An optimization approach for minimum norm and robust partial quadratic eigenvalue assignment problems for vibrating structures

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Abstract

The partial quadratic eigenvalue assignment problem (PQEVAP) concerns the reassignment of a small number of undesirable eigenvalues of a quadratic matrix pencil, while leaving the remaining large number of eigenvalues and the corresponding eigenvectors unchanged. The problem arises in controlling undesirable resonance in vibrating structures and in stabilizing control systems. The solution of this problem requires computations of a pair of feedback matrices. For practical effectiveness, these feedback matrices must be computed in such a way that their norms and the condition number of the closed-loop eigenvector matrix are as small as possible. These considerations give rise to the minimum norm partial quadratic eigenvalue assignment problem (MNPQEVAP) and the robust partial quadratic eigenvalue assignment problem (RPQEVAP), respectively. In this paper we propose new optimization based algorithms for solving these problems. The problems are solved directly in a second-order setting without resorting to a standard first-order formulation so as to avoid the inversion of a possibly ill-conditioned matrix and the loss of exploitable structures of the original model. The algorithms require the knowledge of only the open-loop eigenvalues to be replaced and their corresponding eigenvectors. The remaining open-loop eigenvalues and their corresponding eigenvectors are kept unchanged. The invariance of the large number of eigenvalues and eigenvectors under feedback is guaranteed by a proven mathematical result. Furthermore, the gradient formulas needed to solve the problems by using the quasi-Newton optimization technique employed are computed in terms of the known quantities only. Above all, the proposed methods do not require the reduction of the model order or the order of the controller, even when the underlying finite element model has a very large degree of freedom. These attractive features, coupled with minimal computational requirements, such as solutions of small diagonal Sylvester equations make the proposed algorithms ideally suited for application to large real-life structures. Numerical results show significant improvement in feedback norms and in the condition number of the closed-loop system. Also, the closed-loop eigenvalues have acceptable accuracy.

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Nomenclature

In this paper matrices and vectors have been denoted by bold characters. The following symbols and abbreviations have been used.

$\bar{a}$ complex conjugate of the number a

$\mathbf{A}^H$ Hermitian of the matrix $\mathbf{A}$

$\mathbf{A}^T$ transpose of the matrix $\mathbf{A}$

tr($\mathbf{A}$)   trace of matrix $\mathbf{A}$

det($\mathbf{A}$)   determinant of matrix $\mathbf{A}$

rank($\mathbf{A}$) rank of matrix $\mathbf{A}$

$\mathbf{A} > 0$ the matrix $\mathbf{A}$ is positive definite

$\mathbf{A} \geqslant 0$ the matrix $\mathbf{A}$ is positive semi-definite

$[\mathbf{A}, \mathbf{B}]$ the matrix $\mathbf{A}$ augmented by the matrix $\mathbf{B}$

$\mathbf{I}$ unit matrix

$\mathbf{I}_n$ unit matrix of order ($n \times n$)

$\mathbf{I}_{n \times n}$ unit matrix of order ($n \times n$)

$\mathbf{O}$ null matrix

$\mathbf{O}_{m \times n}$ null matrix of order ($m \times n$)

$P \cap Q$ the intersection of the sets P and Q

$x \in S$ x is an element of set S

$x \notin S$ x is not an element of set S

$\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{2n})$ matrix of right eigenvectors of the open-loop pencil $\mathbf{P}(\lambda)$

$\|\mathbf{X}\|_F$ Frobenius norm of $\mathbf{X}$

$\mathbf{X}_1 = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p)$ matrix of open-loop eigenvectors corresponding to the open-loop eigenvalues $\lambda_1, \ldots, \lambda_p$

$\mathbf{X}_2 = (\mathbf{x}_{p+1}, \ldots, \mathbf{x}_{2n})$ matrix of open-loop eigenvectors corresponding to the open-loop eigenvalues $\lambda_{p+1}, \ldots, \lambda_{2n}$

$\Delta \mathbf{Z}$ the first-order differential of the matrix $\mathbf{Z}$

$\nabla_{\boldsymbol{\Gamma}}(\mathbb{I})$ gradient of $\mathbb{I}$ with respect to Γ

ε tolerance limit for gradient in Algorithm 1

$\boldsymbol{\Lambda} = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{2n})$ the matrix of eigenvalues of the open-loop pencil $\mathbf{P}(\lambda)$

$\boldsymbol{\Lambda}_1 = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p)$ matrix of open-loop eigenvalues to be reassigned

$\boldsymbol{\Lambda}_2 = \text{diag}(\lambda_{p+1}, \ldots, \lambda_{2n})$ matrix of open-loop eigenvalues to remain invariant

$\boldsymbol{\Lambda}'_1 = \text{diag}(\mu_1, \mu_2, \ldots, \mu_p)$ matrix of new eigenvalues to replace those in $\boldsymbol{\Lambda}_1$

diag($a_1, a_2, \ldots, a_n$) the ($n \times n$) diagonal matrix whose diagonal elements starting from upper left hand corner are $a_1, a_2, \ldots, a_n$

iff.   if and only if

Max$_{\text{iter}}$ maximum number of iterations allowed in Algorithm 1

1. Introduction

It is well-known that vibrating structures, such as bridges, high rise buildings, aircrafts, spacecrafts, etc., can be modeled by a second-order linear matrix ordinary differential equation of the form:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{D}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t), \tag{1}$$

where $\mathbf{M}$, $\mathbf{D}$, $\mathbf{K}$ are constant real ($n \times n$) matrices, and $\mathbf{q}(t)$ and $\mathbf{f}(t)$ are real n-vectors. The matrix $\mathbf{M}$ is called the mass matrix, $\mathbf{D}$ is called the damping matrix and $\mathbf{K}$ is called the stiffness matrix. The vector $\mathbf{f}(t)$ represents an external force, t represents time and n is an integer, called the degrees of freedom (dof) of the system. In many applications, the matrices $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{D}$ are symmetric; furthermore, $\mathbf{M}$ is positive definite and $\mathbf{K}$ is positive semi-definite. We make the same assumptions about $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{D}$.

The dynamics of the structures modeled by Eq. (1), are governed by the eigenvalues and eigenvectors of the quadratic matrix polynomial $\mathbf{P}(\lambda)$ (see Refs. [1–3]) where

$$\mathbf{P}(\lambda) \equiv \mathbf{M}\lambda^2 + \mathbf{D}\lambda + \mathbf{K}. \tag{2}$$

The matrix polynomial $\mathbf{P}(\lambda)$ is called the open-loop pencil and its eigenvalues are called the open-loop eigenvalues. If $\mathbf{M}$ is non-singular, $\mathbf{P}(\lambda)$ has $2n$ eigenvalues which are the roots of the equation $\det(\mathbf{P}(\lambda)) = 0$ (see, for example, Ref. [4]). Also, a non-zero vector $\mathbf{x}$ is called the right eigenvector corresponding to the eigenvalue λ of the quadratic matrix pencil $\mathbf{P}(\lambda)$ if $(\lambda^2 \mathbf{M} + \lambda \mathbf{D} + \mathbf{K})\mathbf{x} = 0$.

In some cases we find that a small number of open-loop eigenvalues are undesirable, because their presence has a damaging effect on the system. Undesirable resonance in vibrating structures is a case in point. It is well

known that if a vibrating structure is acted on by external forces that are vibratory in nature whose frequencies are equal to or close to the natural frequencies of the vibrating structure then the vibrations get greatly amplified. This phenomenon is called resonance. Now, the natural frequencies of a vibrating structure, which are a characteristic property of the structure, are closely related to the open-loop eigenvalues. Thus it is the presence of certain open-loop eigenvalues which is responsible for causing resonance. Resonance can have a disastrous effect on structures. Failures of many structures like buildings, bridges, airplane wings and turbines have been attributed to resonance.

Vibrations can be controlled either by using passive control devices such as vibration absorbers or vibration dampers or by applying a feedback control force by means of active controllers. Passive control devices are widely used because of their simplicity, low cost and easiness of use. However they have some practical limitations. A more effective way of controlling resonance or other forms of instability, especially when a large control force is required, is by using active controllers on the structures. Due to the advances in sensors and actuators, the use of active vibration control is now becoming more and more popular. Indeed, many structures in several countries have been built using active controllers. It may be noted that the Kyobashi Seiwa Building built in 1989 in Tokyo was the first building to be constructed with an active control system.

To implement an active control strategy, feedback control gains must be computed in real time and it is crucial that such gains are computed in an efficient and numerically robust way. Suppose that a control force of the form $\mathbf{B}\mathbf{u}(t)$ (see Refs. [3–5]) is applied to the structure. Here $\mathbf{B}$ is a given real $n \times m$ matrix ($m \leqslant n$) and $\mathbf{u}(t)$ is a real m-vector given by

$$\mathbf{u}(t) = \mathbf{F}^{\mathrm{T}}\dot{\mathbf{q}}(t) + \mathbf{G}^{\mathrm{T}}\mathbf{q}(t), \tag{3}$$

and $\mathbf{F}$ and $\mathbf{G}$ are unknown, constant, real, ($n \times m$) matrices called the feedback matrices. We assume that $\mathbf{q}(t)$ and $\dot{\mathbf{q}}(t)$ can be measured. From Eqs. (1) and (3), we obtain the closed-loop system:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + (\mathbf{D} - \mathbf{B}\mathbf{F}^{\mathrm{T}})\dot{\mathbf{q}}(t) + (\mathbf{K} - \mathbf{B}\mathbf{G}^{\mathrm{T}})\mathbf{q}(t) = \mathbf{f}(t). \tag{4}$$

The dynamics of this closed-loop system are determined by the eigenvalues and eigenvectors of the quadratic matrix polynomial:

$$\mathbf{P}_c(\lambda) \equiv \lambda^2\mathbf{M} + \lambda(\mathbf{D} - \mathbf{B}\mathbf{F}^{\mathrm{T}}) + (\mathbf{K} - \mathbf{B}\mathbf{G}^{\mathrm{T}}). \tag{5}$$

The quadratic matrix polynomial $\mathbf{P}_c(\lambda)$ is called the closed-loop pencil and its eigenvalues are called the closed-loop eigenvalues. Now, in order to prevent resonance, the feedback matrices $\mathbf{F}$ and $\mathbf{G}$ are determined such that the closed-loop spectrum is the one that we obtain from the open-loop spectrum by replacing the small number of resonant eigenvalues, by suitably chosen ones while keeping the remaining eigenvalues and associated eigenvectors unchanged. The last property, known as the no spill-over property, guarantees that the large number of unassigned modes will not themselves become resonant or unstable. The problem of finding $\mathbf{F}$ and $\mathbf{G}$ is referred to as the partial quadratic eigenvalue assignment problem (PQEVAP). In this context it may be noted that if rank($[\mathbf{M}\lambda^2 + \mathbf{D}\lambda + \mathbf{K}, \mathbf{B}]$) = n holds for the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_p$ then these p open-loop eigenvalues can be reassigned arbitrarily by choosing $\mathbf{F}$ and $\mathbf{G}$ appropriately (see, for example, Ref. [6]). If this condition is true, we say that the pair ($\mathbf{P}(\lambda), \mathbf{B}$) is partially controllable with respect to these p eigenvalues.

The partial pole assignment problem was first introduced by Porter and Crosley (see Ref. [7]) for the first-order model. The problem was later studied by Datta and Sarkissian (see Ref. [8]), and Saad (see Ref. [9]), and Datta and Saad (see Ref. [10]), also for the first-order system. One way of solving the PQEVAP is to transform the quadratic control problem to a standard first-order state-space problem and then solve the partial pole-placement problem for the first order system (see Ref. [11]). However, there are several computational concerns with this approach. For instance, it would require inversion of the mass matrix, which may be ill-conditioned. Also, this transformation would, in most cases, destroy all the exploitable structures inherent in most practical problems, such as symmetry, definiteness, sparsity, bandedness, etc. Furthermore, since the existing eigenvalue assignment methods are designed for small and dense problems only, the order of the second-order finite element model, which is usually very large, must be reduced, and this process will invariably give rise to instability due to controllability and observability spill-overs. Moreover, the target eigenvalues and eigenvectors computed from a reduced-order model often differ very much from those of the original model. Similarly, the state-of-the-art independent modal space-control (IMSC) approach (see Ref. [4])

for vibration control, also suffers from practical engineering and computational limitations. The basic idea behind IMSC is to decouple the problem into two independent problems, solve these independent problems individually, and then piece the solutions together to get the solution of the original problem. But the open-loop decoupling requires the complete knowledge of the eigenvalues and eigenvectors of the open-loop pencil, whereas the state-of-the-art computational techniques for quadratic eigenvalue problem, such as the Jacobi–Davidson method (see Ref. [12]), are capable of computing only a small number of extremal eigenvalues and eigenvectors of a large quadratic pencil. The closed-loop decoupling, on the other hand, imposes stringent requirements on sensors and actuators (see Ref. [4]). In view of these considerations, for a numerically effective solution of the PQEVAP, it is imperative that the problem be solved in quadratic setting using only the small number of eigenvalues and eigenvectors that are computable or measurable in a vibration laboratory, and without any a priori model reduction. Moreover, in a practical computational setting, in the absence of techniques for computing the whole spectrum and the associated eigenvectors of a quadratic matrix pencil, the no spill-over property must be established with the help of a mathematical theory.

In the multi-input case (i.e. when $m > 1$), the solution of the PQEVAP is not unique. For practical effectiveness, we can take advantage of this fact by determining $\mathbf{F}$ and $\mathbf{G}$ in such a way that not only $\mathbf{P}_c(\lambda)$ has the desired spectrum but the system has some additional desirable features also. An important practical consideration in designing a vibration control system is to ensure robustness, that is insensitivity of the closed-loop eigenvalues to small perturbations in data. To achieve this, the feedback matrices should be computed in such a way that their norms are as small as possible and the closed-loop eigenvector matrix has minimum condition number (see Ref. [13]). Smaller feedback norms also lead to smaller control signals which in turn lead to lesser energy consumption and lesser noise amplification (see Ref. [13]). The problem of finding the feedback matrices such that the closed-loop pencil has the desired spectrum and the feedback norms are as small as possible is known as the minimum norm partial quadratic eigenvalue assignment problem (MNPQEVAP) and the problem of finding the feedback matrices such that the closed-loop pencil has the desired spectrum and the closed-loop eigenvector matrix has minimum condition number is called the robust partial quadratic eigenvalue assignment problem (RPQEVAP).

These are clearly optimization problems. To solve them in an optimization setting, a parameterized family of feedback matrices is generated and then the parametric matrix is determined appropriately. In this paper we use a quasi-Newton optimization algorithm, the Broyden–Fletcher–Goldfarb–Shanno (BFGS) method (see Ref. [14]).

A direct and partial-modal approach for solving the PQEVAP was first proposed in the single-input case by Datta, Elhay and Ram (see Ref. [2]) and then generalized to the multi-input case by Datta and Sarkissian (see Ref. [8]) and by Datta, Elhay and Ram (see Ref. [15]), without using Sylvester equations. However the problems of feedback norm minimization or minimization of the condition number of the closed-loop eigenvector matrix were not considered in these papers. Such problems were considered, for first-order control systems, by Keel, Fleming and Bhattacharyya (see Ref. [16]), Cavin and Bhattacharyya (see Ref. [17]) and Varga (see Ref. [13]). The only paper that has been published so far that deals solely with the robustness issue for PQEVAP is by Qian and Xu (see Ref. [18]). This algorithm is not optimization-based.

The major contributions of this paper are as follows:

- A parameterized algorithm for generating a family of feedback matrices for solving the PQEVAP using Sylvester equations has been developed. The no spill-over property is guaranteed by establishing mathematical results.
- Gradient formulas for the BFGS algorithm used in solving the MNPQEVAP, RPQEVAP and also for the problem of simultaneously reducing the feedback norms and the condition number of the closed-loop eigenvector matrix have been developed.

Also the parametric expressions for feedback matrices and the gradient formulas for the optimization problems are computed in terms of known quantities only, viz., the open-loop eigenvalues that are to be reassigned, their corresponding eigenvectors and the data matrices. The Sylvester equation approach for parameterized solutions turn out to be crucial for the development of the gradient formulas. Numerical experimental results show that the considerable reductions in both the feedback norms and the closed-loop

condition numbers can be achieved by our algorithms. The accuracy of the closed-loop eigenvalues is acceptable as well. Our results on RPQEVAP show that these are quite comparable with those obtained by Qian and Xu.

This paper contains a detailed discussion on results relating to the RPQEVAP only. A preliminary version of this paper has appeared (see Ref. [21]). In this paper the results for MNPQEVAP have been stated without proof. The proofs may be found in Refs. [22,23].

To conclude this section, we make the following observations:

(i) Implementation of the control law in Eq. (3) requires the complete knowledge of the state and velocity vectors. However, in practice some of these may not be measurable. The non-measurable ones must be estimated by constructing suitable observers for a second-order control system. In a recent paper by Carvalho and Datta (see Ref. [24]), a numerically effective algorithm has been proposed for state and velocity vector estimation of a second-order control system via its transformation to a descriptor control system.
(ii) Our results apply only to finite-element models having a finite number of dof. However, it might be possible to use our scheme for vibration control in real-life models such as a beam or a plate that have infinite number of dof, in conjunction with some passive control device. It is also worth mentioning in this context that Datta and Sarkissian (see Ref. [25]) have developed a scheme for feedback control in distributed gyroscopic systems and applied it to partial pole assignment, corresponding to the lowest frequencies of small oscillations of a moving string. Further research on developing such schemes for other distributed parameter systems is in order.

2. Assumptions

The following assumptions, which are quite reasonable in practice, are made:

$$\{\lambda_1, \lambda_2, \ldots, \lambda_p\} \cap \{\lambda_{p+1}, \lambda_{p+2}, \ldots, \lambda_{2n}\} \cap \{\mu_1, \mu_2, \ldots, \mu_p\} = \phi, \tag{6a}$$

$$0 \notin \{\lambda_1, \lambda_2, \ldots, \lambda_p\}, \tag{6b}$$

$$\mathbf{P}(\lambda) \text{ has a complete set of linearly independent eigenvectors}, \tag{6c}$$

$$\text{The pair } (\mathbf{P}(\lambda), \mathbf{B}) \text{ is partially controllable with respect to } \lambda_1, \lambda_2, \ldots, \lambda_p, \tag{6d}$$

$$\text{The sets } \{\lambda_1, \ldots, \lambda_p\} \text{ and } \{\mu_1, \ldots, \mu_p\} \text{ are closed under complex conjugation.} \tag{6e}$$

3. Some eigenvalue–eigenvector properties of the quadratic matrix pencil

In this section we present some useful properties of the quadratic matrix pencils $\mathbf{P}(\lambda)$ and $\mathbf{P}_c(\lambda)$ which we use later in this paper.

Theorem 3.1.

(i) *A scaler $\lambda \in \mathbb{C}$ is an eigenvalue of the quadratic pencil $\mathbf{P}(\lambda) = \mathbf{M}\lambda^2 + \mathbf{D}\lambda + \mathbf{K}$ with right eigenvector $\mathbf{x}$ iff λ is an eigenvalue of the matrix*

$$\mathbf{A} = \begin{pmatrix} \mathbf{O}_{(n \times n)} & \mathbf{I}_{(n \times n)} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{D} \end{pmatrix},$$

with right eigenvector

$$\hat{\mathbf{x}} = \begin{pmatrix} \mathbf{x} \\ \lambda\mathbf{x} \end{pmatrix}.$$

(ii) *The eigenvalues of the closed-loop pencil* $\mathbf{P}_c(\lambda) \equiv \mathbf{M}\lambda^2 + (\mathbf{D} - \mathbf{BF}^T)\lambda + (\mathbf{K} - \mathbf{BG}^T)$ *are the same as the eigenvalues of the matrix* $\widehat{\mathbf{A}} = \mathbf{A} - \widehat{\mathbf{B}}\widehat{\mathbf{F}}$, *where* $\widehat{\mathbf{F}} = [-\mathbf{G}^T, -\mathbf{F}^T]$ *and* $\widehat{\mathbf{B}} = \begin{pmatrix} \mathbf{O}_{(n \times m)} \\ \mathbf{M}^{-1}\mathbf{B} \end{pmatrix}$.

Also, the matrix of right eigenvectors of $\widehat{\mathbf{A}}$ *is the matrix*

$$\mathbf{Y} = \begin{pmatrix} \mathbf{Y}_1 & \mathbf{X}_2 \\ \mathbf{Y}_1\mathbf{\Lambda}_1' & \mathbf{X}_2\mathbf{\Lambda}_2 \end{pmatrix},$$

where $\mathbf{Y}_1 = [\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p]$, *and* $\mathbf{y}_i$ *is the right eigenvector of the pencil* $\mathbf{P}_c(\lambda)$ *corresponding to the eigenvalue* μ_i.

Proof. Since λ is an eigenvalue of the matrix pencil $\mathbf{P}(\lambda)$ with right eigenvector $\mathbf{x}$ we have $(\mathbf{M}\lambda^2 + \mathbf{D}\lambda + \mathbf{K})\mathbf{x} = 0$.

Thus

$$\mathbf{A}\hat{\mathbf{x}} = \begin{pmatrix} \lambda\mathbf{x} \\ -\mathbf{M}^{-1}(\mathbf{K} + \lambda\mathbf{D})\mathbf{x} \end{pmatrix} = \lambda\begin{pmatrix} \mathbf{x} \\ \lambda\mathbf{x} \end{pmatrix}. \tag{7}$$

This proves result (i).

By result (i) it follows that the eigenvalues of $\mathbf{P}_c(\lambda)$ are the same as the eigenvalues of the matrix

$$\begin{pmatrix} \mathbf{O}_{(n \times n)} & \mathbf{I}_{(n \times n)} \\ -\mathbf{M}^{-1}(\mathbf{K} - \mathbf{BG}^T) & -\mathbf{M}^{-1}(\mathbf{D} - \mathbf{BF}^T) \end{pmatrix} = \mathbf{A} - \widehat{\mathbf{B}}\widehat{\mathbf{F}}, \tag{8}$$

and the matrix of right eigenvectors of the matrix $\mathbf{A} - \widehat{\mathbf{B}}\widehat{\mathbf{F}}$ is the matrix

$$\mathbf{Y} = \begin{pmatrix} \mathbf{Y}_1 & \mathbf{X}_2 \\ \mathbf{Y}_1\mathbf{\Lambda}_1' & \mathbf{X}_2\mathbf{\Lambda}_2 \end{pmatrix}. \quad \square$$

3.1. Orthogonality properties of the eigenvectors of a quadratic matrix pencil

Generalizing the well-known orthogonality properties of the eigenvectors of a symmetric matrix and of a symmetric definite linear pencil of the form $(\mathbf{K} - \lambda\mathbf{M})$, three orthogonality relations for the quadratic matrix pencil were derived by Datta, Elhay and Ram (see Ref. [2]). One of these relations, which is stated below, is used in this paper.

Theorem 3.2. *Let* $\mathbf{M} = \mathbf{M}^T > 0$, $\mathbf{K} = \mathbf{K}^T \geqslant 0$ *and suppose* $\mathbf{\Lambda}_1$ *and* $\mathbf{\Lambda}_2$ *have disjoint spectra. Then*

$$\mathbf{\Lambda}_1\mathbf{X}_1^T\mathbf{M}\mathbf{X}_2\mathbf{\Lambda}_2 - \mathbf{X}_1^T\mathbf{K}\mathbf{X}_2 = 0. \tag{9}$$

Proof. See Refs. [2,23]. $\quad \square$

The above orthogonality relation can be rewritten as

Corollary 3.1.

$$\begin{pmatrix} -\mathbf{KX}_1 \\ \mathbf{MX}_1\mathbf{\Lambda}_1 \end{pmatrix}^T \begin{pmatrix} \mathbf{X}_2 \\ \mathbf{X}_2\mathbf{\Lambda}_2 \end{pmatrix} = \mathbf{O}_{(p \times 2n - p)}. \tag{10}$$

This corollary can be used to prove the following result:

Corollary 3.2. *If a* $(2n \times p)$ *matrix* $\mathbf{Q}$ *satisfies* $\mathbf{Q}\begin{pmatrix} \mathbf{X}_2 \\ \mathbf{X}_2\mathbf{\Lambda}_2 \end{pmatrix} = \mathbf{O}_{(p \times 2n - p)}$, *then there exists a* $(p \times p)$ *matrix* $\mathbf{\Psi}$ *such that*

$$\mathbf{Q}^T = \begin{pmatrix} -\mathbf{KX}_1 \\ \mathbf{MX}_1\mathbf{\Lambda}_1 \end{pmatrix}\mathbf{\Psi}. \tag{11}$$

Proof. See Ref. [23]. $\quad \square$

4. A parametric expression for the feedback matrices

As a first step towards solving the RPQEVAP and the MNPQEVAP in an optimization setting, we proceed to obtain a parametric solution of the PQEVAP, in terms of a parametric matrix $\mathbf{\Gamma}$. Then in the following sections we describe how to choose $\mathbf{\Gamma}$ appropriately to solve the MNPQEVAP and RPQEVAP.

Theorem 4.1 (*Parametric expression for feedback matrices*). *Let* $\mathbf{\Gamma} = \{\gamma_1, \gamma_2, \ldots, \gamma_p\} \in \mathbb{C}^{m \times p}$ *be such that if* $\mu_j = \bar{\mu}_k$ *then* $\gamma_j = \bar{\gamma}_k$. *Let* $\mathbf{Z}$ *be the unique solution of the* $(p \times p)$ *Sylvester equation*:

$$\mathbf{\Lambda}_1 \mathbf{Z} - \mathbf{Z} \mathbf{\Lambda}_1' = -\mathbf{\Lambda}_1 \mathbf{X}_1^{\mathrm{T}} \mathbf{B} \mathbf{\Gamma}. \tag{12}$$

Then feedback matrices $\mathbf{F}$ *and* $\mathbf{G}$ *given by*

$$\mathbf{F} = \mathbf{M} \mathbf{X}_1 \mathbf{\Lambda}_1 \mathbf{\Phi}^{\mathrm{T}} \quad and \quad \mathbf{G} = -\mathbf{K} \mathbf{X}_1 \mathbf{\Phi}^{\mathrm{T}}, \tag{13}$$

where $\mathbf{\Phi}$ *is obtained by solving the linear system*:

$$\mathbf{\Phi} \mathbf{Z} = \mathbf{\Gamma} \tag{14}$$

are solutions of the PQEVAP. Moreover, for every non-zero $\mathbf{\Gamma}$ *satisfying the above relation, the matrices* $\mathbf{F}$ *and* $\mathbf{G}$ *are real.*

Proof. The proof has four parts.

Part I (proof of the no-spill-over property): Here we prove that if the matrices $\mathbf{F}$ and $\mathbf{G}$ are chosen as above and $\mathbf{\Phi}$ is any non-zero $(m \times p)$ matrix, then the eigenvalues of the closed-loop pencil would include $\lambda_{p+1}, \lambda_{p+2}, \ldots, \lambda_{2n}$ with corresponding eigenvectors $\mathbf{x}_{p+1}, \mathbf{x}_{p+2}, \ldots, \mathbf{x}_{2n}$.

That is, with these choices of $\mathbf{F}$ and $\mathbf{G}$, there will be no spill-over.

For this we need to show that

$$\mathbf{M} \mathbf{X}_2 \mathbf{\Lambda}_2^2 + (\mathbf{D} - \mathbf{B} \mathbf{F}^{\mathrm{T}}) \mathbf{X}_2 \mathbf{\Lambda}_2 + (\mathbf{K} - \mathbf{B} \mathbf{G}^{\mathrm{T}}) \mathbf{X}_2 = 0. \tag{15}$$

Eq. (15) can be rewritten as

$$\mathbf{M} \mathbf{X}_2 \mathbf{\Lambda}_2^2 + \mathbf{D} \mathbf{X}_2 \mathbf{\Lambda}_2 + \mathbf{K} \mathbf{X}_2 - \mathbf{B} [\mathbf{F}^{\mathrm{T}} \mathbf{X}_2 \mathbf{\Lambda}_2 + \mathbf{G}^{\mathrm{T}} \mathbf{X}_2] = 0. \tag{16}$$

Now, since the matrices $(\mathbf{\Lambda}_1, \mathbf{X}_1)$ and $(\mathbf{\Lambda}_2, \mathbf{X}_2)$ are partial eigenvalue and corresponding eigenvector matrices of the quadratic matrix pencil $\mathbf{P}(\lambda)$, we have

$$\mathbf{M} \mathbf{X}_1 \mathbf{\Lambda}_1^2 + \mathbf{D} \mathbf{X}_1 \mathbf{\Lambda}_1 + \mathbf{K} \mathbf{X}_1 = 0, \tag{17}$$

$$\mathbf{M} \mathbf{X}_2 \mathbf{\Lambda}_2^2 + \mathbf{D} \mathbf{X}_2 \mathbf{\Lambda}_2 + \mathbf{K} \mathbf{X}_2 = 0. \tag{18}$$

So, by virtue of Eq. (18), we need only show

$$\mathbf{B} [\mathbf{F}^{\mathrm{T}} \mathbf{X}_2 \mathbf{\Lambda}_2 + \mathbf{G}^{\mathrm{T}} \mathbf{X}_2] = 0. \tag{19}$$

It is easy to see that this relation follows immediately by substituting the expressions of the feedback matrices $\mathbf{F}$ and $\mathbf{G}$ and then using the orthogonality relation in Theorem 3.2.

Part II (partial assignment of the spectrum): Here we show that if $\mathbf{\Phi}$ is chosen according to the criterion stated in the theorem then the spectrum of the closed-loop pencil will include $\mu_1, \mu_2, \ldots, \mu_p$ and the eigenvectors corresponding to these eigenvalues will be $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p$, which we define later.

Let $\mathbf{Y}_1 = [\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p]$.

In order to show that the closed-loop pencil has the eigenvalues $\mu_1, \mu_2, \ldots, \mu_p$ with $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p$ as the corresponding eigenvectors, we need to show that

$$\mathbf{M} \mathbf{Y}_1 \mathbf{\Lambda}_1'^2 + (\mathbf{D} - \mathbf{B} \mathbf{F}^{\mathrm{T}}) \mathbf{Y}_1 \mathbf{\Lambda}_1' + (\mathbf{K} - \mathbf{B} \mathbf{G}^{\mathrm{T}}) \mathbf{Y}_1 = 0. \tag{20}$$

This means that we have to show that

$$\mathbf{M}\mathbf{Y}_1\mathbf{\Lambda}_1'^2 + \mathbf{D}\mathbf{Y}_1\mathbf{\Lambda}_1' + \mathbf{K}\mathbf{Y}_1 = \mathbf{B}[\mathbf{F}^\mathrm{T}\mathbf{Y}_1\mathbf{\Lambda}_1' + \mathbf{G}^\mathrm{T}\mathbf{Y}_1],$$
$$= \mathbf{B}\mathbf{\Phi}[\mathbf{\Lambda}_1\mathbf{X}_1^\mathrm{T}\mathbf{M}\mathbf{Y}_1\mathbf{\Lambda}_1' - \mathbf{X}_1^\mathrm{T}\mathbf{K}\mathbf{Y}_1],$$
$$= \mathbf{B}\mathbf{\Phi}\mathbf{Z}, \tag{21}$$

where

$$\mathbf{Z} = \mathbf{\Lambda}_1\mathbf{X}_1^\mathrm{T}\mathbf{M}\mathbf{Y}_1\mathbf{\Lambda}_1' - \mathbf{X}_1^\mathrm{T}\mathbf{K}\mathbf{Y}_1. \tag{22}$$

Suppose we choose $\mathbf{\Phi}$ such that $\mathbf{\Phi}\mathbf{Z} = \mathbf{\Gamma}$, where $\mathbf{\Gamma} = \{\gamma_1, \gamma_2, \dots, \gamma_p\}$ is chosen arbitrarily.

Then, we must show that

$$\mathbf{M}\mathbf{Y}_1\mathbf{\Lambda}_1'^2 + \mathbf{D}\mathbf{Y}_1\mathbf{\Lambda}_1' + \mathbf{K}\mathbf{Y}_1 = \mathbf{B}\mathbf{\Gamma}, \tag{23a}$$

or

$$(\mathbf{M}\mu_j^2 + \mathbf{D}\mu_j + \mathbf{K})\mathbf{y}_j = \mathbf{B}\gamma_j \quad \text{for } (j = 1 : p). \tag{23b}$$

By assumption (6a) it follows that μ_j (for $j = 1 : p$) is not an eigenvalue of $\mathbf{P}(\lambda)$. Hence,

$$\det(\mathbf{M}\mu_j^2 + \mathbf{D}\mu_j + \mathbf{K}) \neq 0. \tag{24}$$

This means that $\mathbf{y}_j$'s are uniquely determined by Eq. (23b). Thus for our choice of $\mathbf{F}$, $\mathbf{G}$, $\mathbf{Y}_1$ and $\mathbf{\Phi}$ Eq. (20) is satisfied.

Part III (solution of the Sylvester Eq. (12) is unique): We now prove that $\mathbf{Z}$ is the unique solution of the Sylvester equation $\mathbf{\Lambda}_1\mathbf{Z} - \mathbf{Z}\mathbf{\Lambda}_1' = -\mathbf{\Lambda}_1\mathbf{X}_1^\mathrm{T}\mathbf{B}\mathbf{\Gamma}$.

Pre-multiplying both sides of Eq. (23a) by $-\mathbf{\Lambda}_1\mathbf{X}_1^\mathrm{T}$ we get

$$-\mathbf{\Lambda}_1\mathbf{X}_1^\mathrm{T}\mathbf{M}\mathbf{Y}_1\mathbf{\Lambda}_1'^2 - \mathbf{\Lambda}_1\mathbf{X}_1^\mathrm{T}\mathbf{D}\mathbf{Y}_1\mathbf{\Lambda}_1' - \mathbf{\Lambda}_1\mathbf{X}_1^\mathrm{T}\mathbf{K}\mathbf{Y}_1 = -\mathbf{\Lambda}_1\mathbf{X}_1^\mathrm{T}\mathbf{B}\mathbf{\Gamma}. \tag{25}$$

Now eliminating $\mathbf{D}$ from this equation using Eq. (17) we obtain $-\mathbf{\Lambda}_1\mathbf{X}_1^\mathrm{T}\mathbf{B}\mathbf{\Gamma} = \mathbf{\Lambda}_1\mathbf{Z} - \mathbf{Z}\mathbf{\Lambda}_1'$.

Also, by virtue of assumption (6a) the diagonal matrices $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_1'$ have no eigenvalues in common, thus the Sylvester equation (12) has a unique solution. It may be noted that the Sylvester equation $\mathbf{A}\mathbf{X} - \mathbf{X}\mathbf{B} = \mathbf{C}$ has a unique solution iff. the matrices $\mathbf{A}$ and $\mathbf{B}$ do not have common eigenvalues (see, Ref. [11]).

Part IV (feedback matrices are real): Finally, we show that if $\mathbf{\Gamma}$ is chosen such that if $\mu_j = \bar{\mu}_k$ then $\gamma_j = \bar{\gamma}_k$ then, $\mathbf{F}$ and $\mathbf{G}$ are real.

Since $\{\mu_1, \mu_2, \dots, \mu_p\}$ is a self-conjugate set, $\bar{\mathbf{\Lambda}}_1'$ can be obtained from $\mathbf{\Lambda}_1'$ by interchanging certain pairs of columns and the corresponding rows.

Thus, there exists a permutation matrix $\mathbf{P}_1$ such that

$$\bar{\mathbf{\Lambda}}_1' = \mathbf{P}_1^\mathrm{T}\mathbf{\Lambda}_1'\mathbf{P}_1. \tag{26}$$

Again, by virtue of the condition that if $\mu_j = \bar{\mu}_k$ then $\gamma_j = \bar{\gamma}_k$ it follows that $\bar{\mathbf{\Gamma}}$ can be obtained from $\mathbf{\Gamma}$ by interchanging the same pairs of columns as was done to obtain $\bar{\mathbf{\Lambda}}_1'$ from $\mathbf{\Lambda}_1$. Thus,

$$\bar{\mathbf{\Gamma}} = \mathbf{\Gamma}\mathbf{P}_1. \tag{27}$$

Also, since $\{\lambda_1, \lambda_2, \dots, \lambda_p\}$ is a self-conjugate set, there exists a permutation matrix $\mathbf{P}_2$ such that

$$\bar{\mathbf{\Lambda}}_1 = \mathbf{P}_2^\mathrm{T}\mathbf{\Lambda}_1\mathbf{P}_2. \tag{28}$$

Now, we know that if $\mathbf{A}$ is a real matrix then the eigenvectors of $\mathbf{A}$, associated with complex-conjugate eigenvalues are themselves complex-conjugate.

Thus,

$$\bar{\mathbf{Y}}_1 = \mathbf{Y}_1\mathbf{P}_1 \quad \text{and} \quad \bar{\mathbf{X}}_1 = \mathbf{X}_1\mathbf{P}_2. \tag{29}$$

From above it follows

$$\bar{\mathbf{Z}} = \bar{\mathbf{\Lambda}}_1(\bar{\mathbf{X}}_1)^\mathrm{T}\mathbf{M}\bar{\mathbf{Y}}_1\bar{\mathbf{\Lambda}}_1' - (\bar{\mathbf{X}}_1)^\mathrm{T}\mathbf{K}\bar{\mathbf{Y}}_1 = \mathbf{P}_2^\mathrm{T}\mathbf{Z}\mathbf{P}_1, \tag{30}$$

and

$$\bar{\mathbf{\Phi}} = \mathbf{\Phi}\mathbf{P}_2. \tag{31}$$

From the relations obtained it follows that $\overline{\mathbf{F}} = \mathbf{F}$ and $\overline{\mathbf{G}} = \mathbf{G}$. Hence $\mathbf{F}$ and $\mathbf{G}$ are real.

It may be noted that although there does not exist a numerically verifiable necessary and sufficient condition for the non-singularity of the solution $\mathbf{Z}$ of Eq. (12), for most non-null choices of the matrix $\mathbf{\Gamma}$, the matrix $\mathbf{Z}$ is non-singular (as also noted in Ref. [27]). □

5. Minimizing the feedback norms

Let $\mathbb{I} = \frac{1}{2}\|\mathbf{S}\|_F^2 = \frac{1}{2}[\|\mathbf{F}\|_F^2 + \|\mathbf{G}\|_F^2]$ where $\mathbf{S} = [\mathbf{G}^T\ \mathbf{F}^T]$. Since $\mathbf{F}$ and $\mathbf{G}$ are functions of $\mathbf{\Gamma}$ only, the problem of minimizing $\mathbb{I}$ can be posed as Minimize: $\mathbb{I} = f(\mathbf{\Gamma})$.

This is an unconstrained optimization problem. To solve this problem using the BFGS algorithm we require an analytic expression for the gradient of $\mathbb{I}$ with respect to $\mathbf{\Gamma}$. We denote this gradient by $\nabla_{\mathbf{\Gamma}}(\mathbb{I})$. The gradient formula is obtained in terms of only the known quantities, viz., $\mathbf{\Lambda}_1, \mathbf{\Lambda}_1', \mathbf{X}_1, \mathbf{M}, \mathbf{K}, \mathbf{B}$ and $\mathbf{\Gamma}$.

Below we just state the result without proof. The proof can be found in Refs. [22,23]. For completeness, we first state a result on the solution of a Sylvester equation (see Refs. [28,29]) that was used to derive our gradient expression for MNPQEVAP and which will also be used to obtain the gradient formula for RPQEVAP later in this paper.

Theorem 5.1. *Suppose the Sylvester equation*

$$\mathbf{AX} - \mathbf{XB} = \mathbf{C} \tag{32}$$

has a unique solution. Here $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{X}$ are $(m \times m)$, $(n \times n)$, $(m \times n)$, $(m \times n)$ matrices, respectively. Let $\alpha(t)$ and $\beta(t)$ be coprime monic polynomials having degrees μ and v, respectively, such that $\boldsymbol{\alpha}(\mathbf{A})\mathbf{C} = 0$ and $\mathbf{C}\boldsymbol{\beta}(\mathbf{B}) = 0$. Then a unique solution of the Sylvester equation can be represented in the form

$$\mathbf{X} = \sum_{j=1}^{v}\sum_{i=1}^{\mu}\gamma_{ij}\mathbf{A}^{i-1}\mathbf{C}\mathbf{B}^{j-1}, \tag{33}$$

where (γ_{ij})'s are scalars that are of no significance to us.

Corollary 5.1. *Suppose $m = n$ and the matrices $\mathbf{A}$ and $\mathbf{B}$ have n distinct eigenvalues, with no eigenvalues in common. Then for an arbitrary $n \times n$ matrix $\mathbf{C}_1$, a unique solution of the Sylvester equation*

$$\mathbf{BY} - \mathbf{YA} = \mathbf{C}_1 \tag{34}$$

is given by

$$\mathbf{Y} = \sum_{j=1}^{n}\sum_{i=1}^{n}\gamma_{ij}\mathbf{B}^{j-1}(-\mathbf{C}_1)\mathbf{A}^{i-1}. \tag{35}$$

Theorem 5.2 (*Gradient formula for $\mathbb{I}$*). *Suppose $\mathbf{F}$ and $\mathbf{G}$ are defined as in the previous theorem. Let $\mathbf{S} = [\mathbf{G}^T, \mathbf{F}^T]$, $\mathbf{P} = \mathbf{MX}_1\mathbf{\Lambda}_1$, $\mathbf{Q} = -\mathbf{KX}_1$ and $\mathbf{C} = [\mathbf{Q}^T, \mathbf{P}^T]$. Let $\mathbf{Z}$ satisfy the Sylvester equation*:

$$\mathbf{\Lambda}_1\mathbf{Z} - \mathbf{Z}\mathbf{\Lambda}_1' = -\mathbf{\Lambda}_1\mathbf{X}_1^T\mathbf{B}\mathbf{\Gamma}.$$

Suppose that $\mathbf{Z}$ is invertible and $\mathbf{U}$ satisfies the Sylvester equation:

$$\mathbf{\Lambda}_1'\mathbf{U} - \mathbf{U}\mathbf{\Lambda}_1 = \mathbf{Z}^{-1}\mathbf{CS}^H\mathbf{\Phi}, \tag{36}$$

where $\mathbf{\Phi}\mathbf{Z} = \mathbf{\Gamma}$. Then

$$\text{(i)} \quad \mathbf{S} = \mathbf{\Gamma}\mathbf{Z}^{-1}\mathbf{C}, \tag{37a}$$

$$\text{(ii)} \quad \nabla_{\mathbf{\Gamma}}(\mathbb{I}) = \tfrac{1}{2}[\mathbf{Z}^{-1}\mathbf{CS}^H - \mathbf{U}\mathbf{\Lambda}_1\mathbf{X}_1^T\mathbf{B}]^T. \tag{37b}$$

Proof. See Refs. [22,23]. □

6. A gradient-based method for robust partial quadratic eigenvalue assignment

By the Bauer–Fike theorem, an overall measure of the conditioning of the eigenvalues of the closed-loop matrix is provided by the condition number of the matrix $\mathbf{Y}$. Now, the conditioning of the eigenvalues of this matrix is best when $\mathbf{Y}$ is unitary or orthogonal since in this case the condition number of $\mathbf{Y}$ (with respect to 2-norm) is 1. Thus we seek to determine $\mathbf{\Gamma}$ such that $J = \|(\mathbf{I} - \mathbf{Y}^H\mathbf{Y})^2\|_F^2$ is as small as possible. This measure of robustness was used before by Keel, Fleming and Bhattacharyya (see Ref. [16]) for the first-order model. It worked well in the first-order case and also works well in the quadratic case, as shown by the results of our numerical examples.

If $\mathbf{Y}$ were a unitary or orthogonal matrix then

$$\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_1\mathbf{\Lambda}_1' \end{pmatrix}^H \begin{pmatrix} \mathbf{X}_2 \\ \mathbf{X}_2\mathbf{\Lambda}_2 \end{pmatrix} = \mathbf{O}_{(p\times 2n-p)}, \tag{38}$$

or $\mathbf{Y}_1^H\mathbf{X}_2 + \bar{\mathbf{\Lambda}}_1'\mathbf{Y}_1^H\mathbf{X}_2\mathbf{\Lambda}_2 = \mathbf{O}_{(p\times 2n-p)}$.

We now show that J can be split into two parts J_1 and J_2, one of which is a function of $\mathbf{\Gamma}$ and the other is independent of $\mathbf{\Gamma}$.

From Eq. (38) and Corollary 3.2 we obtain

$$\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_1\mathbf{\Lambda}_1' \end{pmatrix} = \begin{pmatrix} -\mathbf{K}\overline{\mathbf{X}}_1 \\ \mathbf{M}\overline{\mathbf{X}}_1\mathbf{\Lambda}_1 \end{pmatrix}\mathbf{\Psi}_1 = \begin{pmatrix} -\mathbf{K}\overline{\mathbf{X}}_1\mathbf{\Psi}_1 \\ \mathbf{M}\overline{\mathbf{X}}_1\mathbf{\Lambda}_1\mathbf{\Psi}_1 \end{pmatrix}. \tag{39}$$

Hence,

$$\mathbf{Y}_1 = -\mathbf{K}\overline{\mathbf{X}}_1\mathbf{\Psi}_1 = \bar{\mathbf{Q}}\mathbf{\Psi}_1, \tag{40}$$

$$\mathbf{Y}_1\mathbf{\Lambda}_1' = \mathbf{M}\overline{\mathbf{X}}_1\mathbf{\Lambda}_1\mathbf{\Psi}_1 = \bar{\mathbf{P}}\mathbf{\Psi}_1, \tag{41}$$

where $\mathbf{P} = \mathbf{M}\mathbf{X}_1\mathbf{\Lambda}_1$ and $\mathbf{Q} = -\mathbf{K}\mathbf{X}_1$.

Let $\mathbf{W}_1 = \mathbf{I}_p - \mathbf{Y}_1^H\mathbf{Y}_1 - \mathbf{\Lambda}_1'\mathbf{Y}_1^H\mathbf{Y}_1\mathbf{\Lambda}_1$ and $\mathbf{W}_2 = \mathbf{I}_{2n-p} - \mathbf{X}_2^H\mathbf{X}_2 - \bar{\mathbf{\Lambda}}_2\mathbf{X}_2^H\mathbf{X}_2\mathbf{\Lambda}_2$. Then by Eq. (38) we obtain

$$(\mathbf{I} - \mathbf{Y}^H\mathbf{Y}) = \begin{pmatrix} \mathbf{W}_1 & \mathbf{O}_{(p\times 2n-p)} \\ \mathbf{O}_{(2n-p\times p)} & \mathbf{W}_2 \end{pmatrix}. \tag{42}$$

Therefore,

$$(\mathbf{I} - \mathbf{Y}^H\mathbf{Y})^2 = \begin{pmatrix} (\mathbf{W}_1)^2 & \mathbf{O}_{(p\times 2n-p)} \\ \mathbf{O}_{(2n-p\times p)} & (\mathbf{W}_2)^2 \end{pmatrix}. \tag{43}$$

Thus,

$$J = \|(\mathbf{I} - \mathbf{Y}^H\mathbf{Y})^2\|_F^2 = \|(\mathbf{W}_1)^2\|_F^2 + \|(\mathbf{W}_2)^2\|_F^2 = J_1 + J_2 \quad \text{(say)}. \tag{44}$$

It may be noted, Eq. (38) relates $\mathbf{Y}_1$, $\mathbf{X}_2$, $\mathbf{\Lambda}_2$ and $\mathbf{\Lambda}_1'$. Now, $\mathbf{Y}_1 = [\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p]$ is determined by the choice of $\mathbf{\Gamma} = \{\gamma_1, \gamma_2, \ldots, \gamma_p\}$ since $\mathbf{y}_j$'s satisfy the equation $(\mathbf{M}\mu_j^2 + \mathbf{D}\mu_j + \mathbf{K})\mathbf{y}_j = \mathbf{B}\gamma_j$ for $(j = 1 : p)$. $\mathbf{\Gamma}$ is chosen by the user, whereas $(\mathbf{M}\mu_j^2 + \mathbf{D}\mu_j + \mathbf{K})$ and $\mathbf{B}$ involve given quantities only. The matrix $\mathbf{\Lambda}_1'$ is also given. Since one of our objectives is to work without any knowledge of the matrices $\mathbf{X}_2$ and $\mathbf{\Lambda}_2$, Eq. (38) cannot be directly utilized in choosing $\mathbf{\Gamma}$. It is possible that for the optimal $\mathbf{\Gamma}$, Eq. (38) will not be exactly satisfied. If $\mathbf{Y}_1^H\mathbf{X}_2 + \bar{\mathbf{\Lambda}}_1'\mathbf{Y}_1^H\mathbf{X}_2\mathbf{\Lambda}_2 = \mathbf{W}_3$, then the exact expression for J is

$$J = \|\mathbf{W}_4\|_F^2 + \|\mathbf{W}_5\|_F^2 + \|\mathbf{W}_6\|_F^2 + \|\mathbf{W}_7\|_F^2, \tag{45}$$

where $\mathbf{W}_4 = (\mathbf{W}_1)^2 + \mathbf{W}_3\mathbf{W}_3^H$, $\mathbf{W}_5 = (\mathbf{W}_2)^2 + \mathbf{W}_3^H\mathbf{W}_3$, $\mathbf{W}_6 = \mathbf{W}_1\mathbf{W}_3 + \mathbf{W}_3\mathbf{W}_2$, $\mathbf{W}_7 = \mathbf{W}_3^H\mathbf{W}_1 + \mathbf{W}_2\mathbf{W}_3^H$. In this equation $\mathbf{W}_3$ and $\mathbf{W}_2$ are unknown. By assuming Eq. (38) to be true we obtain Eq. (44), which is a simple approximation of this equation and this permits us to obtain the necessary gradients of J in terms of only those quantities that are known.

Now, the matrix $\mathbf{X}_2$ is independent of $\boldsymbol{\Gamma}$; $\boldsymbol{\Lambda}_1'$, $\boldsymbol{\Lambda}_2$ are fixed matrices; and the matrix $\mathbf{Y}_1$ is a function of the parameter $\boldsymbol{\Gamma}$. Thus, J_1 is a function of $\boldsymbol{\Gamma}$ and J_2 is independent of $\boldsymbol{\Gamma}$.

So, $\nabla_{\boldsymbol{\Gamma}}(J_2) = 0$. Hence, by Eq. (44)

$$\nabla_{\boldsymbol{\Gamma}}(J) = \nabla_{\boldsymbol{\Gamma}}(J_1). \tag{46}$$

Also, since J_2 remains invariant, therefore, J is as small as possible when J_1 is as small as possible. Thus in order to determine $\boldsymbol{\Gamma}$ for which J is as small as possible, we determine the $\boldsymbol{\Gamma}$ for which J_1 is as small as possible using the BFGS method. For this we obtain the gradient of J_1 with respect to $\boldsymbol{\Gamma}$.

Theorem 6.1 (*Matrix gradient formula for J_1*). Let $\mathbf{Z}_1 \equiv \mathbf{I}_p - \mathbf{Y}_1^H \mathbf{Y}_1 - \bar{\boldsymbol{\Lambda}}_1 \mathbf{Y}_1^H \mathbf{Y}_1 \bar{\boldsymbol{\Lambda}}_1'$, $\mathbf{Z}_2 \equiv \mathbf{I}_p - \mathbf{Y}_1^H \mathbf{Y}_1 - \boldsymbol{\Lambda}_1' \mathbf{Y}_1^H \mathbf{Y}_1 \boldsymbol{\Lambda}_1$, $\mathbf{Z}_3 \equiv \mathbf{Z}_1^2 \mathbf{Z}_2 + \mathbf{Z}_2^2 \mathbf{Z}_1 + \boldsymbol{\Lambda}_1 \mathbf{Z}_1^2 \mathbf{Z}_2 \boldsymbol{\Lambda}_1' + \bar{\boldsymbol{\Lambda}}_1' \mathbf{Z}_2^2 \mathbf{Z}_1 \bar{\boldsymbol{\Lambda}}_1$, Let $\mathbf{U}_1$ satisfy the Sylvester equation

$$\boldsymbol{\Lambda}_1' \mathbf{U}_1 - \mathbf{U}_1 \boldsymbol{\Lambda}_1 = -\mathbf{Z}_3 \mathbf{Y}_1^H \mathbf{K} \overline{\mathbf{X}}_1 \mathbf{C}_1^{-1}, \tag{47}$$

where $\mathbf{C}_1 = \mathbf{P}^T \bar{\mathbf{P}} + \mathbf{Q}^T \bar{\mathbf{Q}}$, $\mathbf{P} = \mathbf{M} \mathbf{X}_1 \boldsymbol{\Lambda}_1$ and $\mathbf{Q} = -\mathbf{K} \mathbf{X}_1$.

Then,

$$\nabla_{\boldsymbol{\Gamma}}(J_1) = 2[\mathbf{U}_1 \boldsymbol{\Lambda}_1 \mathbf{X}_1^T \mathbf{B}]^T. \tag{48}$$

Proof. From the definition of $J_1 = \|(\mathbf{W}_1)^2\|_F^2$, it follows that $J_1 = \mathrm{tr}[\{(\mathbf{W}_1)^2\}^H (\mathbf{W}_1)^2] = \mathrm{tr}[(\mathbf{W}_1^H)^2 (\mathbf{W}_1)^2]$. Thus $J_1 = \mathrm{tr}[\mathbf{Z}_1^2 \mathbf{Z}_2^2]$. So,

$$\Delta J_1 = \mathrm{tr}[(\Delta \mathbf{Z}_1^2) \mathbf{Z}_2^2 + \mathbf{Z}_1^2 (\Delta \mathbf{Z}_2^2)]$$
$$= 2\mathrm{tr}[\mathbf{Z}_2^2 \mathbf{Z}_1 \Delta \mathbf{Z}_1 + \mathbf{Z}_1^2 \mathbf{Z}_2 \Delta \mathbf{Z}_2]. \tag{49}$$

Again,

$$\Delta \mathbf{Z}_1 = -[\Delta \mathbf{Y}_1^H \mathbf{Y}_1 + \mathbf{Y}_1^H \Delta \mathbf{Y}_1 + \bar{\boldsymbol{\Lambda}}_1 \Delta \mathbf{Y}_1^H \mathbf{Y}_1 \bar{\boldsymbol{\Lambda}}_1' + \bar{\boldsymbol{\Lambda}}_1 \mathbf{Y}_1^H \Delta \mathbf{Y}_1 \bar{\boldsymbol{\Lambda}}_1'], \tag{50}$$

and

$$\Delta \mathbf{Z}_2 = -[\Delta \mathbf{Y}_1^H \mathbf{Y}_1 + \mathbf{Y}_1^H \Delta \mathbf{Y}_1 + \boldsymbol{\Lambda}_1' \Delta \mathbf{Y}_1^H \mathbf{Y}_1 \boldsymbol{\Lambda}_1 + \boldsymbol{\Lambda}_1' \mathbf{Y}_1^H \Delta \mathbf{Y}_1 \boldsymbol{\Lambda}_1]. \tag{51}$$

Substituting Eqs. (51) and (50) in Eq. (49) we get

$$\Delta J_1 = -2\mathrm{tr}[\mathbf{Y}_1 (\mathbf{Z}_1^2 \mathbf{Z}_2 + \mathbf{Z}_2^2 \mathbf{Z}_1 \boldsymbol{\Lambda}_1 \mathbf{Z}_1^2 \mathbf{Z}_2 \boldsymbol{\Lambda}_1' + \bar{\boldsymbol{\Lambda}}_1' \mathbf{Z}_2^2 \mathbf{Z}_1 \bar{\boldsymbol{\Lambda}}_1) \Delta \mathbf{Y}_1^H$$
$$+ (\mathbf{Z}_1^2 \mathbf{Z}_2 + \mathbf{Z}_2^2 \mathbf{Z}_1 + \boldsymbol{\Lambda}_1 \mathbf{Z}_1^2 \mathbf{Z}_2 \boldsymbol{\Lambda}_1' + \bar{\boldsymbol{\Lambda}}_1' \mathbf{Z}_2^2 \mathbf{Z}_1 \bar{\boldsymbol{\Lambda}}_1) \mathbf{Y}_1^H \Delta \mathbf{Y}_1]$$

$$\therefore \Delta J_1 = -2\mathrm{tr}[\mathbf{Y}_1 \mathbf{Z}_3 \Delta \mathbf{Y}_1^H + \mathbf{Z}_3 \mathbf{Y}_1^H \Delta \mathbf{Y}_1] = -2\mathrm{tr}[\mathbf{Y}_1 \mathbf{Z}_3 \Delta \mathbf{Y}_1^H] - 2\mathrm{tr}[\mathbf{Z}_3 \mathbf{Y}_1^H \Delta \mathbf{Y}_1]. \tag{52}$$

We will now show that both the terms $\mathrm{tr}[\mathbf{Z}_3 \mathbf{Y}_1^H \Delta \mathbf{Y}_1]$ and $\mathrm{tr}[\mathbf{Y}_1 \mathbf{Z}_3 \Delta \mathbf{Y}_1^H]$ can be expressed in terms of the quantities $\mathbf{X}_1$, $\mathbf{B}$ and $\Delta \boldsymbol{\Gamma}$.

First, consider $\mathrm{tr}[\mathbf{Z}_3 \mathbf{Y}_1^H \Delta \mathbf{Y}_1]$. Recall,

$$\mathbf{Z} = (\mathbf{M} \mathbf{X}_1 \boldsymbol{\Lambda}_1)^T \mathbf{Y}_1 \boldsymbol{\Lambda}_1' + (\mathbf{K} \mathbf{X}_1)^T \mathbf{Y}_1$$
$$= \mathbf{P}^T \mathbf{Y}_1 \boldsymbol{\Lambda}_1' + \mathbf{Q}^T \mathbf{Y}_1$$
$$= \mathbf{P}^T \bar{\mathbf{P}} \boldsymbol{\Psi}_1 + \mathbf{Q}^T \bar{\mathbf{Q}} \boldsymbol{\Psi}_1. \text{ (using Eqs. (40) and (41)).}$$

Thus,

$$\mathbf{Z} = \mathbf{C}_1 \boldsymbol{\Psi}_1. \tag{53}$$

Hence,

$$\Delta \boldsymbol{\Psi}_1 = \mathbf{C}_1^{-1} \Delta \mathbf{Z}. \tag{54}$$

Thus,

$$\Delta \mathbf{Y}_1 = -\mathbf{K} \overline{\mathbf{X}}_1 \Delta \boldsymbol{\Psi}_1 = -\mathbf{K} \overline{\mathbf{X}}_1 \mathbf{C}_1^{-1} \Delta \mathbf{Z} \tag{55}$$

So, $\mathrm{tr}[\mathbf{Z}_3 \mathbf{Y}_1^H \Delta \mathbf{Y}_1] = -\mathrm{tr}[\mathbf{Z}_3 \mathbf{Y}_1^H \mathbf{K} \overline{\mathbf{X}}_1 \mathbf{C}_1^{-1} \Delta \mathbf{Z}]$.

Now since $\mathbf{Z}$ satisfies the Sylvester equation

$$\mathbf{\Lambda}_1 \mathbf{Z} - \mathbf{Z}\mathbf{\Lambda}_1' = -\mathbf{\Lambda}_1 \mathbf{X}_1^{\mathrm{T}} \mathbf{B}\mathbf{\Gamma},$$

we have

$$\mathbf{\Lambda}_1(\Delta\mathbf{Z}) - (\Delta\mathbf{Z})\mathbf{\Lambda}_1' = -\mathbf{\Lambda}_1 \mathbf{X}_1^{\mathrm{T}} \mathbf{B}(\Delta\mathbf{\Gamma}). \tag{56}$$

The analytical solution of this Sylvester equation is

$$\Delta\mathbf{Z} = \sum_{j=0}^{p-1} \sum_{k=0}^{p-1} \gamma_{jk} (\mathbf{\Lambda}_1)^j \, (-\mathbf{\Lambda}_1 \mathbf{X}_1^{\mathrm{T}} \mathbf{B}\Delta\mathbf{\Gamma})(\mathbf{\Lambda}_1')^k. \tag{57}$$

So,

$$\begin{aligned}
\mathrm{tr}[\mathbf{Z}_3 \mathbf{Y}_1^{\mathrm{H}} \Delta\mathbf{Y}_1] &= -\mathrm{tr}[\mathbf{Z}_3 \mathbf{Y}_1^{\mathrm{H}} \mathbf{K}\overline{\mathbf{X}}_1 \mathbf{C}_1^{-1} \Delta\mathbf{Z}], \\
&= -\mathrm{tr}\left[\sum_{j=0}^{p-1} \sum_{k=0}^{p-1} \gamma_{jk} \mathbf{Z}_4 (\mathbf{\Lambda}_1)^j (-\mathbf{Z}_5)(\mathbf{\Lambda}_1')^k \right], \\
&= \mathrm{tr}\left[\sum_{j=0}^{p-1} \sum_{k=0}^{p-1} \gamma_{jk} (\mathbf{\Lambda}_1')^k \mathbf{Z}_4 (\mathbf{\Lambda}_1)^j \mathbf{Z}_5 \right],
\end{aligned} \tag{58}$$

where $\mathbf{Z}_4 = \mathbf{Z}_3 \mathbf{Y}_1^{\mathrm{H}} \mathbf{K}\overline{\mathbf{X}}_1 \mathbf{C}_1^{-1}$ and $\mathbf{Z}_5 = \mathbf{\Lambda}_1 \mathbf{X}_1^{\mathrm{T}} \mathbf{B}\Delta\mathbf{\Gamma}$.

Since $\mathbf{U}_1$ satisfies the Sylvester equation $\mathbf{\Lambda}_1' \mathbf{U}_1 - \mathbf{U}_1 \mathbf{\Lambda}_1 = -\mathbf{Z}_3 \mathbf{Y}_1^{\mathrm{H}} \mathbf{K}\overline{\mathbf{X}}_1 \mathbf{C}_1^{-1}$, using Corollary 5.1 we can write

$$\mathbf{U}_1 = \sum_{j=0}^{p-1} \sum_{k=0}^{p-1} \gamma_{jk} (\mathbf{\Lambda}_1')^k \mathbf{Z}_4 (\mathbf{\Lambda}_1)^j. \tag{59}$$

So, finally we have

$$\mathrm{tr}[\mathbf{Z}_3 \mathbf{Y}_1^{\mathrm{H}} \Delta\mathbf{Y}_1] = \mathrm{tr}[\mathbf{U}_1 \mathbf{\Lambda}_1 \mathbf{X}_1^{\mathrm{T}} \mathbf{B}\Delta\mathbf{\Gamma}]. \tag{60}$$

Next, consider $\mathrm{tr}[\mathbf{Y}_1 \mathbf{Z}_3 \Delta\mathbf{Y}_1^{\mathrm{H}}]$.

$$\begin{aligned}
\mathrm{tr}[\mathbf{Y}_1 \mathbf{Z}_3 \Delta\mathbf{Y}_1^{\mathrm{H}}] &= \mathrm{tr}[\mathbf{Y}_1 \mathbf{Z}_3 (\Delta\mathbf{Y}_1)^{\mathrm{H}}] \\
&= -\mathrm{tr}[\mathbf{Y}_1 \mathbf{Z}_3 (\Delta\mathbf{Z})^{\mathrm{H}} (\mathbf{C}_1^{-1})^{\mathrm{H}} \mathbf{X}_1^{\mathrm{T}} \mathbf{K}] \quad \text{(using Eq. (55))} \\
&= -\mathrm{tr}[(\mathbf{C}_1^{-1})^{\mathrm{H}} \mathbf{X}_1^{\mathrm{T}} \mathbf{K}\mathbf{Y}_1 \mathbf{Z}_3 (\Delta\mathbf{Z})^{\mathrm{H}}].
\end{aligned} \tag{61}$$

Since, $\mathbf{\Lambda}_1(\Delta\mathbf{Z}) - (\Delta\mathbf{Z})\mathbf{\Lambda}_1' = -\mathbf{Z}_5$. We have $\bar{\mathbf{\Lambda}}_1'(\Delta\mathbf{Z})^{\mathrm{H}} - (\Delta\mathbf{Z})^{\mathrm{H}} \bar{\mathbf{\Lambda}}_1 = (\mathbf{Z}_5)^{\mathrm{H}}$. Also the solution $(\Delta\mathbf{Z})^{\mathrm{H}}$ of this Sylvester equation is

$$(\Delta\mathbf{Z})^{\mathrm{H}} = \sum_{j=0}^{p-1} \sum_{k=0}^{p-1} \delta_{jk} (\bar{\mathbf{\Lambda}}_1')^j \mathbf{Z}_5^{\mathrm{H}} (\bar{\mathbf{\Lambda}}_1)^k. \tag{62}$$

Let $\mathbf{Z}_6 = (\mathbf{C}_1^{-1})^{\mathrm{H}} \mathbf{X}_1^{\mathrm{T}} \mathbf{K}\mathbf{Y}_1 \mathbf{Z}_3$. Then

$$\begin{aligned}
\mathrm{tr}[\mathbf{Y}_1 \mathbf{Z}_3 \Delta\mathbf{Y}_1^{\mathrm{H}}] &= -\mathrm{tr}[\mathbf{Z}_6 (\Delta\mathbf{Z})^{\mathrm{H}}] \\
&= -\mathrm{tr}\left[\sum_{j=0}^{p-1} \sum_{k=0}^{p-1} \delta_{jk} \mathbf{Z}_6 (\bar{\mathbf{\Lambda}}_1')^j \{\mathbf{Z}_5^{\mathrm{H}}\}(\bar{\mathbf{\Lambda}}_1)^k \right] \\
&= -\mathrm{tr}\left[\sum_{j=0}^{p-1} \sum_{k=0}^{p-1} \delta_{jk} (\bar{\mathbf{\Lambda}}_1)^k \{\mathbf{Z}_6\}(\bar{\mathbf{\Lambda}}_1')^j \mathbf{Z}_5^{\mathrm{H}} \right].
\end{aligned} \tag{63}$$

Now, define the matrix $\mathbf{U}_2$ to be the unique solution of the Sylvester equation

$$\bar{\mathbf{\Lambda}}_1 \mathbf{U}_2 - \mathbf{U}_2 \bar{\mathbf{\Lambda}}_1' = -\mathbf{Z}_6. \tag{64}$$

Then

$$\text{tr}[\mathbf{Y}_1\mathbf{Z}_3\Delta\mathbf{Y}_1^{\mathrm{H}}] = -\text{tr}[\mathbf{U}_2(\Delta\boldsymbol{\Gamma})^{\mathrm{H}}\mathbf{B}^{\mathrm{H}}\bar{\mathbf{X}}_1\bar{\boldsymbol{\Lambda}}_1]$$
$$= -\text{tr}[\mathbf{B}^{\mathrm{H}}\bar{\mathbf{X}}_1\bar{\boldsymbol{\Lambda}}_1\mathbf{U}_2(\Delta\boldsymbol{\Gamma})^{\mathrm{H}}]. \tag{65}$$

Thus by substituting Eqs. (60) and (65) in Eq. (52)

$$\Delta J_1 = 2\text{tr}[\mathbf{U}_1\boldsymbol{\Lambda}_1\mathbf{X}_1^{\mathrm{T}}\mathbf{B}(\Delta\boldsymbol{\Gamma}) + \mathbf{B}^{\mathrm{H}}\bar{\mathbf{X}}_1\bar{\boldsymbol{\Lambda}}_1\mathbf{U}_2(\Delta\boldsymbol{\Gamma})^{\mathrm{H}}]. \tag{66}$$

Now, we know that if $\mathbb{J}$ is a function of matrices $\mathbf{G}$ and $\mathbf{L}$ and

$$\Delta\mathbb{J} = \text{tr}[\mathbb{J}_1\Delta\mathbf{G} + \mathbb{J}_2\Delta\mathbf{L}].$$

Then, $\nabla_G\mathbb{J} = \mathbb{J}_1^{\mathrm{T}}$ and $\nabla_L\mathbb{J} = \mathbb{J}_2^{\mathrm{T}}$ (see, for example Ref. [30]).
Hence from Eq. (66) we obtain,

$$\nabla_{\boldsymbol{\Gamma}}(J_1) = 2[\mathbf{U}_1\boldsymbol{\Lambda}_1\mathbf{X}_1^{\mathrm{T}}\mathbf{B}]^{\mathrm{T}}. \tag{67}$$

The above results lead to the following algorithm.

Algorithm 1. A Robust Partial Quadratic Eigenvalue Assignment Algorithm.
  Inputs:

 (i) The matrices $\mathbf{M} = \mathbf{M}^{\mathrm{T}} > 0$, $\mathbf{K} = \mathbf{K}^{\mathrm{T}} \geqslant 0$, $\mathbf{D} = \mathbf{D}^{\mathrm{T}}$.
 (ii) The control matrix $\mathbf{B}$ of order $n \times m$ ($m \leqslant n$).
 (iii) A self-conjugate subset $\{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ of the spectrum of $P(\lambda)$ and corresponding eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p$.
 (iv) A suitably chosen self-conjugate set of complex numbers $\{\mu_1, \ldots, \mu_p\}$.
 (v) The initial metric $\mathbf{D}1$ used in the BFGS method.
 (vi) Tolerance $\in$ and maximum number of iterations, Max_{iter}.

  Output: Real feedback matrices $\mathbf{F}$ and $\mathbf{G}$ such that the condition number of the closed-loop eigenvector matrix is as small as possible.

  Step 0: Form the matrices $\boldsymbol{\Lambda}_1$, $\boldsymbol{\Lambda}_1'$, $\mathbf{X}_1$, $\mathbf{C}_1$. Set $k = 1$.
  Step 1: Choose a matrix $\boldsymbol{\Gamma} = \{\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \ldots, \boldsymbol{\gamma}_p\} \in \mathbb{C}^{m \times p}$ such that if $\mu_j = \bar{\mu}_k$, then $\boldsymbol{\gamma}_j = \bar{\boldsymbol{\gamma}}_k$.
  Step 2: Compute the unique solution $\mathbf{Z}$ of the Sylvester equation $\boldsymbol{\Lambda}_1\mathbf{Z} - \mathbf{Z}\boldsymbol{\Lambda}_1' = -\boldsymbol{\Lambda}_1\mathbf{X}_1^{\mathrm{T}}\mathbf{B}\boldsymbol{\Gamma}$. If $\text{cond}(\mathbf{Z})$ is large, take another $\boldsymbol{\Gamma}$.
  Step 3: Compute $\mathbf{Y}_1 = [\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p]$ where, $\mathbf{y}_j$ satisfies $(\mathbf{M}\mu_j^2 + \mathbf{D}\mu_j + \mathbf{K})\mathbf{y}_j = \mathbf{B}\boldsymbol{\gamma}_j$ for $(j = 1 : p)$, and the solution $\mathbf{U}_1$ of the Sylvester equation: $\boldsymbol{\Lambda}_1'\mathbf{U}_1 - \mathbf{U}_1\boldsymbol{\Lambda}_1 = \mathbf{Z}_3\mathbf{Y}_1^{\mathrm{H}}\mathbf{K}\bar{\mathbf{X}}_1\mathbf{C}_1^{-1}$.
  Step 4: Compute $\mathbf{Grad} = \nabla_{\boldsymbol{\Gamma}}(J_1)$. If $\|\mathbf{Grad}\|_F < \varepsilon$ or if the number of iterations exceed Max_{iter}, go to Step 6. Else, go to Step 5.
  Step 5′: Update $\boldsymbol{\Gamma}$ using BFGS method, set $k = k + 1$ and repeat from Step 2.
  Step 6: Record the minimum value for J_1 obtained thus far and the corresponding value of $\boldsymbol{\Gamma}$. For this $\boldsymbol{\Gamma}$ compute the matrices $\mathbf{F}$ and $\mathbf{G}$ using Theorem 3.1. Stop.
  Computation of updated $\boldsymbol{\Gamma}$ in Step 4:
  The function to be minimized is $J_1 = \|(\mathbf{I}_p - \mathbf{Y}_1^{\mathrm{H}}\mathbf{Y}_1 - \boldsymbol{\Lambda}_1'\mathbf{Y}_1^{\mathrm{H}}\mathbf{Y}_1\boldsymbol{\Lambda}_1)^2\|_F^2$. We denote the current value of $\boldsymbol{\Gamma}$ by $\boldsymbol{\Gamma}_{\text{old}}$ and the updated value of $\boldsymbol{\Gamma}_{\text{old}}$ by $\boldsymbol{\Gamma}_{\text{new}}$. Then $\boldsymbol{\Gamma}_{\text{new}}$ can be obtained as follows:

 (i) Replace $\boldsymbol{\Gamma}_{\text{old}}$ by $\hat{\boldsymbol{\Gamma}} = \boldsymbol{\Gamma}_{\text{old}} + \alpha\mathbf{d}_j$ where $\mathbf{d}_j$ is given by $\mathbf{d}_j = -\mathbf{D}_j\mathbf{Grad}$. Here $\mathbf{Grad}$ represents the current gradient, $\mathbf{D}_j$ is the metric obtained as in the BFGS method and α is a scaler.
 (ii) Obtain the value $\hat{\mathbf{Y}}_1$ of $\mathbf{Y}_1$ corresponding to $\hat{\boldsymbol{\Gamma}}$, as follows: Let $\hat{\mathbf{Y}}_1 = \{\hat{\mathbf{y}}_1, \hat{\mathbf{y}}_2, \ldots, \hat{\mathbf{y}}_p\}$ and $\hat{\boldsymbol{\Gamma}} = \{\hat{\boldsymbol{\gamma}}_1, \hat{\boldsymbol{\gamma}}_2, \ldots, \hat{\boldsymbol{\gamma}}_p\}$ then obtain $\hat{\mathbf{y}}_i$ by solving the Sylvester equation $(\mathbf{M}\mu_i^2 + \mathbf{D}\mu_i + \mathbf{K})\hat{\mathbf{y}}_i = \mathbf{B}\hat{\boldsymbol{\gamma}}_i$ for $i = 1 : p$.
 (iii) Find $l = \min_{\alpha}\|(\mathbf{I}_p - \hat{\mathbf{Y}}_1^{\mathrm{H}}\hat{\mathbf{Y}}_1 - \boldsymbol{\Lambda}_1'\hat{\mathbf{Y}}_1^{\mathrm{H}}\hat{\mathbf{Y}}_1\boldsymbol{\Lambda}_1)^2\|_F^2$. (This is obtained by using the MATLAB function fminbnd.)
 (iv) $\boldsymbol{\Gamma}_{\text{new}} = \boldsymbol{\Gamma}_{\text{old}} + l\mathbf{d}_j$.

Efficiency: The cost of solving each of the $(p \times p)$ Sylvester equations in Steps 2 and 3 is $O(p^3)$ flops. The cost of computing **Grad** in Step 4 is $O(n^2p)$ flops.

7. A gradient based method for simultaneous improvement of feedback norms and condition number of eigenvector matrix

We have so far discussed feedback norm minimization and the minimization of the conditioning of the closed-loop eigenvector matrix separately. The question naturally arises whether these two aspects can be combined in one setting. That is, can the feedback norms and the condition number of the closed-loop eigenvector matrix be improved simultaneously? This problem has been considered by Varga (see Ref. [13]) in the case of the first-order model. Following Varga, we now consider the following objective function for the quadratic pencil:

$$\mathbb{O} = \frac{[(C1)(\alpha)\mathbb{I} + (C2)(1-\alpha)J_1]}{[(C1)(\alpha) + (C2)(1-\alpha)]}, \tag{68}$$

where $\mathbb{I} = \frac{1}{2}\|\mathbf{S}\|_F^2 = \frac{1}{2}[\|\mathbf{F}\|_F^2 + \|\mathbf{G}\|_F^2]$ and $J_1 = \|(\mathbf{I}_p - \mathbf{Y}_1^H\mathbf{Y}_1 - \mathbf{\Lambda}_1'\mathbf{Y}_1^H\mathbf{Y}_1\mathbf{\Lambda}_1)^2\|_F^2$ and $C1, C2$ and α are constants. Note that when $\alpha = 1$, we have the minimum-norm problem and when $\alpha = 0$, we have the robust problem.

The gradient of $\mathbb{O}$ with respect to $\mathbf{\Gamma}$ can be computed as follows (see Ref. [23] for details):

$$\nabla_{\mathbf{\Gamma}}(\mathbb{O}) = \frac{[(C1)(\alpha)\nabla_{\mathbf{\Gamma}}(\mathbb{I}) + (C2)(1-\alpha)\nabla_{\mathbf{\Gamma}}(J_1)]}{[(C1)(\alpha) + (C2)(1-\alpha)]} \tag{69}$$

$\nabla_{\mathbf{\Gamma}}(\mathbb{I})$ is given by Theorem 5.2 and $\nabla_{\mathbf{\Gamma}}(J_1)$ is given by Theorem 5.1, respectively. If the magnitudes of the elements of $\nabla_{\mathbf{\Gamma}}(\mathbb{I})$ and $\nabla_{\mathbf{\Gamma}}(J_1)$ are widely disparate then the constants $C1$ and $C2$ need to be chosen to scale their magnitudes, to prevent one gradient totally dominating the other. After the constants $C1$ and $C2$ have been chosen appropriately for a particular problem, the value of the constant α is varied between 0 and 1 to bring about different amounts of reduction in the feedback norms and the condition number of the eigenvector matrix.

8. Results of numerical experiments

The results on our numerical experiments are presented below. Problem 1 is from Ref. [1], Problem 2 is a benchmark example taken from Ref. [31] and Problem 3 is from Ref. [18]. Tables 1–5 contain, respectively, the results of Problem 1, Problem 2(i), Problem 2(ii), Problem 2(iii) and Problem 3. Table 6 contains results of comparisons of our Algorithm 1 with those of the Qian–Xu method. A graph based on the results of Problem 2(i) is shown in Fig. 1. In the following, the percentage reduction in the Frobenius norm is calculated as follows: Percentage reduction in the Frobenius norm $= 100((IN - FN)/IN)$ where IN is the value of the norm with initial $\mathbf{\Gamma}$ and FN is the value of the norm with optimal $\mathbf{\Gamma}$. The percentage reduction in the condition number is similarly defined.

Table 1
Numerical results for Problem 1.

Results of the three algorithms for Problem 1			Perturbation results for Algorithm 1	
Algorithm	Accuracy	Percentage reduction	Norm of $\Delta\mathbf{K}$	Accuracy with perturbed $\mathbf{K}$
(i) Norm reduction (I)	$O(10^{-14})$	99.99	0.3181	0.1687
(ii) Condition number reduction (CN) (Algorithm 1)	$O(10^{-14})$	99.74	0.0172	0.0109
			0.0024	0.0015
(iii) Simultaneous reduction of norm and condition number for $\alpha = 0.4$, $C1 = 50$, $C2 = 0.5$	$O(10^{-13})$	99.99(I)	$3.0672E{-}004$	$1.9691E{-}004$
		99.28(CN)	$3.5702E{-}005$	$2.2926E{-}005$
			$2.8591E{-}006$	$1.8360E{-}006$

Table 2
Numerical results for Problem 2(i).

Results of the three algorithms for Problem 2(i)			Perturbation results for Algorithm 1	
Algorithm	Accuracy	Percentage reduction	Norm of $\Delta\mathbf{K}$	Accuracy with perturbed $\mathbf{K}$
(i) Norm reduction (I)	$O(10^{-14})$	99.94	0.1101	0.0312
(ii) Condition number reduction (CN) (Algorithm 1)	$O(10^{-14})$	99.66	0.0253	0.0072
			0.0058	0.0017
(iii) Simultaneous reduction of norm and condition number for $\alpha = 0.4$, $C1 = 10^{16}$, $C2 = 1$	$O(10^{-14})$	99.98(I) 98.68(CN)	5.7261E−004 2.5621E−005 5.5808E−006	1.6406E−004 7.3413E−006 1.5991E−006

Table 3
Numerical results for Problem 2(ii).

Results of the three algorithms for Problem 2(ii)			Perturbation results for Algorithm 1	
Algorithm	Accuracy	Percentage reduction	Norm of $\Delta\mathbf{K}$	Accuracy with perturbed $\mathbf{K}$
(i) Norm reduction (I)	$O(10^{-6})$	99.92	1.9438E−001	7.4819E+000
(ii) Condition number reduction (CN) (Algorithm 1)	$O(10^{-5})$	99.39	1.4198 E−002	2.8098E+000
			1.0908E−004	4.5488−001
(iii) Simultaneous reduction of norm and condition number for $\alpha = 0.4$, $C1 = 10^{9}$, $C2 = 1$	$O(10^{-5})$	99.99(I) 98.94(CN)	3.3279−005 2.9898E−006 6.0432−007	2.3029E−001 2.5192−002 5.5007−003

Table 4
Numerical results for Problem 2(iii).

Results of the three algorithms for Problem 2(iii)			Perturbation results for Algorithm 1	
Algorithm	Accuracy	Percentage reduction	Norm of $\Delta\mathbf{K}$	Accuracy with perturbed $\mathbf{K}$
(i) Norm reduction (I)	$O(10^{-2})$	99.97	3.1643E+000	1.2004E+000
(ii) Condition number reduction (CN) (Algorithm 1)	$O(10^{-2})$	99.32	1.4159E−001	3.1010E−001
			3.0841E−002	2.2944E−001
(iii) Simultaneous reduction of norm and condition number for $\alpha = 0.4$, $C1 = 10^{17}$, $C2 = 1$	$O(10^{-2})$	99.99(I) 96.89(CN)	1.2178E−003 2.8063E−004 2.0837E−005	1.0295E−001 9.2421E−002 1.3234E−001

Accuracy = The Frobenius norm of the difference between the desired closed-loop eigenvalues and the actual closed-loop eigenvalues obtained for optimal $\mathbf{\Gamma}$.

Verifying the sensitivity of the closed-loop eigenvalues under small perturbations:

To verify the robustness of our solutions to the partial eigenvalue assignment problem, we perturb the stiffness matrix $\mathbf{K}$ and then compute the closed-loop eigenvalues corresponding to the feedback matrices $\mathbf{F}$ and $\mathbf{G}$ (obtained for the optimal $\mathbf{\Gamma}$ for the unperturbed problem) keeping $\mathbf{M}$, $\mathbf{D}$ and $\mathbf{B}$ unchanged. The rationale behind leaving the mass and stiffness matrices unperturbed is that the mass matrix is usually accurately determined and the damping matrix is hard to estimate in practice. On the other hand, the stiffness matrix $\mathbf{K}$ is

Table 5
Numerical results for Problem 3.

Results of the three algorithms for Problem 3			Perturbation results for Algorithm 1	
Algorithm	Accuracy	Percentage reduction	Norm of $\Delta\mathbf{K}$	Accuracy with perturbed $\mathbf{K}$
(i) Norm reduction (I)	$O(10^{-14})$	99.79	0.0693	0.0143
(ii) Condition number reduction (CN) (Algorithm 1)	$O(10^{-13})$	99.96	0.0182	0.0037
			0.0015	0.0003
(iii) Simultaneous reduction of norm and condition number for $\alpha = 0.4$, $C1 = 10^7$, $C2 = 1$	$O(10^{-15})$	98.93(I) 97.86(CN)	2.6739E−004 2.2863E−005 5.5511E−009	5.4816E−005 4.6869E−006 1.1379E−009

Table 6
Numerical results of the Qian–Xu method.

Prob.	Percentage reduction	Accuracy
1	97.52	5.8042E−010
2(i)	99.61	1.8195
2(ii)	98.56	8.3242E−008
2(iii)	98.14	0.0525
3	99.45	2.1915E−014
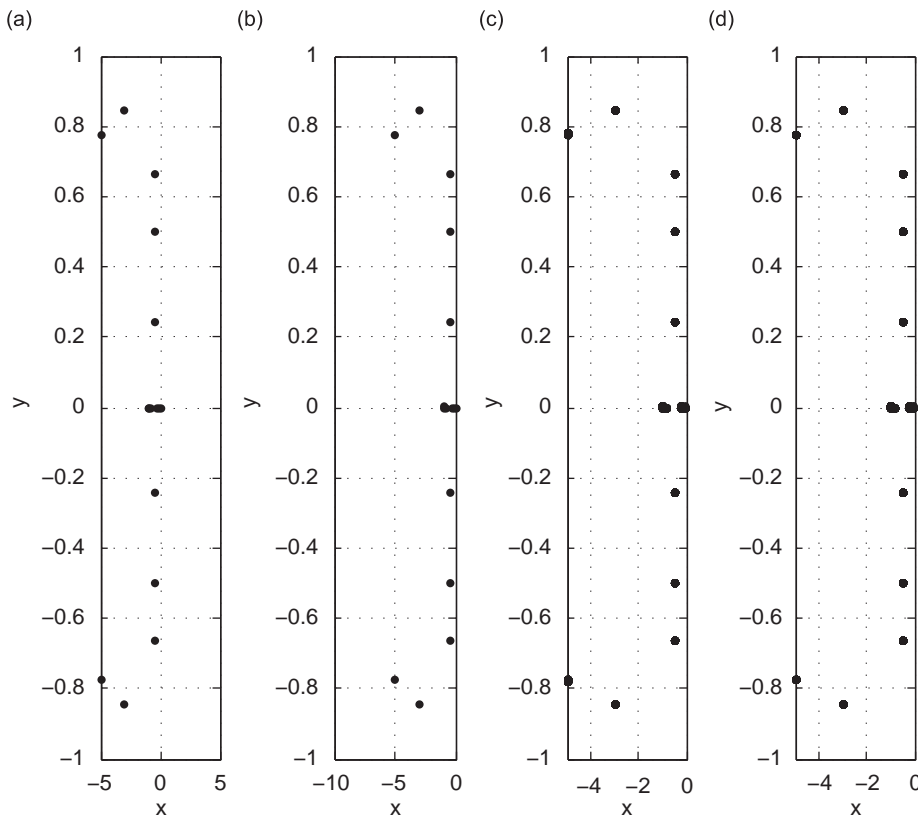
Fig. 1. Closed-loop eigenvalues for Problem 2(i): (a) desired closed-loop eigenvalues, (b) actual closed-loop eigenvalues for optimal $\mathbf{\Gamma}$ for the unperturbed problem, (c) closed-loop eigenvalues under 100 random perturbations of order 0.01 and (d) closed-loop eigenvalues under 100 random perturbations of order 0.001.

often not determined accurately. We then calculate the norm (accuracy with perturbed **K**) of the difference of the matrices of the closed-loop eigenvalues under perturbation obtained as described and the corresponding closed-loop eigenvalues actually obtained for the unperturbed problem. Our results show that small perturbations in K produce only small changes in the closed-loop eigenvalues.

Problem 1.

$$\mathbf{M} = \begin{pmatrix} 1.4685 & 0.7177 & 0.4757 & 0.4311 \\ 0.7177 & 2.6938 & 1.2660 & 0.9676 \\ 0.4757 & 1.2660 & 2.7061 & 1.3918 \\ 0.4311 & 0.9676 & 1.3918 & 2.1876 \end{pmatrix},$$

$$\mathbf{K} = \begin{pmatrix} 1.7824 & 0.0076 & -0.1359 & -0.7290 \\ 0.0076 & 1.0287 & -0.0101 & -0.0493 \\ -0.1359 & -0.0101 & 2.8360 & -0.2564 \\ -0.7290 & -0.0493 & -0.2564 & 1.9130 \end{pmatrix},$$

$$\mathbf{D} = \begin{pmatrix} 1.3525 & 1.2695 & 0.7967 & 0.8160 \\ 1.2695 & 1.3274 & 0.9144 & 0.7325 \\ 0.7967 & 0.9144 & 0.9456 & 0.8310 \\ 0.8160 & 0.7325 & 0.8310 & 1.1536 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0.3450 & 0.4578 \\ 0.0579 & 0.7630 \\ 0.5967 & 0.9990 \\ 0.2853 & 0.3063 \end{pmatrix}.$$

The open-loop eigenvalues are $\{-0.0861 \pm 1.6242i, -0.1748 \pm 1.1922i, -0.4480 \pm 0.2465i, -0.1022 \pm 0.8876i\}$. The first two eigenvalues $\{-0.0861 \pm 1.6242i\}$ were reassigned to $\{-8 \pm 1.6242i\}$, the other eigenvalues were kept unchanged.

Problem 2. $\mathbf{M} = 4\mathbf{I_{n \times n}}, \mathbf{D} = 4\mathbf{I_{n \times n}}$

$$\mathbf{K} = \begin{bmatrix} 1 & -1 & 0 & \ldots & 0 & 0 \\ -1 & 2 & -1 & \ldots & 0 & 0 \\ 0 & -1 & 2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & -1 & 2 & -1 \\ 0 & 0 & \ldots & 0 & -1 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ \vdots & \vdots \\ 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

Case i: With $n = 8$, there are 16 open-loop eigenvalues of which the first four, viz., $\{-0.5 \pm 0.8438i, -0.5 \pm 0.7769i\}$ are reassigned to $\{-3 \pm 0.8438i, -5 \pm 0.7769i\}$, keeping other eigenvalues unchanged.

Case ii: With $n = 10$, there are 20 open-loop eigenvalues of which the first six, viz., $\{-0.5 \pm 0.7375i, -0.5 \pm 0.8518i, -0.5 \pm 0.8090i\}$ are reassigned to $\{-8 \pm .7375i, -8 \pm 0.8518i, -8 \pm 0.8090\}$, keeping other eigenvalues unchanged.

Case iii: With $n = 200$, there are 400 open-loop eigenvalues of which the first six, viz., $\{-0.5 \pm 0.8660i, -0.5 \pm 0.8659i, -0.5 \pm 0.8657i\}$ are reassigned to $\{-1 \pm 0.8660i, -1 \pm 0.8659i, -1 \pm 0.8657i\}$, keeping other eigenvalues unchanged.

Problem 3.

$$\mathbf{K} = \begin{pmatrix} 40 & -40 & 0 \\ -40 & 80 & -40 \\ 0 & -40 & 80 \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \\ 3 & 4 \end{pmatrix}, \quad \mathbf{M} = 10\mathbf{I}_{3,3}, \quad \mathbf{D} = \mathbf{O}_{3,3}.$$

The open-loop eigenvalues are $\{\pm 3.6039i, \pm 2.49399i, \pm 0.8901i\}$. The first two eigenvalues $\{\pm 3.6039i\}$ were reassigned to $\{-1, -2\}$, the other eigenvalues were kept unchanged.

Note on convergence of Algorithm 1: For the five problems the optimal $\mathbf{\Gamma}$ was obtained after 2, 2, 8, 2 and 5 sweeps, respectively. We set $\varepsilon = 10^{-4}$ for Problems 1 and 2(i) and $\varepsilon = 10^{-8}$ for the remaining three problems. In most cases the algorithm stopped when the number of iterations equalled Max_{iter}. The solution is dependent on the choice of the initial $\mathbf{\Gamma}$ and the initial metric $\mathbf{D}_1$. For some choices of $\mathbf{\Gamma}$ and $\mathbf{D}_1$ the condition number did not decrease.

Comparison of Algorithm 1 *with the Qian–Xu method*: The results obtained by using the Qian–Xu method for the five problems considered are presented in Table 6. A comparison of the two algorithms show that they produce almost the same accuracy and condition number reductions also are comparable. However since our method is optimization based, this method is also suitable for large-scale computations using specialized large scale optimization techniques. Furthermore our method can also handle simultaneous minimization of both feedback norms and the closed-loop eigenvector conditioning. It may be noted in the context of comparison that the "accuracy" measure is computed differently in the Qian–Xu paper.

9. Conclusion

The problem of designing a robust active controller for a vibrating structure modeled by a system of second-order matrix differential equations is the one in which a feedback controller has to be constructed such that the feedback matrices have minimum norms and the condition number of the closed-loop eigenvector matrix is as small as possible to ensure that the closed-loop eigenvalues are not sensitive to small perturbations of the data. Mathematically, this leads to minimum-norm and robust partial quadratic eigenvalue assignment problems (MNPQEVAP and RPQEVAP). Basically these problems are optimization problems, and one special advantage of solving these problems in an optimization setting is that an excellent numerical optimization technique, such as the BFGS method, can be profitably used. However, a bottleneck in using this technique is deriving parametric expressions for feedback matrices and developing appropriate gradient formulas. In case of the problems under consideration here, a further computational challenge is to develop such gradient formulas using only a few eigenvalues and the corresponding eigenvectors of the associated quadratic eigenvalue problem, since in practice it is impossible to compute all the eigenvalues and eigenvectors of a large quadratic matrix pencil even by using state-of-the-art computational techniques. In the present paper and in another recent one, (i) parametric expressions for feedback matrices have been derived using Sylvester equations and (ii) appropriate gradient formulas both for minimum-norm and robust eigenvalue assignment problems have been developed using only the small number of eigenvalues that need to be reassigned and the associated eigenvectors. These techniques are, therefore, implementable in practice even for large-scale structures. However, some more work still needs to be done. One of the underlying mathematical problems is how to choose the initial parametric matrix in each algorithm so that convergence can be guaranteed within a reasonable number of steps. It is to be noted in this context that the underlying optimization problems are difficult and further research is needed and currently underway for finding local and global solutions.

Acknowledgments

References

[1] B.N. Datta, D.R. Sarkissian, Theory and computations of some inverse eigenvalue problems for the quadratic pencil, *Contemporary Mathematics* 280 (2001) 221–240.

[2] B.N. Datta, S. Elhay, Y. Ram, Orthogonality and partial pole assignment for the symmetric definite quadratic pencil, *Linear Algebra and its Applications* 257 (1997) 29–48.

[3] F. Tisseur, K. Meerbergen, The quadratic eigenvalue problem, *SIAM Review* 43 (2001) 235–286.

[4] D.J. Inman, *Engineering Vibration*, Pearson Prentice Hall, Upper Saddle River, NJ, 2008.

[5] N.K. Nichols, J. Kautsky, Robust eigenstructure assignment in quadratic matrix polynomials: nonsingular case, *SIAM Journal on Matrix Analysis and Applications* 23 (2001) 77–102.

[6] A.J. Laub, W.F. Arnold, Controllability and observability for multivariate linear second order models, *IEEE Transactions on Automatic Control* 29 (1984) 163–165.

[7] B. Porter, R. Crossley, *Modal Control: Theory and Applications*, Barnes & Noble, New York, 1972.

[8] B.N. Datta, D.R. Sarkissian, Partial eigenvalue assignment in linear systems: existence, uniqueness and numerical solution, *Proceedings of the Mathematical Theory of Networks and Systems*, Notre Dame, August 2002.

[9] Y. Saad, A projection method for partial pole assignment in linear state feedback, *IEEE Transactions on Automatic Control* 33 (1998) 290–297.

[10] B.N. Datta, Y. Saad, Arnoldi methods for large Sylvester-like observer matrix equations and an associated algorithm for partial spectrum assignment, *Linear Algebra and its Applications* 154–156 (1991) 225–244.

[11] B.N. Datta, *Numerical Methods for Linear Control Systems Design and Analysis*, Elsevier Academic Press, Boston, 2003.

[12] G.L. Sleijpen, H.A. van der Vorst, Z. Bai, Jacobi–Davidson algorithms for various eigenproblems: a working document, 1999, available at ⟨www.math.uu.nl/publications/preprints/1114.ps⟩.

[13] A. Varga, A numerically reliable approach to robust pole assignment for descriptor systems, *Future Generation Computer Systems* 19 (2003) 1221–1230.

[14] J. Nocedal, S. Wright, *Numerical Optimization*, second ed., Springer, New York, 2006.

[15] B.N. Datta, S. Elhay, Y. Ram, An algorithm for the partial multi-input pole assignment of a second-order control system, *Proceedings of the IEEE Conference on Decision and Control*, Kobe, Japan, December 1996, pp. 2025–2029.

[16] L.H. Keel, J.A. Fleming, S.P. Bhattacharyya, Minimum norm pole assignment via Sylvester's equations, *Contemporary Mathematics* 47 (1985) 265–272.

[17] R.K. Cavin III, S.P. Bhattacharyya, Robust and well-conditioned eigenstructure assignment via Sylvester's equation, *Optimal Control Applications & Methods* 4 (1983) 205–212.

[18] J. Qian, S. Xu, Robust partial eigenvalue assignment problem for the second-order system, *Journal of Sound and Vibration* 282 (2005) 937–948.

[21] S. Brahma, B.N. Datta, A Sylvester-equation based parametric approach for minimum norm and robust partial quadratic eigenvalue assignment problems, *Proceedings of the IEEE Mediterranean Control Conference*, Athens, Greece, June 2007, pp. 1–6.

[22] S. Brahma, B.N. Datta, A norm-minimizing parametric algorithm for quadratic partial eigenvalue assignment via Sylvester equation, *Proceedings of the IEEE European Control Conference*, Kos, Greece, July 2007, pp. 490–496.

[23] S.K. Brahma, Robust and Minimum Norm Partial Quadratic Eigenvalue Assignment Problems: Theory and Computations, PhD Thesis, Northern Illinois University, 2006.

[24] J. Carvalho, B.N. Datta, A new algorithm for generalized Sylvester equation and its applications to state and velocity estimations in vibrating systems, *Numerical Linear Algebra with Applications*, to appear.

[25] B.N. Datta, D.R. Sarkissian, Feedback control in distributed parameter gyroscopic systems: a solution of the partial eigenvalue assignment, *Mechanical Systems and Signal Processing* 16 (2002) 3–17.

[27] S.P. Bhattacharyya, E. deSouza, Pole assignment via Sylvester's equation, *Systems & Control Letters* 1 (1982) 261–283.

[28] E. de Souza, S.P. Bhattacharyya, Controllability, observability and the solution of $\mathbf{AX} - \mathbf{XB} = \mathbf{C}$, *Linear Algebra and its Applications* 39 (1981) 167–188.

[29] R. Horn, C. Johnson, *Matrix Analysis*, Cambridge University Press, New York, 1991.

[30] A. Hjørungnes, D. Gesbert, D.P. Palomar, Unified theory of complex-valued matrix differentiation, *Proceedings of the IEEE Conference on Acoustics, Speech, and Signal Processing*, Honolulu, Hawaii, USA, April 2007.

[31] P. Benner, A.J. Laub, V. Mehrmann, A collection of bench mark examples for the numerical solution of algebraic Ricatti equations I: continuous time case, Technical Report SPC 95-23, TU Chemnitz-Zwickau, December 1995.