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Lyapunov exponents of parametrically coupled linear two-DOF stochastic systems and related stability problems

Meziane Labou, Tian-Wei Ma*

Department of Civil and Environmental Engineering, University of Hawaii at Manoa, 2540 Dole Street, Honolulu, HI 96822, USA

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Abstract

The almost-sure asymptotic stability of elastic systems subjected to parametric excitation is studied. The excitation consists of a harmonic function on which a stochastic term is superposed. The effect of the parametric action on the stability of a coupled system of differential equations is studied. By means of stochastic transformations of state norm process, the stability boundaries are determined using the stochastic averaging method and a technique due to Khasminskii. As an application, the problem of coupled flexural–torsional instability of a deep rectangular beam in the presence of fluctuating axial loads and end moments is considered.

1. Introduction

Vibrations caused by a variation in the parameters of a system are called parametrically excited or simply parametric vibrations. From the mathematical point of view, the feature common to all parametric vibrations is that they are described by differential equations with coefficients depending explicitly on time. Parametric vibrations in deterministic systems have been investigated in great detail (see for example Refs. [1,2]). Extension of the theory to stochastic systems, whose behavior is described by means of differential equations with coefficients varying stochastically in time, is of interest. Again, there arises the problem of stability of the trivial solutions of these equations. Stability is understood in the stochastic sense, i.e. as stability with respect to probability, to mathematical expectations, to a set of moments of functions, etc.

Parametric resonances arises when certain relations between the frequency of parametric action, v, and the natural frequencies of a system, ω_k, are satisfied, which include $v = 2\omega_k/p$, referred to simple resonances, $v = (\omega_j + \omega_k)/p$, referred to combination sum resonances and $v = (|\omega_j - \omega_k|)/p$, referred to combination differences resonances $(j, k, p = 1, 2, \ldots)$. If the parametric action is a random process with a latent periodicity, analogous resonance phenomena can be expected to occur in the stochastic system [3,4].

The theory of stochastic stability first came into existence mainly in connection with problems of control theory. Extending the classical theory of stability of motion to stochastic systems became necessary.

*Corresponding author.
E-mail address: tianwei@hawaii.edu (T.-W. Ma).

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The mathematical aspects of the theory are treated in Refs. [5,6]. Among numerous applied problems, we cite the problem about the stability of linear stochastic dynamical systems, driven by parametric excitation. A large amount of research has been done in this area. In the case of stationary stochastic excitation, stochastic stability conditions were obtained by, among others, Stratonovich and Romanovskii [7], Weidenhammer [8], and Graefe [9], who showed that, to a first approximation, stability depends only on the excitation spectrum in the neighborhood of the sum of the natural frequencies. Ariaratnam and Tam [10] investigated the effect of parametric action on the moment stability of a damped Mathieu oscillator. The stability in mean square of an oscillator of the same kind, describing the vibration of a propeller blade of a helicopter, has been studied in Ref. [11]. It is found that the turbulent fluctuation of the air speed lead to an additional parametric random action, which according to Ref. [11], may have a stabilizing or a destabilizing effect. The moment stability of a two-dimensional coupled system, driven by parametric excitation was investigated in Ref. [12], using the Stratonovich–Khasminskii theory (SKT) [13]. Here, the variation of the parametric excitation intensity with time is described by the sum of a harmonic function and a stationary random process.

A combination of the stochastic averaging (built on the assumption of light damping and weak excitation of wide-band process) and the Khasminskii's procedure becomes an effective approach to obtain the asymptotic expressions for the largest Lyapunov exponent (as an almost-sure stability indicator). This approach was used by Ariaratnam et al. [14,15] to investigate the stochastic stability of coupled linear systems. The approach was also used by Ariaratnam et al. [16] to analyze the stability of non-gyroscopic viscoelastic systems, in which the integral term arising from the viscoelastic effect was averaged by employing the Larianov's method [17]. The stochastic averaging method, proposed by Zhu et al. in Refs. [18,19], combined with the technique of Khasminskii, has been applied to quasilinear gyroscopic systems under real noise excitation to derive sufficient conditions for the almost-sure asymptotic stability [20].

In this study, the obtained results in Ref. [12] are used in this paper to extend the study further to obtain explicit asymptotic expressions for the largest Lyapunov exponent, using an alternative probabilistic approach to the stability problem [5]. As an application, the problem of coupled flexural–torsional instability of a deep rectangular beam in the presence of fluctuating axial loads and end moments is considered.

2. Formulation

The systems considered are described by differential equations of the form

$$\ddot{q}_i + 2\varepsilon \sum_{j=1}^{2} \beta_{ij}\dot{q}_j + \omega_i^2 \left[\sum_{j=1}^{2} h_{ij} q_j \varepsilon \sin 2\nu t + f(t)\varepsilon^{1/2} \sum_{j=1}^{2} c_{ij} q_j \right] = 0 \quad (i = 1, 2) \tag{1}$$

where q_i are the generalized displacements and ω_i are the natural frequencies of the two subsystems, respectively. Symbol β_{ij} denotes the damping coefficients, h_{ij} the amplitudes of the harmonic excitation, c_{ij} normalization constants and ε is a small parameter to ensure light system damping and weak excitation.

The dynamic stability of systems described by Eq. (1) under deterministic parametric load has been investigated in detail in Refs. [1,2]. There exist situations in which the exciting loads cannot be described adequately in the form of deterministic functions alone, and random fluctuating terms are superimposed. When the excitation, $f(t)$, is taken to be a stationary random process, a probabilistic approach is needed.

The effect of the random parametric excitation on the stability of trivial solutions of system (1) is investigated when the frequency of the harmonic component falls within the region of combination parametric resonance, i.e. $2\nu \approx \omega_1 + \omega_2$.

Considering the case of parametric resonance, i.e. when $p_1 \approx \omega_1$ and $p_2 \approx \omega_2$, and setting

$$\omega_1^2 = p_1^2 + \varepsilon \Delta_1$$

$$\omega_2^2 = p_2^2 + \varepsilon \Delta_2 \tag{2}$$

where $\varepsilon\Delta_i$ ($i = 1, 2$) denotes the amount of detuning. Eq. (1) can be rewritten as

$$\begin{aligned}\ddot{q}_1 + p_1^2 q_1 = & -\varepsilon[2(\beta_{11}\dot{q}_1 + \beta_{12}\dot{q}_2) + (\Delta_1 + \omega_1^2 h_{11}\sin 2\nu t)q_1 \\ & + \omega_1^2 h_{12} q_2 \sin 2\nu t] - \sqrt{\varepsilon}\cdot\omega_1^2(c_{11}q_1 + c_{12}q_2)f(t) \\ \ddot{q}_2 + p_2^2 q_2 = & -\varepsilon[2(\beta_{21}\dot{q}_1 + \beta_{22}\dot{q}_2) + (\Delta_2 + \omega_2^2 h_{22}\sin 2\nu t)q_2 \\ & + \omega_2^2 h_{21} q_1 \sin 2\nu t] - \sqrt{\varepsilon}\cdot\omega_2^2(c_{21}q_1 + c_{22}q_2)f(t)\end{aligned} \tag{3}$$

In this study, ε is assumed to be a small parameter, thus the solution of Eq. (3) will approach a harmonic one; consequently, it is convenient to seek the solution in the form of

$$q_i(t) = z_i \cos p_i t + y_i \sin p_i t, \quad \dot{q}_i(t) = p_i[-z_i \sin p_i t + y_i \cos p_i t] \tag{4}$$

thus,

$$\ddot{q}_i + p_i^2 q_i = p_i[-\dot{z}_i \sin p_i t + \dot{y}_i \cos p_i t], \quad i = 1, 2 \tag{5}$$

and

$$z_i(t) = \frac{1}{\cos p_i t}[q_i - y_i \sin p_i t], \quad \dot{z}_i(t) = -\dot{y}_i(t)\frac{\sin p_i t}{\cos p_i t}, \quad i = 1, 2 \tag{6}$$

Substitution of Eq. (6) into Eq. (5) gives the well-known relations [13]

$$\dot{z}_i = -\frac{1}{p_i}(\ddot{q}_i + p_i^2 q_i)\sin p_i t, \quad \dot{y}_i = \frac{1}{p_i}(\ddot{q}_i + p_i^2 q_i)\cos p_i t \tag{7}$$

We consider that the frequencies of resonance oscillations satisfy the relation $p_1 + p_2 = 2\nu$. The system equations expressed in Eq. (3) may be replaced by the following two pairs of first-order equations.

$$\begin{aligned}\dot{z}_1 = & -\varepsilon\Big[2\beta_{11}\Big(z_1 \sin^2 p_1 t - \frac{y_1}{2}\sin 2p_1 t\Big) + \frac{p_2}{p_1}\beta_{12}(-z_2\cos 2\nu t - y_2 \sin 2\nu t \\ & + z_2 \cos\delta p t - y_2 \sin\delta p t) - \Big(\frac{\Delta_1}{p_1} + \omega_1 h_{11}\sin 2\nu t\Big)\cdot\Big(\frac{z_1}{2}\sin 2p_1 t + y_1 \sin^2 p_1 t\Big) \\ & - \frac{\omega_1}{2}h_{12}\sin 2\nu t(z_2 \sin 2\nu t - y_2 \cos 2\nu t + z_2 \sin\delta p t + y_2 \cos\delta p t)\Big] \\ & - \sqrt{\varepsilon}\cdot\omega_1(c_{11}A_1 + c_{12}A_2)f(t)\end{aligned}$$

$$\begin{aligned}\dot{y}_1 = & -\varepsilon\Big[2\beta_{11}\Big(-\frac{z_1}{2}\sin 2p_1 t - y_1\cos^2 p_1 t\Big) + \frac{p_2}{p_1}\beta_{12}(-z_2 \sin 2\nu t + y_2 \cos 2\nu t \\ & + z_2 \sin\delta p t + y_2 \cos\delta p t) + \Big(\frac{\Delta_1}{p_1} + \omega_1 h_{11}\sin 2\nu t\Big)\cdot\Big(z_1\cos^2 p_1 t + \frac{y_1}{2}\sin^2 p_1 t\Big) \\ & + \frac{\omega_1}{2}h_{12}\sin 2\nu t(z_2\cos 2\nu t + y_2 \sin 2\nu t + z_2\cos\delta p t - y_2 \sin\delta p t)\Big] \\ & - \sqrt{\varepsilon}\cdot\omega_1(c_{11}B_1 + c_{12}B_2)f(t)\end{aligned}$$

$$\begin{aligned}\dot{z}_2 = & -\varepsilon\Big[\frac{p_1}{p_2}\beta_{21}(-z_1\cos 2\nu t - y_1 \sin 2\nu t + z_1\cos\delta p t + y_1 \sin\delta p t) + 2\beta_{22}\Big(z_2 \sin^2 p_2 t \\ & - \frac{y_2}{2}\sin 2p_2 t\Big) + \frac{\omega_2}{2}h_{21}\sin 2\nu t\cdot(-z_1\sin 2\nu t + y_1\cos 2\nu t + z_1 \sin\delta p t - y_1\cos\delta p t) \\ & + \Big(\frac{\Delta_2}{p_2} + \omega_2 h_{22}\sin 2\nu t\Big)\cdot\Big(-\frac{z_2}{2}\sin 2p_2 t - y_2\sin^2 p_2 t\Big)\Big] \\ & - \sqrt{\varepsilon}\cdot\omega_2(c_{21}C_1 + c_{22}C_2)f(t)\end{aligned}$$

$$\dot{y}_2 = -\varepsilon\Big[\frac{p_1}{p_2}\beta_{21}(-z_1\sin 2vt + y_1\cos 2vt - z_1\sin\delta pt + y_1\cos\delta pt) + 2\beta_{22}\Big(-\frac{z_2}{2}\sin 2p_2t + y_2\cos^2 p_2t\Big) + \frac{\omega_2}{2}h_{21}\sin 2vt\cdot(z_1\cos 2vt + y_1\sin 2vt + z_1\cos\delta pt + y_1\sin\delta pt) + \Big(\frac{\Delta_2}{p_2} + \omega_2 h_{22}\sin 2vt\Big)\cdot\Big(z_2\cos^2 p_2t + \frac{y_2}{2}\sin 2p_2t\Big)\Big] - \sqrt{\varepsilon}\cdot\omega_2(c_{21}D_1 + c_{22}D_2)f(t) \tag{8}$$

where

$$A_1 = -\Big(\frac{z_1}{2}\sin 2p_1t + y_1\sin^2 p_1t\Big)$$

$$A_2 = \tfrac{1}{2}(-z_2\sin 2vt + y_2\cos 2vt - z_2\sin\delta pt - y_2\cos\delta pt)$$

$$B_1 = z_1\cos^2 p_1t + \frac{y_1}{2}\sin 2p_1t$$

$$B_2 = \tfrac{1}{2}(z_2\cos 2vt + y_2\sin 2vt + z_2\cos\delta pt - y_2\sin\delta pt)$$

$$C_1 = \tfrac{1}{2}(-z_1\sin 2vt + y_1\cos 2vt + z_1\sin\delta pt - y_1\cos\delta pt)$$

$$C_2 = -\frac{z_2}{2}\sin 2p_2t - y_2\sin^2 p_2t$$

$$D_1 = \tfrac{1}{2}(z_1\cos 2vt + y_1\sin 2vt + z_1\cos\delta pt + y_1\sin\delta pt)$$

$$D_2 = z_2\cos^2 p_2t + \frac{y_2}{2}\sin 2p_2t$$

We assume, that the oscillation frequencies of the two-degrees-of-freedom systems are commensurable, i.e. $n_1\cdot p_1 = n_2\cdot p_2$, where n_1 and n_2 are integers. We can easily show that the fluctuations in two different degrees of freedom have a common period $T = n_1\cdot T_2 + n_2\cdot T_1$, where $T_i = 2\pi/p_i$, and it is possible to directly apply the SKT to standard systems of equations in view of the periodicity of the deterministic functions.

Applying on system (8) the averaging principle of Krylov–Bogolyubov and the SKT [13] leads to the following homogenous Itô equations:

$$\mathrm{d}z_1 = \varepsilon\Big[(-\beta_{11} + d_1)z_1 + \Big(\frac{\Delta_1}{2p_1} - d_2\Big)y_1 + \omega_1\frac{h_{12}}{4}z_2\Big]\mathrm{d}t + \sqrt{\varepsilon}\sum_{j=1}^{4}\sigma_{1j}(z)\,\mathrm{d}w_j$$

$$\mathrm{d}y_1 = \varepsilon\Big[-\Big(\frac{\Delta_1}{2p_1} - d_2\Big)z_1 + (-\beta_{11} + d_1)y_1 - \omega_1\frac{h_{12}}{4}y_2\Big]\mathrm{d}t + \sqrt{\varepsilon}\sum_{j=1}^{4}\sigma_{2j}(z)\,\mathrm{d}w_j$$

$$\mathrm{d}z_2 = \varepsilon\Big[\omega_2\frac{h_{21}}{4}z_1 + (-\beta_{22} + d_3)z_2 + \Big(\frac{\Delta_2}{2p_2} - d_4\Big)y_2\Big]\mathrm{d}t + \sqrt{\varepsilon}\sum_{j=1}^{4}\sigma_{3j}(z)\,\mathrm{d}w_j$$

$$\mathrm{d}y_2 = \varepsilon\Big[-\omega_2\frac{h_{21}}{4}y_1 - \Big(\frac{\Delta_2}{2p_2} - d_4\Big)z_2 + (-\beta_{22} + d_3)y_2\Big]\mathrm{d}t + \sqrt{\varepsilon}\sum_{j=1}^{4}\sigma_{4j}(z)\,\mathrm{d}w_j \tag{9}$$

where $w_j(t)$ $(j = 1, 2\ldots 4)$ are independent Wiener processes of unit intensity and

$$d_1 = \frac{\omega_1}{8}\{\omega_1 c_{11}^2[S(2p_1) - S(0)] + \omega_2 c_{12}c_{21}[S(2v) - S(\delta p)]\}$$

$$d_2 = \frac{\omega_1}{8}\{\omega_1 c_{11}^2\psi(2p_1) + \omega_2 c_{12}c_{21}[\psi(2v) - \psi(\delta p)]\}$$

$$d_3 = \frac{\omega_2}{8}\{\omega_1 c_{12}c_{21}[S(2v) - S(\delta p)] + \omega_2 c_{22}^2[S(2p_2) - S(0)]\}$$

$$d_4 = \frac{\omega_2}{8}\{\omega_1 c_{21}c_{12}[\psi(2v) + \psi(\delta p)] + \omega_2 c_{22}^2\psi(2p_2)\}$$

$$[\sigma\sigma^{\mathrm{T}}]_{11} = \frac{\omega_1^2}{8}\{c_{11}^2(S(2p_1)z_1^2 + c_2 y_1^2) + c_{12}^2(c_1 z_2^2 + c_1 y_2^2)\}$$

$$[\sigma\sigma^{\mathrm{T}}]_{22} = \frac{\omega_1^2}{8}\{c_{11}^2(c_2 z_1^2 + S(2p_1)y_1^2) + c_{12}^2(c_1 z_2^2 + c_1 y_2^2)\}$$

$$[\sigma\sigma^{\mathrm{T}}]_{33} = \frac{\omega_2^2}{8}\{c_{21}^2(c_1 z_1^2 + c_1 y_1^2) + c_{22}^2(S(2p_2)z_2^2 + c_3 y_2^2)\}$$

$$[\sigma\sigma^{\mathrm{T}}]_{44} = \frac{\omega_2^2}{8}\{c_{21}^2(c_1 z_1^2 + c_1 y_1^2) + c_{22}^2(c_3 z_2^2 + S(2p_2)y_2^2)\}$$

$$[\sigma\sigma^{\mathrm{T}}]_{12} = [\sigma\sigma^{\mathrm{T}}]_{21} = -\frac{\omega_1^2}{4}c_{11}^2 S(0)z_1 y_1$$

$$[\sigma\sigma^{\mathrm{T}}]_{13} = [\sigma\sigma^{\mathrm{T}}]_{31} = \frac{\omega_1\omega_2}{8}(c_4 z_1 z_2 + c_5 y_1 y_2)$$

$$[\sigma\sigma^{\mathrm{T}}]_{14} = [\sigma\sigma^{\mathrm{T}}]_{41} = \frac{\omega_1\omega_2}{8}(c_4 z_1 y_2 - c_5 y_1 z_2)$$

$$[\sigma\sigma^{\mathrm{T}}]_{23} = [\sigma\sigma^{\mathrm{T}}]_{32} = \frac{\omega_1\omega_2}{8}(-c_5 z_1 y_2 + c_4 y_1 z_2)$$

$$[\sigma\sigma^{\mathrm{T}}]_{24} = [\sigma\sigma^{\mathrm{T}}]_{42} = \frac{\omega_1\omega_2}{8}(c_5 z_1 z_2 + c_4 y_1 y_2)$$

$$[\sigma\sigma^{\mathrm{T}}]_{34} = [\sigma\sigma^{\mathrm{T}}]_{43} = -\frac{\omega_2^2}{4}c_{22}^2 S(0)z_2 y_2$$

$$c_1 = S(2v) + S(\delta p), \quad c_2 = S(2p_1) + 2S(0)$$

$$c_3 = S(2p_2) + 2S(0), \quad c_4 = c_{12}c_{21}[S(2v) - S(\delta p)]$$

$$c_5 = c_{12}c_{21}[S(2v) + S(\delta p)] + 2c_{11}c_{22}S(0), \quad \delta p = p_1 - p_2$$

Here $S(\omega)$ and $\psi(\omega)$, respectively, denote the cosine and sine power spectral densities of the stochastic process $f(t)$ defined by

$$S(\omega) + \mathrm{i}\psi(\omega) = 2\int_0^\infty E[f(t)f(t+\tau)]\mathrm{e}^{\mathrm{i}\omega\tau}\,\mathrm{d}\tau \tag{10}$$

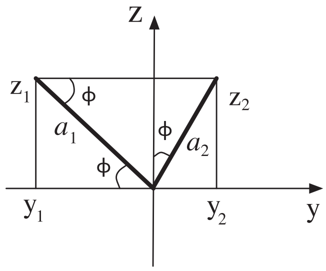

Fig. 1. Cartesian rectangular coordinates.

The Cartesian rectangular coordinates z_1, z_2, y_1, and y_2 are shown in Fig. 1, which are defined as

$$z_1 = a_1 \sin\phi, \quad z_2 = a_2 \cos\phi, \quad a_1^2 = z_1^2 + y_1^2$$
$$y_1 = -a_1 \cos\phi, \quad y_2 = a_2 \sin\phi, \quad a_2^2 = z_2^2 + y_2^2 \tag{11}$$

The stability analysis by the probabilistic approach of trivial solutions of stochastic system (1) is based on the fact that the coefficients of the terms of the right-hand side of Eq. (9) are homogeneous first-order functions of z_1, z_2, y_1, and y_2. Therefore the projection of the amplitude vector (a_1, a_2) on a circle is also a Markovian process. Hence, explicit expression for the largest Lyapunov exponent of the amplitude process may be derived [5]. For this purpose, a further logarithmic polar transformation is applied as follows:

$$\rho = \frac{1}{2}\ln(a_1^2 + a_2^2), \quad \phi = \arctan\left(\frac{a_2}{a_1}\right), \ \ 0 \leqslant \phi \leqslant \frac{\pi}{2} \tag{12}$$

Further making use of Itô's differential rule for four-dimensional diffusion stochastic process [21], the following pair of Itô equations governing ρ and ϕ are obtained:

$$\mathrm{d}\rho = Q(\phi)\,\mathrm{d}t + \Omega(\phi)\,\mathrm{d}w$$
$$\mathrm{d}\phi = \Phi(\phi)\,\mathrm{d}t + \Psi(\phi)\,\mathrm{d}w \tag{13}$$

where $w(t)$ is a Wiener process of unit intensity and

$$Q(\phi) = \alpha_1 \cos^2\phi + \alpha_2 \sin^2\phi + \frac{\varepsilon}{8}(\omega_1 h_{12} + \omega_2 h_{21})\sin^2 2\phi + \varepsilon\frac{\omega_1\omega_2}{8}c_{12}c_{21}S^- + \Psi^2(\phi)$$

$$\begin{aligned}\Phi(\phi) = {} & \frac{1}{2}(\alpha_2 - \alpha_1)\sin 2\phi - \frac{\varepsilon}{4}(\omega_1 h_{12}\sin^2\phi - \omega_2 h_{21}\cos^2\phi)\sin 2\phi \\ & + \frac{\varepsilon}{8}\left(\frac{\omega_1^2}{8}c_{11}^2 S(2p_1) + \frac{\omega_2^2}{8}c_{22}^2 S(2p_2) - \frac{\omega_1\omega_2}{4}c_{12}c_{21}S^-\right)\sin 4\phi \\ & + \frac{\varepsilon}{2}\left(\frac{\omega_1^2}{8}c_{12}^2(\tan\phi)\sin^2\phi + \frac{\omega_2^2}{8}c_{21}^2(\cot\phi)\cos^2\phi\right)S^+\cos 2\phi\end{aligned}$$

$$\begin{aligned}\Psi^2(\phi) = {} & \frac{\varepsilon}{4}\left(\frac{\omega_1^2}{8}c_{11}^2 S(2p_1) + \frac{\omega_2^2}{8}c_{22}^2 S(2p_2) - \frac{\omega_1\omega_2}{4}c_{12}c_{21}S^-\right)\sin^2 2\phi \\ & + \varepsilon\left(\frac{\omega_1^2}{8}c_{12}^2\sin^4\phi + \frac{\omega_2^2}{8}c_{21}^2\cos^4\phi\right)S^+\end{aligned}$$

$$\begin{aligned}\Omega^2(\phi) = {} & \varepsilon\frac{\omega_1^2}{8}c_{11}^2 S(2p_1)\cos^4\phi + \varepsilon\frac{\omega_2^2}{8}c_{22}^2 S(2p_2)\sin^4\phi \\ & + \frac{\varepsilon}{4}\left[\left(\frac{\omega_1^2}{8}c_{12}^2 + \frac{\omega_2^2}{8}c_{21}^2\right)S^+ + \frac{\omega_1\omega_2}{4}c_{12}c_{21}S^-\right]\sin^2 2\phi\end{aligned} \tag{14}$$

where $\alpha_1 = -\varepsilon\beta_{11} + \varepsilon(\omega_1^2/8)c_{11}^2 S(2p_1)$, $\alpha_2 = -\varepsilon\beta_{22} + \varepsilon(\omega_2^2/8)c_{22}^2 S(2p_2)$.

Setting

$$k_{11} = \sqrt{\varepsilon}\omega_1 c_{11}, \quad k_{12} = \sqrt{\varepsilon}\omega_1 c_{12}, \quad k_{21} = \sqrt{\varepsilon}\omega_2 c_{21}, \quad k_{22} = \sqrt{\varepsilon}\omega_2 c_{22}$$
$$\lambda_{12} = \varepsilon\omega_1 h_{12}, \quad \lambda_{21} = \varepsilon\omega_2 h_{21}, \quad \beta_1 = \varepsilon\beta_{11}, \quad \beta_2 = \varepsilon\beta_{22} \tag{15}$$

and, by a suitable scaling of coordinates, it is always possible to take $k_{12} = \pm k_{21} = k > 0$, without loss of generality. Making use of the fact that $p_1 \approx \omega_1$ and $p_2 \approx \omega_2$, expressions in Eq. (14) take the following forms:

$$Q(\phi) = \tfrac{1}{2}[-(2b + \lambda^+)\cos^2 2\phi + (\alpha_1 - \alpha_2)\cos 2\phi + A_1]$$

$$\Psi^2(\phi) = c(1 - a\cos^2 2\phi)$$

$$\Phi(\phi) = \tfrac{1}{2}[(\alpha_2 - \alpha_1 + \lambda^-) + (2b + \lambda^+)\cos 2\phi]\sin 2\phi + \tfrac{1}{8}k^2 S^+ \cot 2\phi$$

$$\Omega^2(\phi) = b\cos^2 2\phi + \tfrac{1}{16}[k_{11}^2 S(2\omega_1) - k_{22}^2 S(2\omega_2)]\cos 2\phi + [b + 2k^2 S(\omega_1 \pm \omega_2)] \tag{16}$$

where

$$\alpha_1 = -\beta_1 + \tfrac{1}{8}k_{11}^2 S(2\omega_1), \quad \alpha_2 = -\beta_2 + \tfrac{1}{8}k_{22}^2 S(2\omega_2), \quad \lambda^{\pm} = \tfrac{1}{4}(\lambda_{21} \pm \lambda_{12})$$

$$A_1 = \alpha_1 + \alpha_2 + \lambda^+ + 2c \pm \tfrac{1}{4}k^2 S^-, \quad c = \tfrac{1}{8}k^2 S^+ + b, \quad a = \frac{b}{c}$$

$$b = \tfrac{1}{8}[\tfrac{1}{4}k_{11}^2 S(2\omega_1) + \tfrac{1}{4}k_{22}^2 S(2\omega_2) - k^2 S(\omega_1 \pm \omega_2)]$$

$$S_{ij}^{\pm} = S(\omega_i + \omega_j) \pm S(\omega_i - \omega_j), \quad i,j = 1,2 \tag{17}$$

in which the upper sign is taken when $k_{12} = k_{21} = k$, and the lower sign when $k_{12} = -k_{21} = k$. Note that parameters b, c and thus a are related to the characteristics of the two individual subsystems, i.e. ω_1, ω_2, the type and strength of the coupling between them, i.e. k, as well as the stochastic property of the excitation, i.e. S. For white-noise excitations, c is always positive and depending on the strength of coupling, b can take any real values. For example, if the coupling is sufficiently strong such that $k^2 > \frac{1}{4}(k_{11}^2 + k_{22}^2)$, $b<0$, and thus $a<0$, whereas in a weakly coupled system, e.g. $k^2 < \frac{1}{4}(k_{11}^2 + k_{22}^2)$, $b>0$, and thus $a>0$.

We consider the nonsingular case, i.e. $\Psi^2(\phi)>0$, then the diffusion process $\phi(t)$ is nonsingular; it has a stationary distribution with a probability density $\mu(\phi)$ being governed by the Fokker–Planck–Kolmogorov equation defined as

$$\frac{1}{2}\frac{d^2}{d\phi^2}(\mu\Psi^2(\phi)) - \frac{d}{d\phi}(\mu\Phi(\phi)) = 0 \tag{18}$$

along with the normalization condition

$$\int_0^{\pi/2} \mu(\phi)\,d\phi = 1 \tag{19}$$

and the periodicity $\mu(0) = \mu(\pi/2)$. Eq. (18) has a unique solution [1] defined by

$$\mu(\phi) = \frac{C}{\Psi^2(\phi)W(\phi)} \tag{20}$$

where

$$W(\phi) = \exp\left\{-2\int^{\phi} \frac{\Phi(x)}{\Psi^2(x)}dx\right\} = \frac{1}{\sin 2\phi}\exp\left\{\frac{1}{2c}\int^{\cos 2\phi} \frac{-\bar{t} + \lambda^+ x}{1 - ax^2}dx\right\} \tag{21}$$

and $\bar{t} = \alpha_1 - \alpha_2 - \lambda^-$. Here C is the normalized constant determined from Eq. (19).

The form of the integral in Eq. (21) depends on the sign of parameter $a = b/c$. Here c is always positive for white-noise excitations, thus, the sign of a is the same as parameter b. In the following, we discuss three forms of integrals which correspond, respectively, to $a>0$, $a=0$, and $a<0$.

- $a<0$

  In this case, the invariant density $\mu(\phi)$ is of the form

$$\mu(\phi) = \frac{C\sin 2\phi}{\Psi^2(\phi)}(1 - a\cos^2 2\phi)^{\lambda^+/4b}\exp\left[\frac{\bar{t}}{2\sqrt{-bc}}\arctan(\sqrt{-a}\cos 2\phi)\right] \tag{22}$$

and

$$C = \frac{2\sqrt{-bc}}{\int_{-\tau_1}^{\tau_1} \frac{1}{(\cos t)^{\lambda^+/2b}} \exp\left(\frac{\bar{t}}{2\sqrt{-bc}} t\right) \mathrm{d}t}, \quad \tau_1 = \arctan\sqrt{-a} \tag{23}$$

For this case, a typical plot of the density $\mu(\phi)$ is shown in Fig. 2, for $S(2\omega_1) = S(2\omega_2) = S(\omega_1 + \omega_2) = S_0$. The largest Lyapunov exponent of system (9) is given by

$$\bar{a} = E[Q(\phi)] = \int_0^{\pi/2} Q(\phi)\mu(\phi)\,\mathrm{d}\phi \tag{24}$$

Substituting from Eqs. (16) and (22) into Eq. (24) yields the following expression for the Lyapunov exponent for $a<0$:

$$\bar{a} = \frac{C}{4\sqrt{-bc}} \int_{-\tau_1}^{\tau_1} \frac{F(t)}{(\cos t)^{\lambda^+/2b}} \exp\left(\frac{\bar{t}}{2\sqrt{-bc}} t\right) \mathrm{d}t \tag{25}$$

where

$$F(t) = \frac{1}{a}(2b + \lambda^+)\tan^2 t + \frac{1}{\sqrt{-a}}(\alpha_1 - \alpha_2)\tan t + A_1 \tag{26}$$

The stability boundary is define as $\bar{a} = 0$. It is clear that the normalization constant $C>0$. Using the intermediate value theorem for integrals, the stability boundary can be written as $F(t_s) = 0$, where $t_s \in [-\tau_1, \tau_1]$. Clearly, t_s is a function of $\bar{t} = \alpha_1 - \alpha_2 - \lambda^-$. Using Eq. (17), the stability boundary becomes

$$\left(1 + \frac{\tan t_s}{\sqrt{-a}}\right)\alpha_1 + \left(1 - \frac{\tan t_s}{\sqrt{-a}}\right)\alpha_2 + 2c\sec^2 t_s + \lambda^+\left(\frac{\tan^2 t_s}{a} + 1\right) \pm \frac{1}{8}k^2 S^- = 0 \tag{27}$$

As t_s is an implicit function of α_1 and α_2, the relation between α_1 and α_2 is also implicit in general. The stability boundary can only be analyzed qualitatively if without numerical simulations. According to Eq. (17), α_1 and α_2 are related to the level of viscous damping of the system. Higher level of damping in the system will have more stabilizing effect, thus it is expected that on the stability boundary, $\mathrm{d}\alpha_2/\mathrm{d}\alpha_1<0$. Furthermore, it can be shown that when $\alpha_1 \to -\infty$, i.e. damping $\beta_1 \to \infty$, $t_s \to -\tau_1$; similarly, when $\alpha_2 \to -\infty$ (i.e. $\beta_2 \to \infty$, $t_s \to \tau_1$, see Appendix). Considering the former case ($\alpha_1 \to -\infty$), Eq. (27) can be written as

$$\alpha_2 = b - c - \frac{1}{2}\lim_{\substack{t_s \to -\tau_1 \\ \alpha_1 \to -\infty}} \left(1 + \frac{\tan t_s}{\sqrt{-a}}\right)\alpha_1 \mp \frac{1}{16}k^2 S^- \tag{28}$$

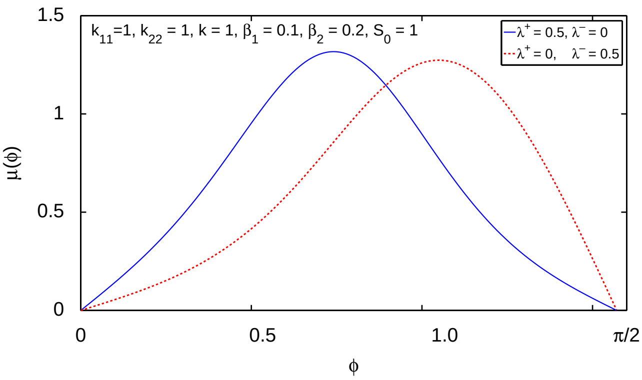

Fig. 2. Probability density $\mu(\phi)$.

which determines an asymptotic line of the stability boundary. Following the same procedure, the asymptotic line for the case when $\alpha_2 \to -\infty$ and $t_s \to \tau_1$ can be found as

$$\alpha_1 = b - c - \frac{1}{2} \lim_{\substack{t_s \to \tau_1 \\ \alpha_1 \to -\infty}} \left(1 - \frac{\tan t_s}{\sqrt{-a}}\right)\alpha_2 \mp \frac{1}{16} k^2 S^- \tag{29}$$

To determine the limits in Eqs. (28) and (29) analytically is not possible in general as t_s is an unknown implicit function of α_1 and α_2. However, as observed from intensive numerical simulations, for white-noise excitations, the asymptotical lines of the stability boundary are determined by

$$\alpha_1 = \alpha_2 = 0 \tag{30}$$

Thus, the limits may take the same finite value as

$$\lim_{\substack{t_s \to -\tau_1 \\ \alpha_1 \to -\infty}} \left(1 + \frac{\tan t_s}{\sqrt{-a}}\right)\alpha_1 = \lim_{\substack{t_s \to \tau_1 \\ \alpha_1 \to -\infty}} \left(1 - \frac{\tan t_s}{\sqrt{-a}}\right)\alpha_2 = 2b - 2c \tag{31}$$

- $a > 0$

In this case, the invariant density $\mu(\phi)$ is of the form

$$\mu(\phi) = \frac{C \sin 2\phi}{\Psi^2(\phi)} (1 - a\cos^2 2\phi)^{\lambda^+/4b} \exp\left[\frac{\bar{t}}{2\sqrt{bc}} \operatorname{arctanh}(\sqrt{a}\cos 2\phi)\right] \tag{32}$$

where

$$C = \frac{2\sqrt{bc}}{\displaystyle\int_{-\tau_2}^{\tau_2} \frac{1}{(\cosh t)^{\lambda^+/2b}} \exp\left(\frac{\bar{t}}{2\sqrt{bc}} t\right) \mathrm{d}t} \tag{33}$$

and $\tau_2 = \operatorname{arctanh}\sqrt{a}$. Substituting Eqs. (16) and (32) into Eq. (24) yields the following expression for the Lyapunov exponent for the case when $a > 0$:

$$\bar{a} = \frac{C}{4\sqrt{bc}} \int_{-\tau_2}^{\tau_2} F(t) \frac{1}{(\cosh t)^{\lambda^+/2b}} \exp\left(\frac{\bar{t}}{2\sqrt{bc}} t\right) \mathrm{d}t \tag{34}$$

where

$$F(t) = -\frac{1}{a}(2b + \lambda^+)\tanh^2 t + \frac{1}{\sqrt{a}}(\alpha_1 - \alpha_2)\tanh t + A_1 \tag{35}$$

Using the intermediate value theorem for integrals, the stability boundary is determined by

$$\left(1 + \frac{\tanh t_s}{\sqrt{a}}\right)\alpha_1 + \left(1 - \frac{\tanh t_s}{\sqrt{a}}\right)\alpha_2 + 2c \cdot \operatorname{sech}^2 t_s + \lambda^+\left(1 - \frac{\tanh^2 t_s}{a}\right) \pm \frac{1}{8} k^2 S^- = 0 \tag{36}$$

where $t_s \in [-\tau_1, \tau_1]$.
Following the same procedure shown in Appendix, it can be shown that on the stability boundary, when $\alpha_1 \to -\infty$, $t_s \to -\tau_1$; whereas when $\alpha_2 \to -\infty$, $t_s \to \tau_1$. The asymptotic lines of the two cases are

$$\alpha_1 = b - c - \frac{1}{2} \lim_{\substack{t_s \to \tau_1 \\ \alpha_2 \to -\infty}} \left(1 - \frac{\tanh t_s}{\sqrt{a}}\right)\alpha_2 \mp \frac{1}{8} k^2 S^-$$

$$\alpha_2 = b - c - \frac{1}{2} \lim_{\substack{t_s \to -\tau_1 \\ \alpha_1 \to -\infty}} \left(1 + \frac{\tanh t_s}{\sqrt{a}}\right)\alpha_1 \mp \frac{1}{8} k^2 S^- \tag{37}$$

As in the previous case when $a < 0$, the two limits in Eq. (37) are not found analytically in this study. Observations made in numerical simulations suggested that for the case of white noise, the asymptotic lines of the stability boundary take the same form as expressed in Eq. (30), thus

$$\lim_{\substack{t_s\to-\tau_1\\ \alpha_1\to-\infty}}\left(1+\frac{\tanh t_s}{\sqrt{-a}}\right)\alpha_1=\lim_{\substack{t_s\to\tau_1\\ \alpha_1\to-\infty}}\left(1-\frac{\tanh t_s}{\sqrt{-a}}\right)\alpha_2=2b-2c \tag{38}$$

- $a=0$

In this case, the invariant density $\mu(\phi)$ is

$$\mu(\phi)=\frac{C}{c}\sin 2\phi\exp\left(\frac{\bar{t}}{2c}\cos 2\phi\right) \tag{39}$$

where

$$C=\frac{\bar{t}}{2}\operatorname{csch}\left(\frac{\bar{t}}{2c}\right) \tag{40}$$

Substituting Eqs. (16) and (39) into Eq. (24) yields the following expression for the Lyapunov exponent for this case:

$$\bar{a}=\frac{1}{2}\left\{\left[4\frac{\lambda^+}{\bar{t}}c+(\alpha_1-\alpha_2)\right]\coth\left(\frac{\bar{t}}{2c}\right)-8\frac{\lambda^+}{\bar{t}^2}c^2+(\alpha_1+\alpha_2)-2\frac{\lambda^-}{\bar{t}}c\pm\frac{1}{4}k^2S^-\right\} \tag{41}$$

The two asymptotic lines of the stability boundary can be found as

$$\alpha_1=\mp\tfrac{1}{8}k^2S^-,\quad \alpha_2=\mp\tfrac{1}{8}k^2S^- \tag{42}$$

Again, in this case, for white-noise excitations, the asymptotic lines of the stability boundary are the same as the previous cases, i.e. $\alpha_1=\alpha_2=0$.

3. Application: flexural–torsional instability of a deep rectangular beam

As an application, the problem of coupled flexural–torsional instability of a deep rectangular beam in the presence of fluctuating axial loads and end moments is considered. The beam is simply supported as shown in Fig. 3.

The governing equations for the coupled flexural and torsional motion of the beam can be written as [22]

$$-EI_x\frac{\partial^4 v_0}{\partial z^4}+P\frac{\partial^2 v_0}{\partial z^2}+M\frac{\partial^2\theta_0}{\partial z^2}-\rho A\frac{\partial^2 v_0}{\partial t^2}-D_v\frac{\partial v_0}{\partial t}=0$$

$$\left[GJ+P\frac{I_P}{A}\right]\frac{\partial^2\theta_0}{\partial z^2}+M\frac{\partial^2 v_0}{\partial z^2}-\rho I_P\frac{\partial^2\theta_0}{\partial t^2}-D_\theta\frac{\partial\theta_0}{\partial t}=0 \tag{43}$$

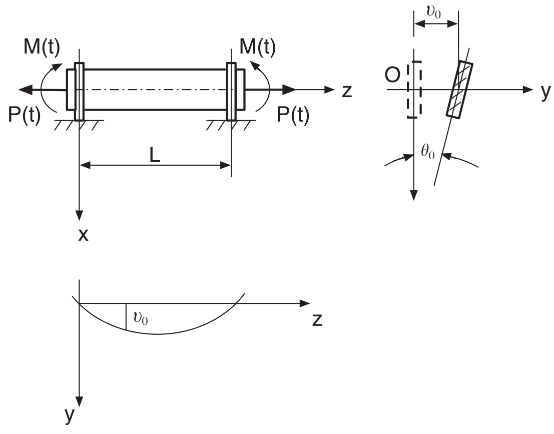

Fig. 3. Loaded rectangular beam in flexural–torsional deformation.

where EI_x and GJ denote the relevant flexural and torsional rigidities of the cross section of the beam, respectively; D_v and D_θ the viscous damping coefficients for flexural and torsional deformations, respectively; ρ the mass density of the material; A the area of the cross section; I_p the polar moment of inertia; and $P(t), M(t)$ are random axial force and member-end moment, respectively. $v_0(z,t)$ and $\theta_0(z,t)$ are the flexural and torsional deformations, respectively, with the following boundary conditions:

$$v_0(0,t) = v_0(L,t) = 0$$

$$\frac{\partial^2 v_0}{\partial z^2}(0,t) = \frac{\partial^2 v_0}{\partial z^2}(L,t) = 0$$

$$\theta_0(0,t) = \theta_0(L,t) = 0 \tag{44}$$

In this example, the fundamental modes of flexural and torsional vibration are assumed to be dominant, thus

$$v_0(z,t) = v(t)\sin\frac{\pi z}{L}, \quad \theta_0(z,t) = \theta(t)\sin\frac{\pi z}{L} \tag{45}$$

Plugging Eq. (45) into Eq. (43) and defining $l = \pi/L$ yield,

$$\ddot{v} + \frac{D_v}{\rho A}\dot{v} + \frac{EI_x}{\rho A}l^4 v + \frac{P(t)}{\rho A}l^2 v + \frac{M(t)}{\rho A}l^2\theta = 0$$

$$\ddot{\theta} + \frac{D_\theta}{\rho I_P}\dot{\theta} + \frac{GJ}{\rho I_P}l^2\theta + \frac{P(t)}{\rho A}l^2\theta + \frac{M(t)}{\rho I_P}l^2 v = 0 \tag{46}$$

Changing to new variables

$$v = \frac{x_1}{\sqrt{\omega_1\omega_2}}, \quad \theta = \frac{x_2}{r_p\sqrt{\omega_1\omega_2}} \tag{47}$$

where $\omega_1 = (\pi^2/L^2)\sqrt{EI_x/\rho A}$ and $\omega_2 = (\pi/L)\sqrt{GJ/\rho I_P}$ are the fundamental frequencies of the flexural and torsional modes, respectively, and $r_p = \sqrt{I_P/A}$ is the polar radius of gyration of the cross section, Eq. (46) can be rewritten as

$$\ddot{x}_1 + \frac{D_v}{\rho A}\dot{x}_1 + \omega_1^2 x_1 + \omega_1^2\left[\frac{1}{\gamma}\frac{M(t)}{M_{cr}}x_2 + \frac{P(t)}{P_{cr}}x_1\right] = 0$$

$$\ddot{x}_2 + \frac{D_\theta}{\rho I_P}\dot{x}_2 + \omega_2^2 x_2 + \omega_2^2\left[\gamma\frac{M(t)}{M_{cr}}x_1 + \gamma^2\frac{P(t)}{P_{cr}}x_2\right] = 0 \tag{48}$$

where

$$P_{cr} = l^2 EI_x, \quad M_{cr} = l\sqrt{EI_x GJ}, \quad \gamma = \frac{\omega_1}{\omega_2} = \frac{\pi}{L}\sqrt{\frac{EI_x}{GJ}}r_p \tag{49}$$

Let $q_1 = x_1$, $q_2 = x_2\sqrt{\gamma}$, $2\varepsilon\beta_{11} = D_v/\rho A$, $2\varepsilon\beta_{22} = D_\theta/\rho I_P$, $P(t)/P_{cr} = \sqrt{\varepsilon}f(t)$, $M(t)/M_{cr} = \varepsilon h\sin 2\nu t + \sqrt{\varepsilon}f(t)$. The system equation (48) can be written as

$$\ddot{q}_1 + 2\varepsilon\beta_{11}\dot{q}_1 + \omega_1^2 q_1 + \omega_1^2\left[\varepsilon\frac{h}{\sqrt{\gamma}}(\sin 2\nu t)q_2 + \left(q_1 + \frac{1}{\sqrt{\gamma}}q_2\right)\sqrt{\varepsilon}f(t)\right] = 0$$

$$\ddot{q}_2 + 2\varepsilon\beta_{22}\dot{q}_2 + \omega_2^2 q_2 + \omega_2^2[\varepsilon\sqrt{\gamma}h(\sin 2\nu t)q_1 + (\sqrt{\gamma}q_1 + \gamma^2 q_2)\sqrt{\varepsilon}f(t)] = 0 \tag{50}$$

which takes the exact same form as system equation (1) with the following coefficients:

$$\beta_{12} = \beta_{21} = h_{11} = h_{22} = 0$$

$$c_{11} = 1, \quad c_{12} = \frac{1}{\sqrt{\gamma}}, \quad c_{21} = \sqrt{\gamma}, \quad c_{22} = \gamma^2$$

$$h_{12} = \frac{h}{\sqrt{\gamma}}, \quad h_{21} = \sqrt{\gamma} h \tag{51}$$

Thus, the coefficients of Eq. (15) are obtained as

$$k_{11} = \sqrt{\varepsilon}\omega_1, \quad k_{22} = \gamma k_{11}, \quad k_{12} = k_{21} = k = \frac{k_{11}}{\sqrt{\gamma}} = \sqrt{\varepsilon \omega_1 \omega_2}$$

$$\lambda = \varepsilon \omega_1 h, \quad \lambda_{21} = \lambda_{12} = \frac{\lambda}{\sqrt{\gamma}}, \quad \beta_1 = \varepsilon \beta_{11}, \quad \beta_2 = \varepsilon \beta_{22} \tag{52}$$

Note that in this case, the strength of coupling is proportional to the geometrical average of the two fundamental natural frequencies, i.e. $\sqrt{\omega_1 \omega_2}$.

Letting $h_\varepsilon = \varepsilon h$ and $f_\varepsilon(t) = (f(t)/\sqrt{h})\,(h \neq 0)$, Eq. (50) can be further rewritten as

$$\ddot{q}_1 + 2\beta_1 \dot{q}_1 + \omega_1^2 q_1 + \omega_1^2 \left[\frac{h_\varepsilon}{\sqrt{\gamma}} (\sin 2\nu t) q_2 + \left(q_1 + \frac{1}{\sqrt{\gamma}} q_2 \right) \sqrt{h_\varepsilon} f_\varepsilon(t) \right] = 0$$

$$\ddot{q}_2 + 2\beta_2 \dot{q}_2 + \omega_2^2 q_2 + \omega_2^2 [\sqrt{\gamma} h_\varepsilon (\sin 2\nu t) q_1 + (\sqrt{\gamma} q_1 + \gamma^2 q_2) \sqrt{h_\varepsilon} f_\varepsilon(t)] = 0 \tag{53}$$

Here we study the effect of random parametric excitation on system stability when the system is under combination parametric resonance, i.e. $2\nu \approx \omega_1 + \omega_2$.

If excitation $f(t)$ is white noise of intensity S_0, $f_\varepsilon(t)$ is also a white noise of intensity $S_\varepsilon = S_0/h$; thus $\alpha_1 = -\beta_1 + \frac{1}{8}\omega_1^2 h_\varepsilon S_\varepsilon$ and $\alpha_2 = -\beta_2 + \frac{1}{8}\gamma^2 \omega_1^2 h_\varepsilon S_\varepsilon$ (see Eq. (17)). Furthermore, the constants a, b and c in Eq. (15) are

$$b = \frac{\omega_1 \omega_2 h_\varepsilon S_\varepsilon}{32} (\gamma^3 + \gamma - 4), \quad c = \frac{\omega_1 \omega_2 h_\varepsilon S_\varepsilon}{32} (\gamma^3 + \gamma + 4), \quad a = \frac{(\gamma^3 + \gamma - 4)}{(\gamma^3 + \gamma + 4)} \tag{54}$$

In this case, constant a is only dependent on the ratio of the two fundamental frequencies, i.e. γ. Particularly, $a<0$ if $\gamma<1.3788$; $a>0$ if $\gamma>1.3788$ and $a=0$ if $\gamma = 1.3788$.

Without loss of generality, we assume that $S_\varepsilon = 1$ and let $\hat{\alpha}_1 = \beta_1 - \frac{1}{8}\omega_1^2 h_\varepsilon S_\varepsilon = -\alpha_1$ and $\hat{\alpha}_2 = \beta_2 - \frac{1}{8}\gamma^2 \omega_1^2 h_\varepsilon S_\varepsilon = -\alpha_2$. A large number of numerical simulations were carried out to evaluate the effect of combination parametric resonance on the system stability. It was observed that the stability regions of all the cases considered take similar shape. Typical results are summarized in Figs. 4–7. It is seen that while the analytical expressions of the Lyapunov exponent are different for the cases where the ratio of the two fundamental frequencies (i.e. $\gamma = \omega_1/\omega_2$) are in different regions (see Eqs. (25), (34), and (41)), the resulted stability boundaries are similar in shape. Particularly, there are asymptotical lines for all the stability regions

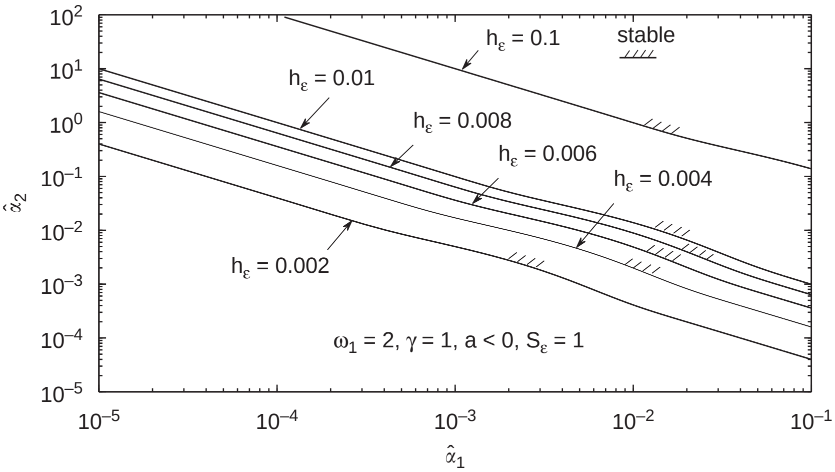

Fig. 4. Stability region for $a<0$ with varying h_ε.

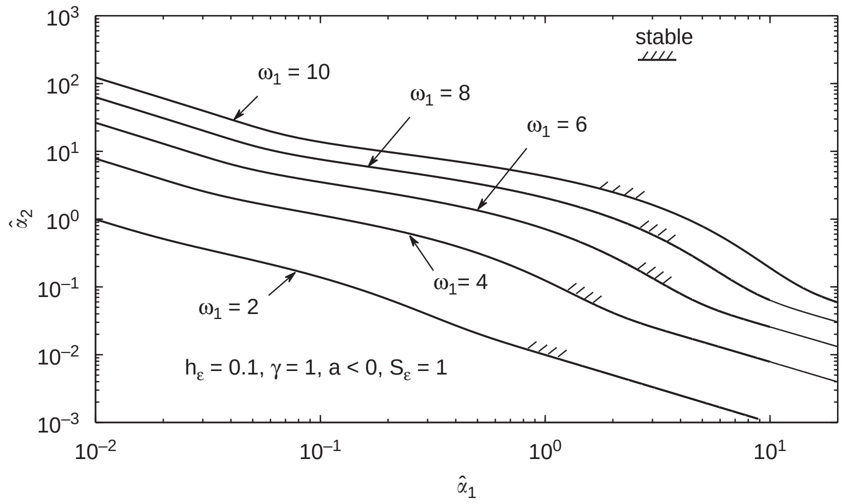

Fig. 5. Stability region for $a<0$ with varying ω_1.

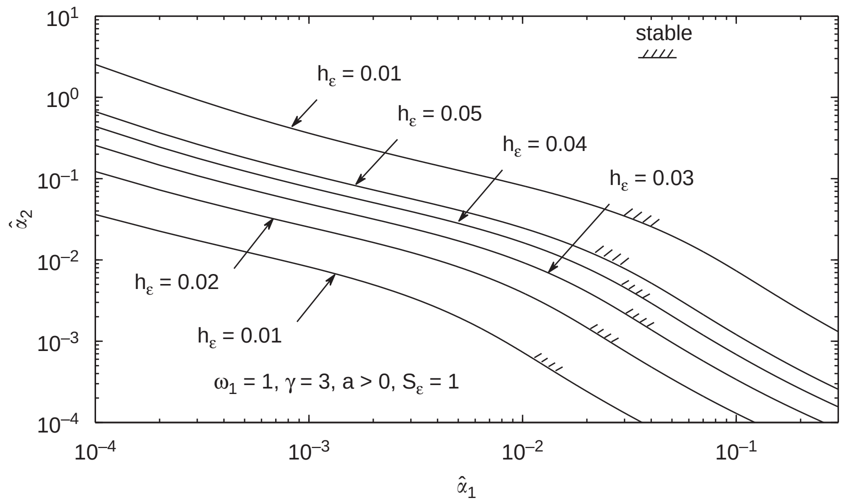

Fig. 6. Stability region for $a>0$ with varying h_ε.

obtained, which can be estimated using the numerical results obtained. It was observed that for all the cases considered, the relation between $\hat{\alpha}_1$ and $\hat{\alpha}_2$ on the stability boundary tends to be linear in the logarithm scale when either of them is sufficiently large. This observation suggests that $\ln\hat{\alpha}_1 \propto -\ln\hat{\alpha}_2$ for sufficiently large $\hat{\alpha}_1$ or $\hat{\alpha}_2$, i.e. in such cases, $\hat{\alpha}_1 \propto 1/\hat{\alpha}_2$, which implies that the asymptotical lines are defined by $\hat{\alpha}_1 = -\alpha_1 = 0$, $\hat{\alpha}_2 = -\alpha_2 = 0$, or $\beta_1 = \frac{1}{8}k_{11}^2 S_0$, $\beta_2 = \frac{1}{8}k_{22}^2 S_0$. Note that for the case where $\gamma = 1.7388 (a = 0)$, the asymptotically lines can be found analytically as $\alpha_1 = \alpha_2 = 0$ for white-noise excitations (see Eq. (42)). Using expressions in Eq. (17), the asymptotical lines of the stability region in the (β_1, β_2) plane are

$$\beta_1 = \tfrac{1}{8}k_{11}^2 S_0, \quad \beta_2 = \tfrac{1}{8}k_{22}^2 S_0 \tag{55}$$

It is also observed that as the excitation level (i.e. h_ε) or the fundamental frequencies, ω_1 and/or ω_2 increase, higher levels of viscous damping are required for the system to remain stable.

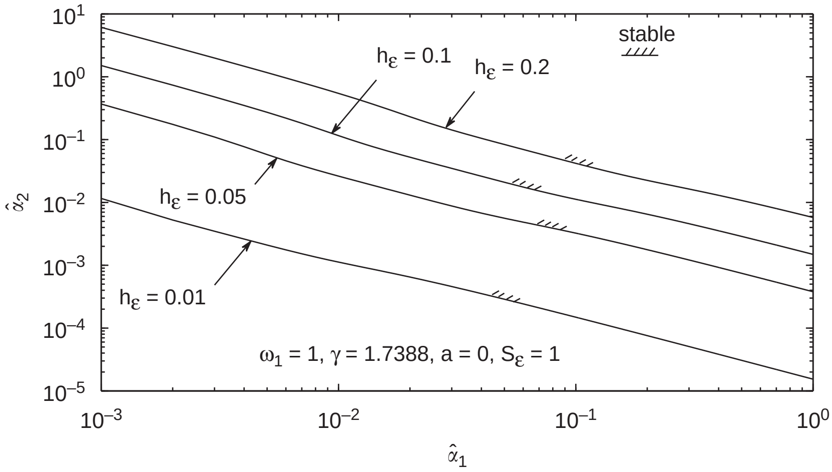

Fig. 7. Stability region for $a = 0$ with varying h_ε.

4. Concluding remarks

In this study, the almost-sure asymptotic stability of elastic systems subjected to parametric excitations has been investigated. The stability region of a linear two-degree-of-freedom coupled system can be obtained using the combination of the stochastic averaging method and the Khasminskii's formulation. It is noted that the stochastic averaging method is built on the assumption of light damping and weak excitation of wide-band process in order that the system response approaches a harmonic one. A numerical example of a deep rectangular beam subjected to fluctuating axial loads and end moments has been considered in order to demonstrate the effectiveness of the proposed method and its implementation to real-world applications. The parametric excitation consists of a white-noise action and a harmonic function. The coupled flexural–torsional instability has been investigated. It has been shown that depending on the ratio between the two fundamental frequencies, i.e. $\gamma = \omega_1/\omega_2$, the analytical expression of the Lyapunov exponent of the system takes different form. The resulted stability boundaries are, however, similar in shape. There also appears to be an asymptotical minimal damping level for each of the flexural and tensional modes that is required for the system to be stable. Such limits are independent of the strength of the coupling and are proportional to the power density of the excitation as well as the square of the natural frequencies of the individual subsystems.

The proposed method may be applied to a variety of engineering systems. For instance, it may be used to study the effect of additional random internal pressure on the load capacity of elastic cylindrical shells, and to study the stability of a helicopter rotor blade subjected to turbulent fluctuations.

Acknowledgments

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Appendix

Consider the case where $a<0$. Using Eq. (26), the stability boundary can be written as

$$\frac{1}{a}(2b+\lambda^+)\tan^2 t_s + \frac{1}{\sqrt{-a}}(\alpha_1-\alpha_2)\tan t_s + A_1 = 0 \tag{A.1}$$

Letting $z = \tan t_s$ and using the definition of A_1 (see Eq. (17)), the following expression is readily obtained:

$$z = \frac{1}{2}\frac{-B \pm \sqrt{B^2 - 4AC}}{A} \tag{A.2}$$

where $A = (1/a)(2b + \lambda^+)$, $B = (1/\sqrt{-a})(\alpha_1 - \alpha_2)$, and $C = \alpha_1 + \alpha_2 + \lambda^+ + 2c \pm \frac{1}{4}k^2 S^-$.

Ignoring the solution $z = \frac{1}{2}(-B - \sqrt{B^2 - 4AC})/A$, which leads to unrealistic results, the following expressions are obtained:

$$\lim_{\alpha_1 \to -\infty} z = -\sqrt{-a} = \tan(-\tau_1)$$

$$\lim_{\alpha_2 \to -\infty} z = \sqrt{-a} = \tan \tau_1 \tag{A.3}$$

Replacing $\tan t_s$ with $\tanh t_s$ in Eq. (A.1) gives the stability boundary for the case of $a > 0$. The same results can be obtained.

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