}^{\mathrm{T}}\mathbf{M}\mathbf{U} = \mathbf{I}, \tag{6}$$

$$\mathbf{U}^{\mathrm{T}}\mathbf{K}\mathbf{U} = \mathbf{\Omega}. \tag{7}$$

Define a modal transformation by

$$\mathbf{q} = \sum_{i=1}^{n} p_i(t)\mathbf{u}_i = \mathbf{U}\mathbf{p}. \tag{8}$$

In terms of the principal coordinate $\mathbf{p}$, the equation of damped free vibration takes the canonical form

$$\ddot{\mathbf{p}} + \mathbf{D}\dot{\mathbf{p}} + \mathbf{\Omega}\mathbf{p} = \mathbf{0}, \tag{9}$$

where the symmetric matrix

$$\mathbf{D} = \mathbf{U}^{\mathrm{T}}\mathbf{C}\mathbf{U} \tag{10}$$

is referred to as the modal damping matrix. Note that the mass matrix $\mathbf{M}$ and the stiffness matrix $\mathbf{K}$ have been diagonalized by modal transformation. Thus an undamped system can always be decoupled by modal analysis. Any coupling in a linear system occurs ultimately through damping.

2.2. Inadequacy of classical modal analysis

A system is classically damped if it can be decoupled by classical modal analysis, whereby the modal damping matrix $\mathbf{D}$ in Eq. (10) is diagonal. In Section 97 of "The Theory of Sound" in 1894, Rayleigh [1] asserted that a system is classically damped if

$$\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K} \tag{11}$$

for some scalar constants α and β. This requirement, referred to as proportional damping, is sufficient but not necessary for classical damping. In 1965, Caughey and O'Kelly [8] established that a necessary and sufficient condition under which a system is classically damped is

$$\mathbf{C}\mathbf{M}^{-1}\mathbf{K} = \mathbf{K}\mathbf{M}^{-1}\mathbf{C}. \tag{12}$$

There is, of course, no particular reason why condition (12) should be satisfied. In general, a linear dynamical system is non-classically damped and it cannot be decoupled by classical modal analysis.

2.3. Inadequacy of state space approach

Classical modal analysis utilizes a real transformation (8). Foss and others [9–11] extended classical modal analysis to a process of complex modal analysis in the state space to treat non-classically damped systems. However, the state-space approach has never appealed to practicing engineers. There are several reasons for this situation. A common excuse is that the state-space approach is computationally more involved because the dimension of the state space is twice the number of degrees of freedom. Another reason is that complex modal analysis still cannot decouple all non-classically damped systems. A condition of non-defective eigenvectors in the state space must be satisfied in order for complex modal analysis to achieve complete decoupling. More importantly, there is little physical insight associated with different elements of complex modal analysis. Classical modal analysis is amenable to physical interpretation. For example, each normal mode $\mathbf{u}_i$ represents a physical profile of vibration. Even the eigenvalue problem (3) can be interpreted geometrically as the problem of finding the principal axes of an n-dimensional ellipsoid.

3. Problem statement

The "classical decoupling problem" is concerned with the elimination of coordinate coupling in damped linear dynamical systems. It is a well-trodden problem that has attracted the attention of many researchers in the past century. In "The Theory of Sound" in 1894, Rayleigh [1] already expounded on the significance of system decoupling and introduced the concept of proportional damping. Over the years, various types of decoupling approximation were employed in the analysis of damped systems [12–21]. Different indices of coupling were also introduced to quantify coordinate coupling [22–29]. However, a solution to the "classical decoupling problem" has not been reported in the open literature.

Mathematically, the "classical decoupling problem" is equivalent to the problem of simultaneous conversion of **M**, **C**, and **K** into diagonal forms. Ma and Caughey [30] showed that no time-invariant linear transformations in the configuration space will decouple every damped linear system. Even partial decoupling, i.e. simultaneous conversion of **M**, **C**, and **K** into upper triangular forms, is not ensured with time-invariant linear transformations [31]. As a consequence, any universal decoupling transformation in the configuration space, if it exists, must be at least time-varying or even nonlinear.

3.1. The quadratic eigenvalue problem

Postulate that system (1) has a solution of the form

$$\mathbf{q} = \mathbf{v}e^{\lambda t}, \tag{13}$$

where $\mathbf{v}$ is an n-dimensional column vector of unspecified constants and λ is a scalar parameter. Upon substitution into Eq. (1), a quadratic (nonlinear) eigenvalue problem [32,33]

$$(\mathbf{M}\lambda^2 + \mathbf{C}\lambda + \mathbf{K})\mathbf{v} = \mathbf{0} \tag{14}$$

is obtained. There are $2n$ eigenvalues λ_i but there cannot be more than n linearly independent eigenvectors $\mathbf{v}_i$ where $i = 1, \ldots, 2n$. The eigenvalues are the roots of the polynomial equation

$$\det(\mathbf{M}\lambda^2 + \mathbf{C}\lambda + \mathbf{K}) = 0. \tag{15}$$

Since the coefficients of the above polynomial are real, any complex roots must occur in complex conjugate pairs. The corresponding eigenvectors are also complex conjugates. In addition, the real parts of all roots must be negative because energy is dissipated by damping.

The general solution of system (1) can be expressed in terms of the eigenvalues λ_i and the corresponding eigenvectors $\mathbf{v}_i$. If the eigenvalues are distinct, the solution is a linear combination of the form

$$\mathbf{q} = \sum_{i=1}^{2n} c_i \mathbf{v}_i e^{\lambda_i t}, \tag{16}$$

where c_i are $2n$ constants to be obtained from initial conditions.

3.2. Oscillatory nature of solution

Pre-multiply Eq. (14) by the complex conjugate transpose $\mathbf{v}^*$ to obtain

$$\mathbf{v}^*\mathbf{M}\mathbf{v}\lambda^2 + \mathbf{v}^*\mathbf{C}\mathbf{v}\lambda + \mathbf{v}^*\mathbf{K}\mathbf{v} = 0. \tag{17}$$

The roots of the above quadratic equation are given by

$$\lambda = \frac{-\mathbf{v}^*\mathbf{C}\mathbf{v} \pm \sqrt{(\mathbf{v}^*\mathbf{C}\mathbf{v})^2 - 4(\mathbf{v}^*\mathbf{M}\mathbf{v})(\mathbf{v}^*\mathbf{K}\mathbf{v})}}{2\mathbf{v}^*\mathbf{M}\mathbf{v}}. \tag{18}$$

The response of system (1) is purely oscillatory if all eigenvalues are complex, which is guaranteed if the discriminant

$$\Delta(\mathbf{x}) = (\mathbf{x}^*\mathbf{C}\mathbf{x})^2 - 4(\mathbf{x}^*\mathbf{M}\mathbf{x})(\mathbf{x}^*\mathbf{K}\mathbf{x}) < 0 \tag{19}$$

for arbitrary n-dimensional vector $\mathbf{x} \neq \mathbf{0}$. In this case,

$$\mathrm{Re}[\lambda] = \alpha = -\frac{\mathbf{v}^* \mathbf{C} \mathbf{v}}{2\mathbf{v}^* \mathbf{M} \mathbf{v}} < 0, \tag{20}$$

because $\mathbf{M}$ and $\mathbf{C}$ are positive definite.

3.3. Assumptions and possible relaxation

It will be assumed that damping is light so that the $2n$ eigenvalues λ_i of the quadratic eigenvalue problem (14) are: (a) complex and (b) distinct. These assumptions are made to streamline the introduction of new concepts and they can be readily relaxed. First, some eigenvalues will be real if assumption (a) is relaxed. This issue will be resolved in a subsequent paper [34] on the decoupling of systems in non-oscillatory free vibration and in forced vibration. Second, assumption (b) can be relaxed effortlessly to the less restrictive condition that eigenvectors associated with repeated eigenvalues are independent. There will be little change to the exposition of decoupling. For example, Eq. (16) remains the general solution of system (1) as long as there is a full complement of independent eigenvectors associated with each repeated eigenvalue.

When an eigenvalue is repeated m times and a full complement of m independent eigenvectors cannot be found, the eigenvalue problem (14) is said to be defective. As an example, Eq. (14) must be defective if any eigenvalue is repeated more than n times [32]. Relaxation of assumption (b) to include defective eigenvalue problems is of a purely theoretical nature but is still relatively straightforward. However, physical insight is obscured due to the occurrence of Jordan sub-matrices in many equations. A numerical example will be provided to demonstrate how defective problems can be decoupled. Should the methodology expounded in this paper be accepted for use, a thorough treatment of defective problems will become more deserving.

Perhaps an alternative viewpoint of the significance of defective eigenvalue problems should be brought up. It is recognized that the probability of a randomly chosen (uniformly distributed in a bounded continuous domain) square matrix being singular is zero. Since a square matrix is singular if and only if its column vectors are dependent, the probability that a set of randomly chosen column vectors being dependent, therefore, is zero. The eigenvectors of Eq. (14) are continuous functions of $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ (except on a set of measure zero). If these symmetric and positive definite matrices are random (uniformly distributed), the resulting eigenvectors are also random. Therefore, loosely speaking, the probability that the eigenvalue problem (14) being defective is zero.

4. Damped modes of vibration

As explained earlier, it is assumed for convenience that all eigenvalues of Eq. (14) are complex and distinct. The eigenvalues λ_j and the corresponding eigenvectors $\mathbf{v}_j$ occur in n pairs of complex conjugates. Let

$$\lambda_j = \alpha_j + \mathrm{i}\omega_j, \tag{21}$$

$$\mathbf{v}_j = [r_{j1}\mathrm{e}^{-\mathrm{i}\phi_{j1}} \;\; r_{j2}\mathrm{e}^{-\mathrm{i}\phi_{j2}} \;\; \cdots \;\; r_{jn}\mathrm{e}^{-\mathrm{i}\phi_{jn}}]^{\mathrm{T}} \tag{22}$$

constitute an eigensolution, where r_{jk}, ϕ_{jk} are real parameters and $j,k = 1,\ldots,n$. Then $\overline{\lambda}_j = \alpha_j - i\omega_j$ and $\overline{\mathbf{v}}_j$ also constitute an eigensolution. Note that $\alpha_j < 0$ and each eigenvector $\mathbf{v}_j$ can only be determined within an arbitrary multiplicative constant.

The two eigensolutions $\mathbf{v}_j\mathrm{e}^{\lambda_j t}$ and $\overline{\mathbf{v}}_j\mathrm{e}^{\overline{\lambda}_j t}$ combine to generate a damped mode of vibration defined by the linear combination

$$\mathbf{s}_j(t) = a_j\mathbf{v}_j\mathrm{e}^{(\alpha_j+\mathrm{i}\omega_j)t} + b_j\overline{\mathbf{v}}_j\mathrm{e}^{(\alpha_j-\mathrm{i}\omega_j)t} \tag{23}$$

for $j = 1,\ldots,n$. If damped harmonic vibration with frequency ω_j is physically realizable, $\mathbf{s}_j(t)$ must be real. This implies that $b_j = \overline{a}_j$ and, as a result,

$$\mathbf{s}_j(t) = 2\,\mathrm{Re}[a_j\mathbf{v}_j\mathrm{e}^{(\alpha_j+\mathrm{i}\omega_j)t}] = 2\mathrm{e}^{\alpha_j t}\,\mathrm{Re}[a_j\mathbf{v}_j\mathrm{e}^{\mathrm{i}\omega_j t}]. \tag{24}$$

Write $2a_j$ in polar form so that

$$2a_j = C_j e^{-i\theta_j}, \tag{25}$$

where C_j and θ_j are real. It follows that:

$$\mathbf{s}_j(t) = C_j e^{\alpha_j t}\,\mathrm{Re}[\mathbf{v}_j e^{i(\omega_j t-\theta_j)}]. \tag{26}$$

Substitute Eq. (22) into (26) to obtain

$$\mathbf{s}_j(t) = C_j e^{\alpha_j t}\begin{bmatrix} r_{j1}\cos(\omega_j t-\theta_j-\phi_{j1}) \\ r_{j2}\cos(\omega_j t-\theta_j-\phi_{j2}) \\ \vdots \\ r_{jn}\cos(\omega_j t-\theta_j-\phi_{jn}) \end{bmatrix}. \tag{27}$$

In the literature, the $2n$ eigensolutions $\mathbf{v}_j e^{\lambda_j t}$ of Eq. (14) are sometimes referred to as the complex modes in configuration space [7,35]. The n damped modes $\mathbf{s}_j(t)$ defined herein are real, physically excitable, and are essentially the same as the real parts of the conjugate complex modes. These real modes $\mathbf{s}_j(t)$ coincide with the classical natural modes for an undamped or classically damped system. It can be deduced from Eq. (16) that

$$\mathbf{q}(t) = \sum_{j=1}^{n}(a_j\mathbf{v}_j e^{\lambda_j t} + \overline{a}_j\overline{\mathbf{v}}_j e^{\overline{\lambda}_j t}) = \sum_{j=1}^{n}\mathbf{s}_j(t). \tag{28}$$

Thus the general solution of system (1) is simply a superposition of n damped modes of vibration.

4.1. Classically damped modes

A classically damped system possesses a full set of classical natural modes [8]. Both λ_j and $\overline{\lambda}_j$ are associated with a real eigenvector $\mathbf{v}_j$, which coincides with a classical natural mode, say $\mathbf{u}_j$, of the symmetric eigenvalue problem (3). In other words, $\phi_{j1} = \phi_{j2} = \cdots = \phi_{jn} = 0$ in Eq. (22) and

$$\mathbf{u}_j = \mathbf{v}_j = \overline{\mathbf{v}}_j = [r_{j1}\ \ r_{j2}\ \ \cdots\ \ r_{jn}]^{\mathrm{T}}. \tag{29}$$

Thus a classically damped mode of vibration has the functional form

$$\mathbf{s}_j(t) = C_j e^{\alpha_j t}\cos(\omega_j t-\theta_j)\mathbf{u}_j. \tag{30}$$

The above expression represents synchronous motion in which all system components perform harmonic motion with the same damped frequency ω_j, passing through their equilibrium positions at the same instant of time. For an undamped system, $\alpha_j = 0$ and $\lambda_j = i\omega_j$, implying that the amplitude of modal vibration does not decrease and that ω_j is a natural frequency.

4.2. Non-classically damped modes

In a non-classically damped system, the eigenvectors $\mathbf{v}_j$ are complex. In each damped mode of vibration all system components perform harmonic vibration with identical frequency ω_j and with the same exponential decay α_j. However, elements of the modal vector $\mathbf{s}_j(t)$ in Eq. (27) generally possess different phase angles ϕ_{jk}. As a consequence, the system components do not pass through their equilibrium positions at the same time. On the other hand, the phase difference between any two elements in $\mathbf{s}_j(t)$ is constant. Thus the order in which the system components pass through their equilibrium positions remains unchanged. After one complete cycle the components return to positions separated by the same phase angles as in the beginning of the cycle. There exists an unchanging pattern from cycle to cycle while the motion decays exponentially. Although the pattern is well defined it is not as easy to recognize as a classical natural mode. Each non-classically damped mode $\mathbf{s}_j(t)$,

for arbitrarily selected C_j and θ_j, can indeed be excited with the initial conditions

$$\mathbf{q}(0) = C_j \begin{bmatrix} r_{j1}\cos(\theta_j + \phi_{j1}) \\ r_{j2}\cos(\theta_j + \phi_{j2}) \\ \vdots \\ r_{jn}\cos(\theta_j + \phi_{jn}) \end{bmatrix}, \tag{31}$$

$$\dot{\mathbf{q}}(0) = \alpha_j C_j \begin{bmatrix} r_{j1}\cos(\theta_j + \phi_{j1}) \\ r_{j2}\cos(\theta_j + \phi_{j2}) \\ \vdots \\ r_{jn}\cos(\theta_j + \phi_{jn}) \end{bmatrix} + \omega_j C_j \begin{bmatrix} r_{j1}\sin(\theta_j + \phi_{j1}) \\ r_{j2}\sin(\theta_j + \phi_{j2}) \\ \vdots \\ r_{jn}\sin(\theta_j + \phi_{jn}) \end{bmatrix}. \tag{32}$$

Finally, in an undamped or classically damped mode there may be non-clamped positions at which the displacement is zero at all times. These positions, referred to as nodes, are fixed in space. In contrast, the positions of zero displacement (instantaneous nodes) in a non-classically damped mode drift around in space, as will be illustrated in a numerical example. There are generally not any fixed positions for mounting additional equipment without disturbing a non-classically damped mode.

5. The mechanics of decoupling of oscillatory systems

What is the essential difference between a classically damped mode and a non-classically damped mode? In a classically damped mode of vibration, the various components of a system are either in phase or out of phase with each other, and they pass through their equilibrium positions at the same instant of time. This is not the case for a non-classically damped mode. Indeed, a classically damped mode has the property that $\phi_{j1} = \phi_{j2} = \cdots = \phi_{jn} = 0$ while not all phase angles ϕ_{jk} are zero in a non-classically damped mode. If suitable phase shifts are introduced into each non-classically damped mode so that all components are either in phase or out of phase, it is possible to transform a non-classically damped system into one with classical damping. A basic objective of a process termed phase synchronization is to do just that.

5.1. Phase synchronization of damped modes

Define a phase shift of the damped mode $\mathbf{s}_j(t)$ into $\mathbf{y}_j(t)$ by

$$\mathbf{y}_j(t) = \begin{bmatrix} y_{j1}(t) \\ y_{j2}(t) \\ \vdots \\ y_{jn}(t) \end{bmatrix} = \begin{bmatrix} s_{j1}(t + \phi_{j1}/\omega_j) \\ s_{j2}(t + \phi_{j2}/\omega_j) \\ \vdots \\ s_{jn}(t + \phi_{jn}/\omega_j) \end{bmatrix}. \tag{33}$$

It follows from Eq. (27) that

$$\mathbf{y}_j(t) = C_j e^{\alpha_j t}\cos(\omega_j t - \theta_j) \begin{bmatrix} r_{j1}e^{\alpha_j\phi_{j1}/\omega_j} \\ r_{j2}e^{\alpha_j\phi_{j2}/\omega_j} \\ \vdots \\ r_{jn}e^{\alpha_j\phi_{jn}/\omega_j} \end{bmatrix}. \tag{34}$$

A system that possesses $\mathbf{y}_j(t)$ as its damped modes of vibration is classically damped. What is the equation of motion of such a system? To answer this question, let

$$\mathbf{z}_j = [r_{j1}e^{\alpha_j\phi_{j1}/\omega_j} \;\; r_{j2}e^{\alpha_j\phi_{j2}/\omega_j} \;\; \cdots \;\; r_{jn}e^{\alpha_j\phi_{jn}/\omega_j}]^{\mathrm{T}} \tag{35}$$

for $j=1,\ldots,n$. If $\mathbf{z}_1,\ldots,\mathbf{z}_n$ are linearly independent, define a real modal matrix $\mathbf{Z}$ with $\mathbf{z}_j$ as its columns such that

$$\mathbf{Z}=[\mathbf{z}_1|\mathbf{z}_2|\cdots|\mathbf{z}_n]. \tag{36}$$

If $\mathbf{z}_1,\ldots,\mathbf{z}_n$ are not linearly independent, choose a maximal subset of linearly independent vectors $\mathbf{z}_1,\ldots,\mathbf{z}_n$ with $m<n$. By orthogonalization, determine $n-m$ linearly independent vectors $\mathbf{w}_1,\ldots,\mathbf{w}_{n-m}$ that are orthogonal to the linear span of $\mathbf{z}_1,\ldots,\mathbf{z}_m$. Instead of Eq. (36), define

$$\mathbf{Z}=[\mathbf{z}_1|\cdots|\mathbf{z}_m|\mathbf{w}_1|\cdots|\mathbf{w}_{n-m}]. \tag{37}$$

Using either Eq. (36) or (37) whichever appropriate, the columns of $\mathbf{Z}$ will be linearly independent. Let

$$\mathbf{D}_1=-\mathrm{diag}[\lambda_j+\overline{\lambda}_j]=-\mathrm{diag}[2\alpha_1,2\alpha_2,\ldots,2\alpha_n], \tag{38}$$

$$\mathbf{\Omega}_1=\mathrm{diag}[\lambda_j\overline{\lambda}_j]=\mathrm{diag}[\alpha_1^2+\omega_1^2,\alpha_2^2+\omega_2^2,\ldots,\alpha_n^2+\omega_n^2]. \tag{39}$$

A system that possesses $\mathbf{y}_j(t)$ as its damped modes satisfies the equation

$$\mathbf{M}_1\ddot{\mathbf{q}}_1+\mathbf{C}_1\dot{\mathbf{q}}_1+\mathbf{K}_1\mathbf{q}_1=\mathbf{0}, \tag{40}$$

where the coefficient matrices are given by

$$\mathbf{M}_1=\mathbf{Z}^{-\mathrm{T}}\mathbf{Z}^{-1}, \tag{41}$$

$$\mathbf{C}_1=\mathbf{Z}^{-\mathrm{T}}\mathbf{D}_1\mathbf{Z}^{-1}, \tag{42}$$

$$\mathbf{K}_1=\mathbf{Z}^{-\mathrm{T}}\mathbf{\Omega}_1\mathbf{Z}^{-1} \tag{43}$$

are symmetric and positive definite. It may be readily verified that

$$\mathbf{C}_1\mathbf{M}_1^{-1}\mathbf{K}_1=\mathbf{K}_1\mathbf{M}_1^{-1}\mathbf{C}_1 \tag{44}$$

and therefore system (40) is classically damped. Eqs. (41)–(43) can be rewritten as

$$\mathbf{Z}^{\mathrm{T}}\mathbf{M}_1\mathbf{Z}=\mathbf{I}, \tag{45}$$

$$\mathbf{Z}^{\mathrm{T}}\mathbf{C}_1\mathbf{Z}=\mathbf{D}_1, \tag{46}$$

$$\mathbf{Z}^{\mathrm{T}}\mathbf{K}_1\mathbf{Z}=\mathbf{\Omega}_1, \tag{47}$$

demonstrating clearly that $\mathbf{Z}$, $\mathbf{D}_1$, $\mathbf{\Omega}_1$ are, respectively, the modal, modal damping, and spectral matrices associated with system (40). In summary, system (1) has been transformed into a classically damped system (40) by phase synchronization of its damped modes of vibration.

5.2. System decoupling

The classically damped system (40) can of course be decoupled by modal analysis. Let

$$\mathbf{q}_1=\mathbf{Z}\mathbf{p}. \tag{48}$$

In terms of the principal or modal coordinate $\mathbf{p}$, Eq. (40) becomes

$$\ddot{\mathbf{p}}+\mathbf{D}_1\dot{\mathbf{p}}+\mathbf{\Omega}_1\mathbf{p}=\mathbf{0}. \tag{49}$$

The above equation clearly admits the solution

$$\mathbf{p}(t)=\begin{bmatrix}p_1(t)\\p_2(t)\\\vdots\\p_n(t)\end{bmatrix}=\begin{bmatrix}C_1e^{\alpha_1 t}\cos(\omega_1 t-\theta_1)\\C_2e^{\alpha_2 t}\cos(\omega_2 t-\theta_2)\\\vdots\\C_ne^{\alpha_n t}\cos(\omega_n t-\theta_n)\end{bmatrix}. \tag{50}$$

What is the transformation that generates the original solution $\mathbf{q}(t)$ of system (1) once the decoupled solution $\mathbf{p}(t)$ of Eq. (49) is obtained? Observe that the phase transformation (33) has an inverse such that

$$\mathbf{s}_j(t) = \begin{bmatrix} s_{j1}(t) \\ s_{j2}(t) \\ \vdots \\ s_{jn}(t) \end{bmatrix} = \begin{bmatrix} y_{j1}(t-\phi_{j1}/\omega_j) \\ y_{j2}(t-\phi_{j2}/\omega_j) \\ \vdots \\ y_{jn}(t-\phi_{jn}/\omega_j) \end{bmatrix}. \tag{51}$$

Combine Eqs. (34), (35), and (50) to obtain

$$\mathbf{y}_j(t) = C_j e^{\alpha_j t}\cos(\omega_j t - \theta_j)\mathbf{z}_j = p_j(t)\mathbf{z}_j. \tag{52}$$

It follows from Eqs. (51) and (52) that

$$\mathbf{s}_j(t) = \begin{bmatrix} p_j(t-\phi_{j1}/\omega_j) r_{j1} e^{\alpha_j \phi_{j1}/\omega_j} \\ p_j(t-\phi_{j2}/\omega_j) r_{j2} e^{\alpha_j \phi_{j2}/\omega_j} \\ \vdots \\ p_j(t-\phi_{jn}/\omega_j) r_{jn} e^{\alpha_j \phi_{jn}/\omega_j} \end{bmatrix}. \tag{53}$$

As given in Eq. (28), $\mathbf{q}(t)$ is a superposition of the damped modes $\mathbf{s}_j(t)$. Thus

$$\mathbf{q}(t) = \sum_{j=1}^{n} \mathrm{diag}[p_j(t-\phi_{j1}/\omega_j), p_j(t-\phi_{j2}/\omega_j), \ldots, p_j(t-\phi_{jn}/\omega_j)]\mathbf{z}_j. \tag{54}$$

The real transformation above is an extension of the classical modal transformation (8). The overall decoupling procedure may be regarded as a two-stage process. By phase synchronization, the original $(\mathbf{M},\mathbf{C},\mathbf{K})$ system is first transformed into a classically damped $(\mathbf{M}_1,\mathbf{C}_1,\mathbf{K}_1)$ system. In the second stage the $(\mathbf{M}_1,\mathbf{C}_1,\mathbf{K}_1)$ system is converted into the $(\mathbf{I},\mathbf{D}_1,\mathbf{\Omega}_1)$ system by classical modal analysis. While the eigenvalues of $(\mathbf{M},\mathbf{C},\mathbf{K})$,

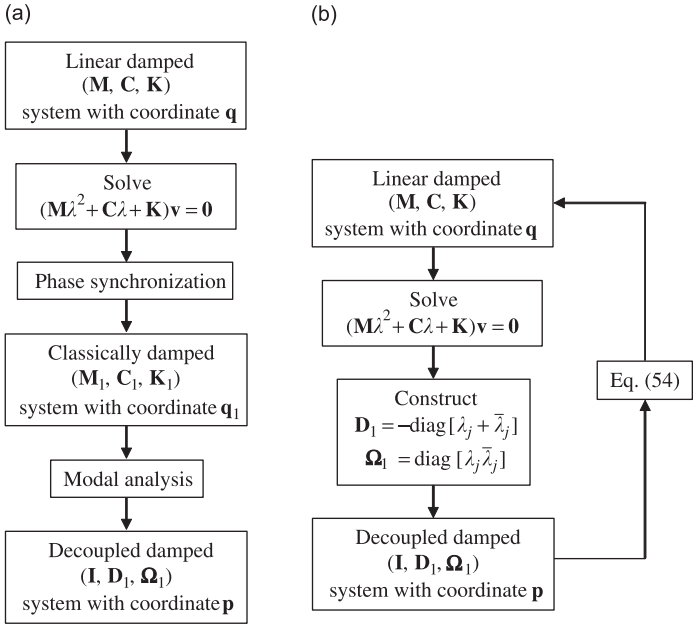

Fig. 1. Flowcharts for decoupling linear systems in oscillatory free vibration: (a) the mechanics of decoupling and (b) algorithm for decoupling and for response calculation.

($\mathbf{M}_1$,$\mathbf{C}_1$,$\mathbf{K}_1$), and ($\mathbf{I}$,$\mathbf{D}_1$,$\mathbf{\Omega}_1$) systems are identical, their eigenvectors are different. A flowchart depicting the mechanics of decoupling is shown in Fig. 1(a).

5.3. Transformation of initial conditions

The initial values of systems (1) and (49) are transformed in accordance to

$$\mathbf{q}(0) = \sum_{j=1}^{n} \mathrm{diag}[p_j(-\phi_{j1}/\omega_j), p_j(-\phi_{j2}/\omega_j), \ldots, p_j(-\phi_{jn}/\omega_j)]\mathbf{z}_j, \tag{55}$$

$$\dot{\mathbf{q}}(0) = \sum_{j=1}^{n} \mathrm{diag}[\dot{p}_j(-\phi_{j1}/\omega_j), \dot{p}_j(-\phi_{j2}/\omega_j), \ldots, \dot{p}_j(-\phi_{jn}/\omega_j)]\mathbf{z}_j. \tag{56}$$

It is thus necessary to start $p_j(t)$ at a negative time $t_j = -\max_k(\phi_{jk}/\omega_j)$. This situation arises because phase synchronization "pushes back" the element $s_{jk}(t)$ by an interval of length ϕ_{jk}/ω_j. To connect the initial conditions of $\mathbf{q}(t)$ and $\mathbf{p}(t)$ at the same instant $t = 0$, write Eq. (28) in the form

$$\mathbf{q}(t) = \sum_{j=1}^{n} (a_j \mathbf{v}_j e^{\lambda_j t} + \overline{a}_j \overline{\mathbf{v}}_j e^{\overline{\lambda}_j t}) = \mathbf{V} e^{\mathbf{\Lambda} t}\mathbf{a} + \overline{\mathbf{V}} e^{\overline{\mathbf{\Lambda}} t}\overline{\mathbf{a}}, \tag{57}$$

where

$$\mathbf{V} = [\mathbf{v}_1 | \mathbf{v}_2 | \cdots | \mathbf{v}_n], \tag{58}$$

$$\mathbf{\Lambda} = \mathrm{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n], \tag{59}$$

$$\mathbf{a} = [a_1 \;\; a_2 \;\; \cdots \;\; a_n]^{\mathrm{T}}. \tag{60}$$

As a state equation,

$$\begin{bmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{V} & \overline{\mathbf{V}} \\ \mathbf{V}\mathbf{\Lambda} & \overline{\mathbf{V}\mathbf{\Lambda}} \end{bmatrix} \begin{bmatrix} e^{\mathbf{\Lambda} t} & \mathbf{0} \\ \mathbf{0} & e^{\overline{\mathbf{\Lambda}} t} \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \overline{\mathbf{a}} \end{bmatrix}, \tag{61}$$

which implies that

$$\begin{bmatrix} \mathbf{q}(0) \\ \dot{\mathbf{q}}(0) \end{bmatrix} = \begin{bmatrix} \mathbf{V} & \overline{\mathbf{V}} \\ \mathbf{V}\mathbf{\Lambda} & \overline{\mathbf{V}\mathbf{\Lambda}} \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \overline{\mathbf{a}} \end{bmatrix}. \tag{62}$$

On the other hand, Eq. (50) yields

$$p_j(t) = C_j e^{\alpha_j t} \cos(\omega_j t - \theta_j), \tag{63}$$

$$\dot{p}_j(t) = \alpha_j C_j e^{\alpha_j t} \cos(\omega_j t - \theta_j) - \omega_j C_j e^{\alpha_j t} \sin(\omega_j t - \theta_j). \tag{64}$$

Recall Eq. (25) to obtain

$$p_j(0) = C_j \cos\theta_j = \mathrm{Re}[2a_j] = a_j + \overline{a}_j, \tag{65}$$

$$\dot{p}_j(0) = a_j C_j \cos\theta_j + \omega_j C_j \sin\theta_j = \mathrm{Re}[2\lambda_j a_j] = \lambda_j a_j + \overline{\lambda}_j \overline{a}_j. \tag{66}$$

In the form of a state equation,

$$\begin{bmatrix} \mathbf{p}(0) \\ \dot{\mathbf{p}}(0) \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{\Lambda} & \overline{\mathbf{\Lambda}} \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \overline{\mathbf{a}} \end{bmatrix}. \tag{67}$$

Combine Eqs. (62) and (67) to result in

$$\begin{bmatrix} \mathbf{p}(0) \\ \dot{\mathbf{p}}(0) \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{\Lambda} & \overline{\mathbf{\Lambda}} \end{bmatrix} \begin{bmatrix} \mathbf{V} & \overline{\mathbf{V}} \\ \mathbf{V\Lambda} & \overline{\mathbf{V\Lambda}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{q}(0) \\ \dot{\mathbf{q}}(0) \end{bmatrix}. \tag{68}$$

It can be readily shown that the overall transformation matrix in the above expression is real and invertible. In fact, Eq. (68) can be written explicitly as

$$\mathbf{p}(0) = 2\,\mathrm{Re}[(\mathbf{V\Lambda V}^{-1}\overline{\mathbf{V}} - \overline{\mathbf{V\Lambda}})^{-1}\mathbf{V\Lambda V}^{-1}]\mathbf{q}(0) - 2\,\mathrm{Re}[(\mathbf{V\Lambda V}^{-1}\overline{\mathbf{V}} - \overline{\mathbf{V\Lambda}})^{-1}]\dot{\mathbf{q}}(0), \tag{69}$$

$$\dot{\mathbf{p}}(0) = 2\,\mathrm{Re}[\mathbf{\Lambda}(\mathbf{V\Lambda V}^{-1}\overline{\mathbf{V}} - \overline{\mathbf{V\Lambda}})^{-1}\mathbf{V\Lambda V}^{-1}]\mathbf{q}(0) - 2\,\mathrm{Re}[\mathbf{\Lambda}(\mathbf{V\Lambda V}^{-1}\overline{\mathbf{V}} - \overline{\mathbf{V\Lambda}})^{-1}]\dot{\mathbf{q}}(0). \tag{70}$$

There is a subtle difference between the above expressions and Eqs. (55) and (56): when the initial conditions of systems (1) and (49) are connected at the same instant, the displacements and velocities can no longer be separated in different equations.

5.4. Exact response by decoupling

After the initial values $\mathbf{p}(0)$ and $\dot{\mathbf{p}}(0)$ have been determined, the exact solution $\mathbf{p}(t)$ of the decoupled system (49) can be written as

$$p_j(t) = \mathrm{e}^{\alpha_j t}\left[p_j(0)\cos\,\omega_j t + \frac{\dot{p}_j(0) - \alpha_j p_j(0)}{\omega_j}\sin\,\omega_j t\right] \tag{71}$$

for $j = 1, \ldots, n$. The original solution $\mathbf{q}(t)$ of system (1) can then be obtained by substituting Eq. (71) into (54). Quantities such as α_j, ω_j, ϕ_{jk}, $\mathbf{z}_j$ that appear in the exact response are all obtained by solution of the quadratic eigenvalue problem (14). A flowchart outlining the algorithm of solution by decoupling is shown in Fig. 1(b). Perhaps it should be stated that response calculation is probably not the most important reason for system decoupling. It is the possibility, for example, of modal reduction (using the real damped modes) and of an investigation of energy distribution among independent coordinates that would make decoupling worthwhile. These practical issues will be pursued elsewhere in the future.

5.5. Reduction to classical modal analysis

The method of phase synchronization is a direct generalization of classical modal analysis. If system (1) is undamped or classically damped, each eigenvector $\mathbf{v}_j$ is real and the phase angles $\phi_{j1} = \phi_{j2} = \cdots = \phi_{jn} = 0$. In this case the phase transformation (33) reduces to the identity transformation so that $(\mathbf{M},\mathbf{C},\mathbf{K})$ and $(\mathbf{M}_1,\mathbf{C}_1,\mathbf{K}_1)$ are the same. Moreover, Eq. (54) simplifies to

$$\mathbf{q}(t) = \sum_{j=1}^{n} p_j(t)\mathbf{z}_j = \sum_{j=1}^{n} p_j(t)\mathbf{u}_j, \tag{72}$$

which coincides with the classical modal transformation (8).

Finally, it should be pointed out that decoupling by phase synchronization is not a unique procedure. Phase synchronization involves the damped modes, which are defined in terms of the eigenvectors $\mathbf{v}_j$ of Eq. (14). However, each eigenvector of Eq. (14) can only be determined up to an arbitrary multiplicative constant. The eigenvectors $\mathbf{v}_j$ may be normalized for convenience in accordance with, for example,

$$2\lambda_j \mathbf{v}_j^{\mathrm{T}}\mathbf{M}\mathbf{v}_j + \mathbf{v}_j^{\mathrm{T}}\mathbf{C}\mathbf{v}_j = 2\mathrm{i}\omega_j. \tag{73}$$

The above normalization reduces to normalization with respect to the mass matrix $\mathbf{M}$ for an undamped or classically damped system [6]. Any normalization can only specify the magnitude of $\mathbf{v}_j$ but its sign is still arbitrary. With or without normalization, the equations of phase synchronization are not unique although the decoupling process is always valid. For instance, if $\mathbf{y}_j(t)$ in Eq. (33) is a synchronized damped mode, so is $-\mathbf{y}_j(t)$. This issue of non-uniqueness should not be surprising since a similar situation exists in classical modal

analysis: the modal transformation (8) is not unique because any classical normal mode $\mathbf{u}_j$ in Eq. (8) may be replaced by $-\mathbf{u}_j$.

5.6. Comparison with Hamiltonian approach

In theory, it is always possible to investigate a non-classically damped system with the Hamiltonian approach, whereby the second-order Eq. (1) is recast into a first-order system of dimension $2n$. The state space representation is one such approach. Every procedure and every equation in configuration space can in principle be cast in state space. If λ_j and $\mathbf{v}_j$ are the $2n$ complex eigenvalues and eigenvectors of Eq. (14), it can be shown that the eigenvalues and eigenvectors of the state companion matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix} \tag{74}$$

are, respectively, λ_j and $[\mathbf{v}_j^{\mathrm{T}} \;\; \lambda_j\mathbf{v}_j^{\mathrm{T}}]^{\mathrm{T}}$. Even the decoupling transformation (54) can be written as

$$\begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{\Lambda} & \overline{\mathbf{\Lambda}} \end{bmatrix} \begin{bmatrix} \mathbf{V} & \overline{\mathbf{V}} \\ \mathbf{V\Lambda} & \overline{\mathbf{V}}\,\overline{\mathbf{\Lambda}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{bmatrix}. \tag{75}$$

Unlike Eq. (54), the displacements and velocities can no longer be separated in the above expression. Perhaps it should be emphasized that the decoupling methodology of this paper has been developed through intuition: by phase synchronization of the real and physically excitable damped modes of vibration. It is not obvious how Eq. (75) can be independently derived without recourse to configuration space. Similar observations may be made in the $2n$-dimensional phase space (with generalized momenta replacing the velocities).

5.7. Efficiency of solution by decoupling

Although system solution is probably not the most important reason for decoupling, it may still be instructive to compare the efficiency of solution of Eq. (1) by direct numerical integration and by decoupling. One measure of the performance of an algorithm is the number of floating point operations (flops) required to evaluate the response at m time points within a given time window. The flops associated with three procedures will be compared. (a) In direct numerical integration, a standard procedure is to rewrite Eq. (1) in first-order form using the state companion matrix in Eq. (74). The state equation is then discretized, and the resulting system of $2n$ coupled difference equations is solved by matrix computations [36]. This procedure involves one-time computation of the exponential matrix $\exp(\mathbf{A}\Delta t)$, where Δt is the sampling time, and one matrix–vector multiplication at each step. The estimate of flops for response calculation at m instants is [37–40]

$$N_1 = 160n^3 + 8mn^2, \tag{76}$$

where n is the number of degrees of freedom of system (1) and $m \gg n$ in general.

In solving Eq. (1) by decoupling, two alternative procedures may be used. (b) It is possible to evaluate the m responses by directly invoking Eqs. (54) and (71). This procedure involves one-time solution of Eqs. (14) and (68), plus evaluation of Eqs. (54) and (71) at each step. The estimate of flops for this procedure is [33,38–40]

$$N_2 = 213n^3 + 2mn^2. \tag{77}$$

(c) Another method of solution is to decouple Eq. (1) through solution of Eq. (14) followed by direct integration of each decoupled equation (this procedure should be used in forced vibration). If each decoupled equation is solved numerically with the same algorithm used in procedure (a) for direct integration of Eq. (1), the estimate of flops is

$$N_3 = 213n^3 + 2mn^2 + 8mn + 1280n. \tag{78}$$

The variations of N_1, N_2, and N_3 with n are illustrated in Fig. 2(a) for a window containing $m = 10^5$ instants. It is observed that the curves associated with N_2, N_3 agree within the line thickness and that procedures (b) and

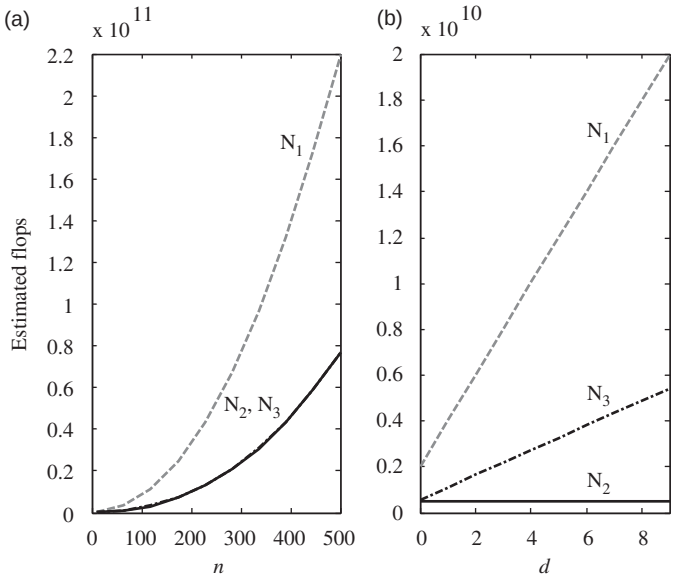

Fig. 2. Comparison of efficiency of three methods of solution: direct numerical integration N_1 (- - -); decoupling by invoking Eqs. (54) and (71) N_2 (—); and decoupling followed by numerical integration of the decoupled equations N_3 (- · · ·). (a) Estimated flops to evaluate the response at $m = 10^5$ instants vs. degree of freedom n. The curves associated with N_2 and N_3 agree within the line thickness. (b) Estimated flops vs. starting time d of window, where the initial conditions are prescribed at $t = 0$, and $m = 10^5$, $n = 50$.

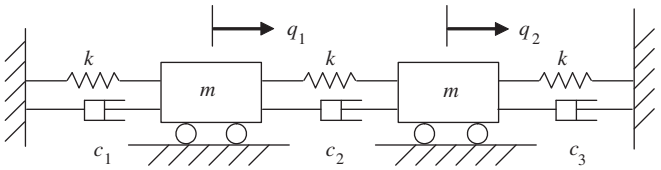

Fig. 3. The mass–spring–damper system of Example 1.

(c) are more efficient than (a). In fact, the estimate of flops shown in Fig. 2(a) is very conservative for two reasons. First, N_3 has been estimated by using the same sampling time in the integration of all decoupled equations. If an optimal sampling time is individually chosen for each decoupled equation, N_3 may decrease substantially. Second, Fig. 2(a) is generated by using a window of $m = 10^5$ points that begins from $t = 0$, the time at which initial values are prescribed. For any window that begins from a time $d \gg 0$, numerical integration must still start from the initial time $t = 0$. A large number of iterations may be required over the interval $0 < t < d$ before the window of interest is reached. Thus for $d \gg 0$, N_1 and N_3 increase appreciably while N_2 remains constant. This situation is depicted in Fig. 2(b), in which N_1, N_2, and N_3 are plotted against d. It is observed that N_1 increases more rapidly than N_3. Based upon Fig. 2(b), it may be stated that solution by decoupling generally reduces the flops and economizes on both core memory and computing time.

6. Illustrative examples

Three examples will be given to illustrate damped modes of vibration as well as the process of decoupling by phase synchronization. Complete details are given in Example 1 to provide physical insight and to reinforce the mathematical concepts expounded earlier.

Example 1. Consider a mass–spring–damper system governed by an equation of the type (1), with

$$\mathbf{M} = m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix}, \quad \mathbf{K} = k\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \tag{79}$$

and initial conditions

$$\mathbf{q}(0) = [1 \;\; 2]^{\mathrm{T}}, \quad \dot{\mathbf{q}}(0) = [-1 \;\; 1]^{\mathrm{T}}. \tag{80}$$

The system is shown in Fig. 3. For convenience, let $m = k = 1$. Three different cases will be examined.

(a) System is undamped: $c_1 = c_2 = c_3 = 0$. Solution of the symmetric eigenvalue problem (3) yields, upon normalization with respect to the mass matrix,

$$\mathbf{\Omega} = \mathrm{diag}[1, \leftrightarrows 3], \quad \mathbf{U} = [\mathbf{u}_1|\mathbf{u}_2] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{81}$$

The general solution is a superposition of two natural modes of vibration such that

$$\mathbf{q}(t) = \sum_{j=1}^{2} \mathbf{s}_j(t) = C_1 \cos(t - \theta_1)\mathbf{u}_1 + C_2 \cos(\sqrt{3}t - \theta_2)\mathbf{u}_2. \tag{82}$$

The constants C_1, C_2, θ_1, and θ_2 are determined by the initial conditions. As shown in Fig. 4, the system components in each mode are either in phase or out of phase so that modal vibration appears truly synchronous. The system can be decoupled by classical modal analysis.

(b) Classically damped system: $c_1 = c_2 = c_3 = 0.1$. Since $\mathbf{C} = 0.1\mathbf{K}$, the system is proportionally damped. The general solution is given by

$$\mathbf{q}(t) = \sum_{j=1}^{2} \mathbf{s}_j(t) = C_1 \mathrm{e}^{-0.05t} \cos(1.00t - \theta_1)\mathbf{u}_1 + C_2 \mathrm{e}^{-0.15t} \cos(1.73t - \theta_2)\mathbf{u}_2. \tag{83}$$

As shown in Fig. 5, the system components in each mode are again either in phase or out of phase but, in contrast to case (a), they decay exponentially. The system can still be decoupled by classical modal analysis.

(c) Non-classically damped system: $c_1 = 0.6$, $c_2 = c_3 = 0.1$. Since condition (12) is not satisfied, the system is non-classically damped and it cannot be decoupled by classical modal analysis. Solution of the quadratic

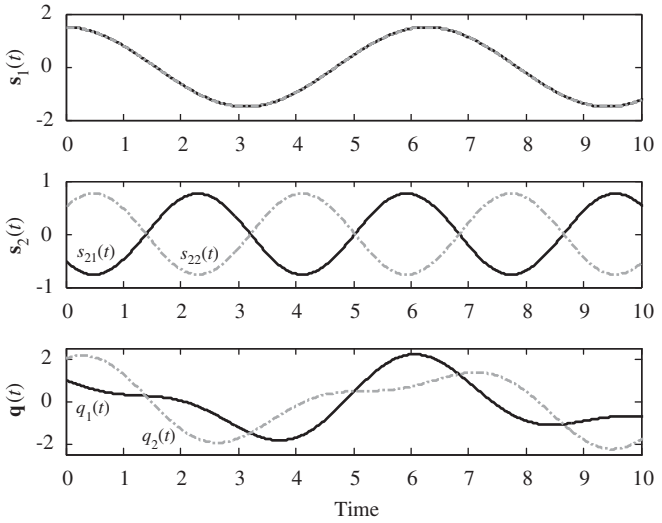

Fig. 4. Natural modes and undamped free response of Example 1(a) shown in three parts. (a) First natural mode of vibration $\mathbf{s}_1(t)$ with first element $s_{11}(t)$ (—) and second element $s_{12}(t)$ (- - -). (b) Second natural mode of vibration $\mathbf{s}_2(t)$ with first element $s_{21}(t)$ (—) and second element $s_{22}(t)$ (- - -). (c) Free response $\mathbf{q}(t)$ with first element $q_1(t)$ (—) and second element $q_2(t)$ (- - -).

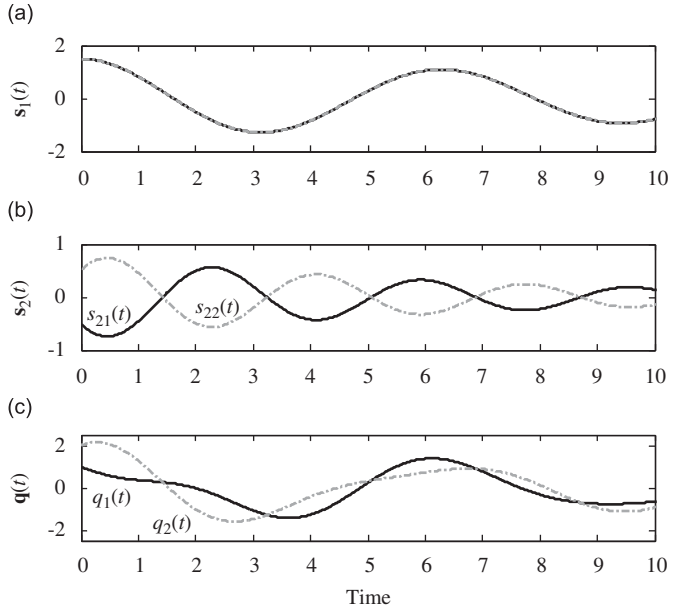

Fig. 5. Classically damped modes and free response of Example 1(b) shown in three parts. (a) First classically damped mode $\mathbf{s}_1(t)$ with first element $s_{11}(t)$ (—) and second element $s_{12}(t)$ (- · · ·). (b) Second classically damped mode $\mathbf{s}_2(t)$ with first element $s_{21}(t)$ (—) and second element $s_{22}(t)$ (- · · ·). (c) Free response $\mathbf{q}(t)$ with first element $q_1(t)$ (—) and second element $q_2(t)$ (- · · ·).

eigenvalue problem (14) yields

$$\lambda_1 = \overline{\lambda}_3 = -0 + .181.00i, \quad \mathbf{v}_1 = \overline{\mathbf{v}}_3 = \begin{bmatrix} 0.74\mathrm{e}^{-\mathrm{i}7.38^\circ} \\ -0.72\mathrm{e}^{-\mathrm{i}172.51^\circ} \end{bmatrix}, \tag{84}$$

$$\lambda_2 = \overline{\lambda}_4 = -0 + .271.68i, \quad \mathbf{v}_2 = \overline{\mathbf{v}}_4 = \begin{bmatrix} -0.73\mathrm{e}^{-\mathrm{i}167.13^\circ} \\ -0.73\mathrm{e}^{-\mathrm{i}12.68^\circ} \end{bmatrix}, \tag{85}$$

where for convenience, the eigenvectors are normalized in accordance with Eq. (73). From Eq. (27), the two non-classically damped modes of vibration are given by

$$\mathbf{s}_1(t) = C_1\mathrm{e}^{-0.18t}\begin{bmatrix} 0.74\cos(1.00t - \theta_1 - 7.38^\circ) \\ -0.72\cos(1.00t - \theta_1 - 172.51^\circ) \end{bmatrix}, \tag{86}$$

$$\mathbf{s}_2(t) = C_2\mathrm{e}^{-0.27t}\begin{bmatrix} -0.73\cos(1.68t - \theta_2 - 167.13^\circ) \\ -0.73\cos(1.68t - \theta_2 - 12.68^\circ) \end{bmatrix}. \tag{87}$$

The general solution is a superposition of these two real damped modes. As can be easily seen in Fig. 6, there is indeed a constant phase difference between the two components in each mode. In phase synchronization, $\mathbf{s}_j(t)$ are transformed into $\mathbf{y}_j(t)$ in accordance with Eq. (33) such that

$$\mathbf{y}_1(t) = C_1\mathrm{e}^{-0.18t}\cos(1.00t - \theta_1)\mathbf{z}_1, \tag{88}$$

$$\mathbf{y}_2(t) = C_2\mathrm{e}^{-0.27t}\cos(1.68t - \theta_2)\mathbf{z}_2. \tag{89}$$

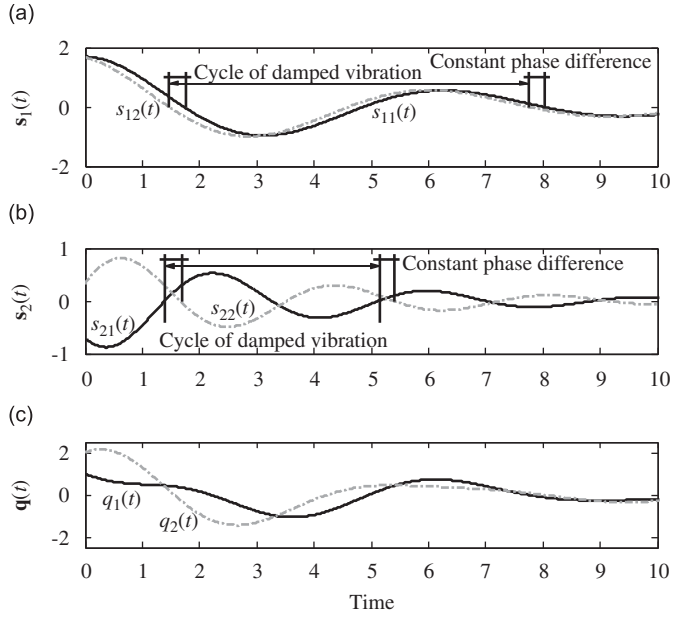

Fig. 6. Non-classically damped modes and free response of Example 1(c) shown in three parts. (a) First non-classically damped mode $\mathbf{s}_1(t)$ with first element $s_{11}(t)$ (—) and second element $s_{12}(t)$ (- · - ·). (b) Second non-classically damped mode $\mathbf{s}_1(t)$ with first element $s_{21}(t)$ (—) and second element $s_{22}(t)$ (- · - ·). (c) Free response $\mathbf{q}(t)$ with first element $q_1(t)$ (—) and second element $q_2(t)$ (- · - ·).

It can be verified that

$$\mathbf{Z} = [\mathbf{z}_1|\mathbf{z}_2] = \begin{bmatrix} 0.72 & -0.46 \\ -0.42 & -0.70 \end{bmatrix}, \tag{90}$$

$$\mathbf{D}_1 = -\mathrm{diag}[\lambda_j + \overline{\lambda}_j] = \mathrm{diag}[0.36, 0.54], \tag{91}$$

$$\mathbf{\Omega}_1 = \mathrm{diag}[\lambda_j \overline{\lambda}_j] = \mathrm{diag}[1.03, 2.90]. \tag{92}$$

Upon decoupling, the equation of motion becomes $\ddot{\mathbf{p}} + \mathbf{D}_1\dot{\mathbf{p}} + \mathbf{\Omega}_1\mathbf{p} = \mathbf{0}$. Using Eq. (68), the initial conditions of the decoupled system are

$$\mathbf{p}(0) = [2.32 \quad -0.71]^{\mathrm{T}}, \quad \dot{\mathbf{p}}(0) = [-0.50 \quad 2.09]^{\mathrm{T}}. \tag{93}$$

The solution $\mathbf{q}(t)$ of the original system can be readily recovered from solution $\mathbf{p}(t)$ of the decoupled system by Eq. (54). It can be checked that $\mathbf{q}(t)$, whether generated by decoupling or by direct numerical solution of the original equation of motion, is the same.

As mentioned earlier, the positions of zero displacement in a non-classically damped mode are not fixed and they drift around in space. To examine the loci of these instantaneous nodes, set up a rectilinear x-coordinate along the line of motion of the masses in Fig. 3. Locate the origin $x = 0$ at the static equilibrium of the left mass and locate $x = 1$ at the static equilibrium of the right mass. In cases (a) and (b), the first mode does not have any node while the second mode possesses one node at $x = 0.5$. In case (c), it can be checked that the first non-classically damped mode possesses an instantaneous node at

$$x_1(t) = \left[1 + \frac{0.72}{0.74}\frac{\cos(1.00t - 172.51^\circ)}{\cos(1.00t - 7.38^\circ)}\right]^{-1}, \tag{94}$$

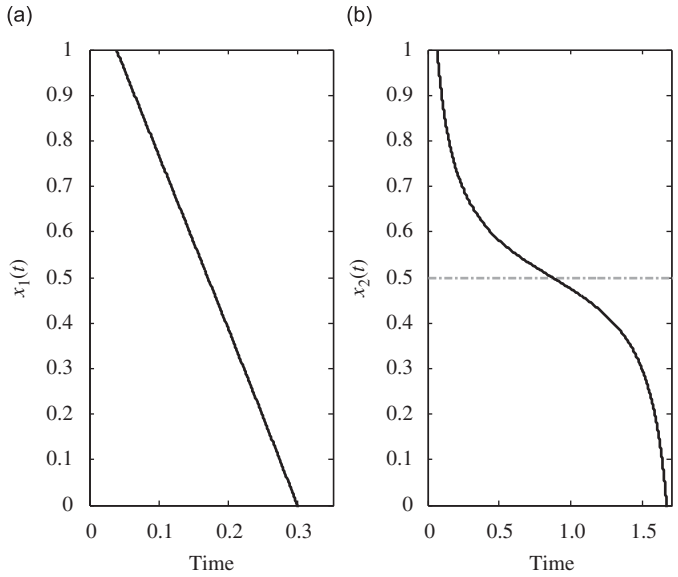

Fig 7. Loci of instantaneous nodes of non-classically damped modes of Example 1(c) shown in two parts. (a) Locus in first mode (—). (b) Locus in second mode (—). Also shown for comparison is the nodal path if damping is absent or classical (- · - · -).

while the second non-classically damped mode possesses an instantaneous node at

$$x_2(t) = \left[1 - 1.00\frac{\cos(1.68t - 12.68^\circ)}{\cos(1.68t - 167.13^\circ)}\right]^{-1}. \tag{95}$$

The loci of the instantaneous nodes are plotted in Fig. 7 over a cycle. In general, there are not any fixed positions for mounting additional equipment without disturbing a non-classically damped mode.

Example 2. A four-degree-of-freedom system of the form (1) is defined by $\mathbf{M} = \mathbf{I}$,

$$\mathbf{C} = 0.1\begin{bmatrix} 7 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}, \tag{96}$$

with initial conditions

$$\mathbf{q}(0) = [1 \;\; 2 \;\; 3 \;\; 4]^{\mathrm{T}}, \quad \dot{\mathbf{q}}(0) = [-1 \;\; 1 \;\; 2 \;\; -3]^{\mathrm{T}}. \tag{97}$$

Since condition (12) is not satisfied, the system is non-classically damped and it cannot be decoupled by modal analysis. However, the system can be decoupled by the method of phase synchronization as outlined in the flowcharts of Fig. 1. The decoupled system has the form $\ddot{\mathbf{p}} + \mathbf{D}_1\dot{\mathbf{p}} + \mathbf{\Omega}_1\mathbf{p} = \mathbf{0}$ where

$$\mathbf{D}_1 = \mathrm{diag}[0.11, 0.33, 0.45, 0.41], \tag{98}$$

$$\mathbf{\Omega}_1 = \mathrm{diag}[0.39, 1.41, 2.55, 3.55]. \tag{99}$$

Using Eq. (68), the initial conditions of the decoupled system are

$$\mathbf{p}(0) = [-4.76 \;\; -2.09 \;\; 0.97 \;\; -0.64]^{\mathrm{T}}, \quad \dot{\mathbf{p}}(0) = [0.14 \;\; -0.79 \;\; 0.67 \;\; -2.95]^{\mathrm{T}}. \tag{100}$$

The solution $\mathbf{q}(t)$ of the original system can be readily recovered from solution $\mathbf{p}(t)$ of the decoupled system by Eq. (54).

Example 3. Defective systems will now be tackled. Recall that the probability of a randomly chosen eigenvalue problem of form (14) being defective is zero. Nonetheless, defective or degenerate systems were

considered by a number of authors [41–45]. Consider a non-classically damped system governed by an equation of the type (1), with

$$\mathbf{M}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\quad \mathbf{C}=\frac{1}{3}\begin{bmatrix}4 & -\sqrt{5}\\ -\sqrt{5} & 8\end{bmatrix},\quad \mathbf{K}=\begin{bmatrix}1 & 0\\ 0 & 4\end{bmatrix}, \tag{101}$$

and initial conditions

$$\mathbf{q}(0)=[1\ \ 2]^{\mathrm{T}},\quad \dot{\mathbf{q}}(0)=[-1\ \ 2]^{\mathrm{T}}. \tag{102}$$

Solution of the quadratic eigenvalue problem (14) yields two repeated eigenvalues

$$\lambda_1=-1+i,\quad \overline{\lambda}_1=-1-i. \tag{103}$$

The algebraic multiplicity of each eigenvalue is two. However, there is only one eigenvector $\mathbf{v}_1=[1\ \ -0.22+\mathrm{i}0.67]^{\mathrm{T}}$ associated with λ_1 and also only one eigenvector $\overline{\mathbf{v}}_1$ associated with $\overline{\lambda}_1$. To decouple this defective system, a generalized theory of phase synchronization will be described concisely.

Suppose the quadratic eigenvalue problem (14) has $N\leqslant n$ pairs of complex conjugate eigenvalues. Let m_k be the multiplicity of λ_k. Then m_k is also the multiplicity of $\overline{\lambda}_k$ so that $m_1+m_2+\cdots+m_N=n$. If $\mathbf{J}_k$ is a Jordan block of order m_k associated with $\lambda_k=\alpha_k+\mathrm{i}\omega_k$, $\overline{\mathbf{J}}_k$ must be a Jordan block associated with $\overline{\lambda}_k=\alpha_k-\mathrm{i}\omega_k$. Let

$$\mathbf{V}_k=[\mathbf{v}_1^k|\mathbf{v}_2^k|\cdots|\mathbf{v}_{m_k}^k] \tag{104}$$

be a matrix of order $n\times m_k$ whose columns are made up of generalized eigenvectors $\mathbf{v}_j^k$ that constitute a Jordan chain of length m_k corresponding to λ_k. In contrast to Eq. (28), the general solution of system (1) now takes the form

$$\mathbf{q}(t)=\sum_{k=1}^{N}(\mathbf{V}_k\mathrm{e}^{\mathbf{J}_k t}\mathbf{a}_k+\overline{\mathbf{V}}_k e^{\overline{\mathbf{J}}_k t}\overline{\mathbf{a}}_k)=\sum_{k=1}^{N}2\mathrm{Re}[\mathbf{V}_k\mathrm{e}^{\mathbf{J}_k t}\mathbf{a}_k]=\sum_{k=1}^{N}\mathbf{s}_k(t), \tag{105}$$

where $\mathbf{a}_k$ are constant vectors to be obtained from initial conditions. In analogy to Eqs. (22) and (25), write

$$\mathbf{v}_j^k=[r_{j1}^k\mathrm{e}^{-\mathrm{i}\phi_{j1}^k}\ \ r_{j2}^k\mathrm{e}^{-\mathrm{i}\phi_{j2}^k}\ \ \cdots\ \ r_{jn}^k\mathrm{e}^{-\mathrm{i}\phi_{jn}^k}]^{\mathrm{T}}, \tag{106}$$

$$2\mathbf{a}_k=[C_1^k\mathrm{e}^{-\mathrm{i}\theta_1^k}\ \ C_2^k\mathrm{e}^{-\mathrm{i}\theta_2^k}\ \ \cdots\ \ C_{m_k}^k\mathrm{e}^{-\mathrm{i}\theta_{m_k}^k}]^{\mathrm{T}}. \tag{107}$$

It can be shown that the ith element of $\mathbf{s}_k(t)$ is given by

$$s_{ki}(t)=\sum_{j=1}^{m_k}r_{ji}^k\sum_{l=j}^{m_k}C_l^k\mathrm{e}^{\alpha_k t}\cos(\omega_k t-\theta_l^k-\phi_{ji}^k)\frac{t^{(l-j)}}{(l-j)!} \tag{108}$$

for $i=1,\ldots,n$. The above expression is an extension of Eq. (27). Upon phase synchronization of only the terms

$$C_l^k\mathrm{e}^{\alpha_k t}\cos(\omega_k t-\theta_l^k-\phi_{ji}^k)=p_l(t-\phi_{ji}^k/\omega_k)\mathrm{e}^{\alpha_k\phi_{ji}^k/\omega_k}, \tag{109}$$

m_k identical single-degree-of-freedom systems are obtained. However, the m_k systems have different initial conditions. This process of phase synchronization can be repeated for all N pairs of complex conjugate eigenvalues, resulting in an n-degree-of-freedom decoupled system. The decoupling procedure outlined in the flowchart of Fig. 1(a) remains valid for defective systems. The modal matrix $\mathbf{Z}$ of the classically damped $(\mathbf{M}_1,\mathbf{C}_1,\mathbf{K}_1)$ system that arises during decoupling consists of columns given by

$$\mathbf{z}_j^k=[r_{j1}^k\mathrm{e}^{\alpha^k\phi_{j1}^k/\omega_k}\ \ r_{j2}^k\mathrm{e}^{\alpha^k\phi_{j2}^k/\omega_k}\ \ \cdots\ \ r_{jn}^k\mathrm{e}^{\alpha^k\phi_{jn}^k/\omega_k}]^{\mathrm{T}} \tag{110}$$

for $j=1,\ldots,m_k$ and $k=1,\ldots,N$. Clearly, the above expression is a generalization of Eq. (35). Eq. (54) that connects $\mathbf{p}(t)$ with $\mathbf{q}(t)$ can be generalized to the form

$$\mathbf{q}(t)=\sum_{k=1}^{N}\sum_{j=1}^{m_k}\left(\sum_{l=j}^{m_k}\frac{t^{(l-j)}}{(l-j)!}\mathrm{diag}[p_l(t-\phi_{j1}^k/\omega_k),\ldots,p_l(t-\phi_{jn}^k/\omega_k)]\right)\mathbf{z}_j^k. \tag{111}$$

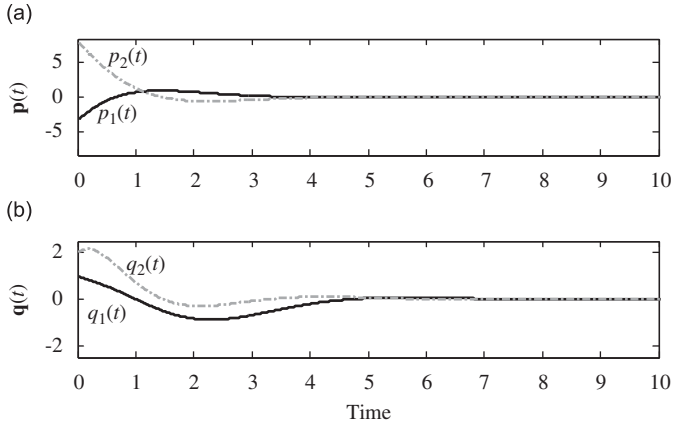

Fig. 8. Free response of Example 3. (a) Free response $\mathbf{p}(t)$ of decoupled system with first element $p_1(t)$ (—) and second element $p_2(t)$ (- - - -). (a) Free response $\mathbf{q}(t)$ of the original defective system with first element $q_1(t)$ (—) and second element $q_2(t)$ (- - - -).

Initial conditions between $\mathbf{p}(t)$ with $\mathbf{q}(t)$ are still connected by Eq. (68) provided that Eqs. (58) and (59) are replaced, respectively, by

$$\mathbf{V} = [\mathbf{V}_1 \ \ \mathbf{V}_2 \ \ \cdots \ \ \mathbf{V}_N], \tag{112}$$

$$\mathbf{\Lambda} = \mathrm{diag}[\mathbf{J}_1, \mathbf{J}_2, \ldots, \mathbf{J}_N]. \tag{113}$$

Indeed, application of phase synchronization to defective systems is straightforward but laborious. However, physical insight is obscured due to the occurrence of Jordan blocks in many equations. Should the methodology expounded in this paper be accepted for use, a thorough treatment of defective problems will become more deserving.

In the present numerical example, $N = 1$ and

$$\mathbf{J}_1 = \begin{bmatrix} -1+i & 1 \\ 0 & -1+i \end{bmatrix}, \quad \mathbf{V}_1 = \begin{bmatrix} 0.53\mathrm{e}^{-\mathrm{i}18.43^\circ} & -0.83\mathrm{e}^{-\mathrm{i}126.87^\circ} \\ -0.37\mathrm{e}^{-\mathrm{i}90^\circ} & 0.37\mathrm{e}^{-\mathrm{i}90^\circ} \end{bmatrix}, \tag{114}$$

where the columns of $\mathbf{V}_1$ are not subjected to any normalization scheme. It can be verified that

$$\mathbf{Z} = \begin{bmatrix} 0.38 & -0.09 \\ -0.08 & 0.08 \end{bmatrix}, \tag{115}$$

$$\mathbf{D}_1 = 2\mathbf{I}, \quad \mathbf{\Omega}_1 = 2\mathbf{I}. \tag{116}$$

Upon decoupling, the equation of motion becomes $\ddot{\mathbf{p}} + \mathbf{D}_1\dot{\mathbf{p}} + \mathbf{\Omega}_1\mathbf{p} = \mathbf{0}$, with

$$\mathbf{p}(0) = [-3.47 \ \ 8.15]^{\mathrm{T}}, \quad \dot{\mathbf{p}}(0) = [16.10 \ \ -9.05]^{\mathrm{T}}. \tag{117}$$

The decoupled system can be readily solved and the solution $\mathbf{q}(t)$ of the original system can be recovered from $\mathbf{p}(t)$ by Eq. (111). The result is plotted in Fig. 8. It can be checked that $\mathbf{q}(t)$, whether generated by Eq. (111) or by direct numerical solution of the equation of motion, is the same.

7. Conclusions

The purpose of this paper is to extend classical modal analysis to decouple any viscously damped linear system in oscillatory free vibration. Based upon an exposition of the mechanics of viscous damping, a real time-varying transformation has been constructed in the configuration space to decouple oscillatory free vibration. The decoupling procedure devised herein possesses tremendous physical insight (such as real and

physically excitable damped modes) and it also lends itself to numerical computations. Major findings of this paper are summarized in the following statements.

1. In a non-classically damped mode of vibration all system components perform harmonic vibration with an identical frequency but with different phase angles. In each cycle these components pass through their equilibrium positions in the same order separated by constant phase differences.
2. A non-classically damped system can be transformed into one with classical damping by synchronizing the phase angles in its damped modes so that all components are either in phase or out of phase.
3. Any viscously damped linear system in oscillatory free vibration can be completely decoupled. Flowcharts outlining the decoupling procedure have been given in Fig. 1. The solution of the original ($\mathbf{M},\mathbf{C},\mathbf{K}$) system can be recovered from that of the decoupled ($\mathbf{I},\mathbf{D}_1,\mathbf{\Omega}_1$) system by the real transformation (54). Their initial conditions are connected by Eqs. (55) and (56) or, equivalently, by Eq. (68).
4. The method of decoupling by phase synchronization reduces to classical modal analysis for systems that are undamped or classically damped.

To streamline the introduction of new concepts, it has been assumed that all eigenvalues of the quadratic eigenvalue problem (14) are complex and distinct. As explained previously, these assumptions can be readily relaxed so that any damped linear dynamical system can be decoupled. This paper constitutes the first part of a complete solution to the "classical decoupling problem" of linear systems. Non-oscillatory free vibration and forced vibration will be treated in a future paper [34]. In non-oscillatory systems there will not be any real oscillations to synchronize while in forced vibration a new decoupling transformation will be required.

System decoupling plays a fundamental role not only in linear vibrations but also in diverse areas such as quantum mechanics, mathematical economics, and computational science. Upon decoupling, a system can be regarded as composing of independent single-degree-of-freedom components. This not only provides an efficient means of evaluating the system response but also greatly facilitates qualitative analysis. Among other things, it is hoped that this paper would point to directions along which further research efforts should be made. One such direction is obvious. The symmetry of $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ has not been used directly in phase synchronization. Thus the method devised in this paper may be further extended to decouple certain systems with non-symmetric coefficient matrices. The study of other issues, such as primary-secondary systems, numerical algorithms for decoupling, and modal reduction using the real damped modes, is also worthwhile in a subsequent course of investigation.

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References

[1] J.W. Strutt Lord Rayleigh, *The Theory of Sound*, Vol. I, Dover, New York, 1945 (Reprint of the 1894 edition).
[2] R.W. Clough, S. Mojtahedi, Earthquake response analysis considering non-proportional damping, *Earthquake Engineering and Structural Dynamics* 4 (1976) 489–496.
[3] H.C. Tsai, J.M. Kelly, Non-classical damping in dynamic analysis of base-isolated structures with internal equipment, *Earthquake Engineering and Structural Dynamics* 16 (1988) 29–43.
[4] H.C. Tsai, J.M. Kelly, Seismic response of the superstructure and attached equipment in a base-isolated building, *Earthquake Engineering and Structural Dynamics* 18 (1989) 551–564.
[5] T. Itoh, Damped vibration mode superposition method for dynamic response analysis, *Earthquake Engineering and Structural Dynamics* 2 (1973) 47–57.
[6] A. Sestieri, S.R. Ibrahim, Analysis of errors and approximations in the use of modal co-ordinates, *Journal of Sound and Vibration* 177 (2) (1994) 145–157.
[7] L. Meirovitch, *Analytical Methods in Vibrations*, Macmillan, New York, 1967.

[8] T.K. Caughey, M.E.J. O'Kelly, Classical normal modes in damped linear dynamic systems, *ASME Journal of Applied Mechanics* 32 (1965) 583–588.
[9] K.A. Foss, Co-ordinates which uncouple the equations of motion of damped linear dynamic systems, *ASME Journal of Applied Mechanics* 25 (1958) 361–364.
[10] A.S. Velestos, C.E. Ventura, Modal analysis of non-classically damped linear systems, *Earthquake Engineering and Structural Dynamics* 14 (1986) 217–243.
[11] F.R. Vigneron, A natural mode model and modal identities for damped linear dynamic structures, *ASME Journal of Applied Mechanics* 53 (1986) 33–38.
[12] D.L. Cronin, Approximation for determining harmonically excited response of nonclassically damped systems, *ASME Journal of Engineering for Industry* 98 (1976) 43–47.
[13] K.R. Chung, C.W. Lee, Dynamic reanalysis of weakly non-proportionally damped systems, *Journal of Sound and Vibration* 111 (1) (1986) 37–50.
[14] S.M. Shahruz, F. Ma, Approximate decoupling of the equations of motion of linear underdamped systems, *ASME Journal of Applied Mechanics* 55 (1988) 716–720.
[15] F.E. Udwadia, R.S. Esfandiari, Nonclassically damped dynamic systems: an iterative approach, *ASME Journal of Applied Mechanics* 57 (1990) 423–433.
[16] S.F. Felszeghy, On uncoupling and solving the equations of motion of vibrating linear discrete systems, *ASME Journal of Applied Mechanics* 60 (1993) 456–462.
[17] J.H. Hwang, F. Ma, On the approximate solution of nonclassically damped linear systems, *ASME Journal of Applied Mechanics* 60 (1993) 695–701.
[18] A. Bhaskar, Estimates of errors in the frequency response of non-classically damped systems, *Journal of Sound and Vibration* 184 (1) (1995) 59–72.
[19] W.K. Gawronski, *Dynamics and Control of Structures: A Modal Approach*, Springer, New York, 1998.
[20] J.K. Knowles, On the approximation of damped linear dynamical systems, *Structural Control and Health Monitoring* 13 (2006) 324–335.
[21] A.K. Chopra, *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, third ed., Prentice-Hall, Upper Saddle River, NJ, 2007.
[22] G. Prater, R. Singh, Quantification of the extent of non-proportional viscous damping in discrete vibratory systems, *Journal of Sound and Vibration* 104 (1) (1986) 109–125.
[23] S.S. Nair, R. Singh, Examination of the validity of proportional approximations and two further numerical indices, *Journal of Sound and Vibration* 104 (2) (1986) 348–350.
[24] J. Bellos, D.J. Inman, Frequency response of nonproportionally damped, lumped parameter, linear dynamic systems, *ASME Journal of Vibration and Acoustics* 112 (1990) 194–201.
[25] K. Xu, T. Igusa, Dynamic characteristics of non-classically damped structures, *Earthquake Engineering and Structural Dynamics* 20 (1991) 1127–1144.
[26] I.W. Park, J.S. Kim, F. Ma, Characteristics of modal coupling in nonclassically damped systems under harmonic excitation, *ASME Journal of Applied Mechanics* 61 (1994) 77–83.
[27] M. Tong, Z. Liang, G.C. Lee, An index of damping non-proportionality for discrete vibratory systems, *Journal of Sound and Vibration* 174 (1) (1994) 37–55.
[28] K. Liu, M.R. Kujath, W. Zheng, Quantification of non-proportionality of damping in discrete vibratory systems, *Computers and Structures* 77 (5) (2000) 557–569.
[29] K. Liu, M.R. Kujath, W. Zheng, Evaluation of damping non-proportionality using identified modal information, *Mechanical Systems and Signal Processing* 15 (1) (2001) 227–242.
[30] F. Ma, T.K. Caughey, Analysis of linear nonconservative vibrations, *ASME Journal of Applied Mechanics* 62 (1995) 685–691.
[31] T.K. Caughey, F. Ma, Complex modes and solvability of nonclassically damped linear systems, *ASME Journal of Applied Mechanics* 60 (1993) 26–28.
[32] P. Lancaster, *Lambda-Matrices and Vibrating Systems*, Pergamon Press, Oxford, UK, 1966.
[33] F. Tisseur, K. Meerbergen, The quadratic eigenvalue problem, *SIAM Review* 43 (2) (2001) 235–286.
[34] F. Ma, M. Morzfeld, A. Imam, The decoupling of damped linear systems in non-oscillatory free vibration and in forced vibration, *Journal of Sound and Vibration* (2009), in preparation.
[35] J. Woodhouse, Linear damping models for structural vibration, *Journal of Sound and Vibration* 215 (3) (1998) 547–569.
[36] W.L. Brogan, *Modern Control Theory*, third ed., Prentice-Hall, Upper Saddle River, NJ, 1991.
[37] C. Moler, C. Van Loan, Nineteen dubious ways to compute the exponential of a matrix, *SIAM Review* 20 (4) (1978) 801–836.
[38] E. Anderson, Z. Bai, C. Bischof, S. Blackford, J. Demmel, J. Dongorra, J. Du Croz, A. Greenbaum, S. Hammerling, A. McKenney, D. Sorensen, *LAPACK Users' Guide*, third ed., SIAM, Philadelphia, PA, 1999.
[39] J.W. Demmel, *Applied Numerical Linear Algebra*, SIAM, Philadelphia, PA, 1997.
[40] G.H. Golub, C.F. Van Loan, *Matrix Computations*, third ed., Johns Hopkins University Press, Baltimore, MD, 1996.
[41] U. Prells, M.I. Friswell, A relationship between defective systems and unit-rank modification of classical damping, *ASME Journal of Vibration and Acoustics* 122 (2) (2000) 180–183.
[42] D.J. Inman, *Vibration with Control*, Wiley, Hoboken, NJ, 2006.
[43] D.E. Newland, *Mechanical Vibration Analysis and Computation*, Longman, Essex, UK, 1989.
[44] I. Gohberg, P. Lancaster, L. Rodman, *Matrix Polynomials*, Academic Press, New York, 1982.
[45] P. Lancaster, M. Tismenetsky, *The Theory of Matrices*, second ed., Academic Press, New York, 1985.