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An innovative eigenvalue problem solver for free vibration of uniform Timoshenko beams by using the Adomian modified decomposition method

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Abstract

This paper deals with free vibration problems of uniform Timoshenko beams under various supporting boundary conditions. The technique we have used is based on applying the Adomian modified decomposition method (AMDM) to our vibration problems. Doing some simple mathematical operations on the method, we can obtain *i*th natural frequencies and mode shapes one at a time. The computed results agree well with those analytical and numerical results given in the literature. These results indicate that the present analysis is accurate, and provides a unified and systematic procedure which is simpler and more straightforward than the other modal analysis.

1. Introduction

The vibration of beams is important in many situations of engineering practice, such as mechanical, civil, and aerospace engineering. The vibration problems of beams have been treated according to the classical Euler–Bernoulli beam theory. However, if the effects of shear deformation and rotary inertia are considered, the Timoshenko beam theory is required. The free vibration of a uniform Timoshenko beam under various boundary conditions has been studied by many authors via many different methods [1–9]. Recently, Posiadala [10] studied the free vibrations of uniform Timoshenko beams with attachments by using the Lagrange multiplier formalism. Ho and Chen [11] presented the analysis of general elastically restrained non-uniform beams using differential transform. Karami et al. [12] presented a differential quadrature element method for vibration of shear deformable beams with general boundary conditions. Lee and Schultz [13] presented the pseudospectral method for eigenvalue analysis of Timoshenko beams. Ferreira and Fasshauer [14] studied the computation of natural frequencies of shear deformable beams by an RBF function–pseudospectral method.

In this study, a new computed approach called Adomian modified decomposition method (AMDM) is introduced to solve the free vibration problems. The concept of AMDM was first proposed by Adomian and

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was applied to solve linear and nonlinear initial/boundary-value problems in physics [15–17]. Using the AMDM, Hsu et al. [18] and Lai et al. [19,20] have proposed the method to solve the free vibration problems of Euler–Bernoulli beams. In this paper, one can extend the Hsu and Lai study and consider the free vibration problems of uniform Timoshenko beams with a tip mass and elastically end constraints. Using the AMDM, the two coupled governing differential equations become two recursive algebraic equations and the boundary conditions at the right end become simple algebraic frequency equations which are suitable for symbolic computation. Moreover, after some simple algebraic operations on these frequency equations any ith natural frequency can be obtained. Finally, some problems of free vibration of uniform Timoshenko beams are solved and showed excellent agreement with the published results to verify the accuracy and efficiency of the present method.

2. The principle of AMDM

In order to solve vibration problems of Timoshenko beams by the Adomian modified decomposition method the basic theory is stated in brief in this section. Consider the system of second-order differential equations consisting of two equations in two unknown functions $u_1(x)$ and $u_2(x)$.

$$F\mathbf{u}(x) = \mathbf{g}(x), \tag{1}$$

where F represents a general nonlinear ordinary differential operator involving both linear and nonlinear parts, that is $F\mathbf{u}(x)$ can be decomposed into

$$F\mathbf{u}(x) = L\mathbf{u}(x) + R\mathbf{u}(x) + N\mathbf{u}(x), \tag{2}$$

where $L\mathbf{u}(x) + R\mathbf{u}(x)$ are the linear terms in $F\mathbf{u}(x)$, L is an invertible operator, which is taken as the highest-order derivative, that is $L = \mathrm{d}^2/\mathrm{d}x^2$, and R is the remainder of the linear operator, and $N\mathbf{u}(x)$ represents the nonlinear terms in $F\mathbf{u}(x)$. Thus $R\mathbf{u}(x)$ can be decomposed into

$$R\mathbf{u}(x) = \mathbf{P}(x)\mathbf{u}''(x) + \mathbf{B}(x)\mathbf{u}'(x) + \mathbf{D}(x)\mathbf{u}(x), \tag{3}$$

where the 2×2 coefficient matrices $\mathbf{P}(x)$, $\mathbf{B}(x)$, and $\mathbf{D}(x)$ are the functions of x only, and the vector $\mathbf{u}(x)$ of two unknown functions and the vector $\mathbf{g}(x)$ of given functions are defined as

$$\mathbf{u}(x) = \begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix}, \quad \mathbf{u}'(x) = \begin{bmatrix} u_1'(x) \\ u_2'(x) \end{bmatrix}, \quad \mathbf{u}''(x) = \begin{bmatrix} u_1''(x) \\ u_2''(x) \end{bmatrix}, \tag{4}$$

and

$$\mathbf{g}(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \end{bmatrix}. \tag{5}$$

Thus, Eq. (1) can be written as

$$L\mathbf{u}(x) + \mathbf{P}(x)\mathbf{u}''(x) + \mathbf{B}(x)\mathbf{u}'(x) + \mathbf{D}(x)\mathbf{u}(x) + N\mathbf{u}(x) = \mathbf{g}(x). \tag{6}$$

Eq. (6) corresponds to an initial value problem or a boundary-value problem. Solving for $L\mathbf{u}(x)$, one can obtain

$$\mathbf{u}(x) = \mathbf{\Phi}(x) + L^{-1}\mathbf{g}(x) - L^{-1}[\mathbf{P}(x)\mathbf{u}''(x)] - L^{-1}[\mathbf{B}(x)\mathbf{u}'(x)] - L^{-1}[\mathbf{D}(x)\mathbf{u}(x)] - L^{-1}[N\mathbf{u}(x)], \tag{7}$$

where $\mathbf{\Phi}(x) = \mathbf{u}(0) + \mathbf{u}'(0)x$ is determined by the initial conditions of the system and the operator L^{-1} may be regarded as a twice definite integration from 0 to x and defined as $L^{-1} = \int_0^x \int_0^x \cdots \mathrm{d}x\,\mathrm{d}x$. In order to solve the system (7) by the AMDM, one can decompose $\mathbf{u}(x)$ into the infinite sum of convergent series

$$\mathbf{u}(x) = \begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{\infty} c_{1,k}x^k \\ \sum_{k=0}^{\infty} c_{2,k}x^k \end{bmatrix} = \sum_{k=0}^{\infty} \begin{bmatrix} c_{1,k} \\ c_{2,k} \end{bmatrix} x^k = \sum_{k=0}^{\infty} \mathbf{c}_k x^k, \tag{8}$$

where the coefficient vectors $\mathbf{c}_k$ are expressed as

$$\mathbf{c}_k = \begin{bmatrix} c_{1,k} \\ c_{2,k} \end{bmatrix} \tag{9}$$

and the given vector $\mathbf{g}(x)$ and the coefficient matrices $\mathbf{P}(x)$, $\mathbf{B}(x)$, and $\mathbf{D}(x)$ can be also decomposed as

$$\mathbf{g}(x) = \sum_{k=0}^{\infty} \mathbf{g}_k x^k; \quad \mathbf{P}(x) = \sum_{k=0}^{\infty} \mathbf{P}_k x^k; \quad \mathbf{B}(x) = \sum_{k=0}^{\infty} \mathbf{B}_k x^k; \quad \mathbf{D}(x) = \sum_{k=0}^{\infty} \mathbf{D}_k x^k, \tag{10}$$

where the vector $\mathbf{g}_k$ and the three matrices $\mathbf{P}_k$, $\mathbf{B}_k$, and $\mathbf{D}_k$ are constants. By using the theorem of Cauchy product, one can decompose the three terms $\mathbf{P}(x)\mathbf{u}''(x)$, $\mathbf{B}(x)\mathbf{u}'(x)$, and $\mathbf{D}(x)\mathbf{u}(x)$ in Eq. (7) into the following expressions:

$$\mathbf{P}(x)\mathbf{u}''(x) = \sum_{k=0}^{\infty} \mathbf{P}_k x^k \sum_{k=0}^{\infty} (k+2)(k+1)\mathbf{c}_{k+2} x^k = \sum_{k=0}^{\infty} x^k \sum_{m=0}^{k} (m+2)(m+1)\mathbf{P}_{k-m}\mathbf{c}_{m+2} = \sum_{k=0}^{\infty} \overline{\mathbf{p}}_k x^k, \tag{11}$$

$$\mathbf{B}(x)\mathbf{u}'(x) = \sum_{k=0}^{\infty} \mathbf{B}_k x^k \sum_{k=0}^{\infty} (k+1)\mathbf{c}_{k+1} x^k = \sum_{k=0}^{\infty} x^k \sum_{m=0}^{k} (m+1)\mathbf{B}_{k-m}\mathbf{c}_{m+1} = \sum_{k=0}^{\infty} \overline{\mathbf{b}}_k x^k, \tag{12}$$

$$\mathbf{D}(x)\mathbf{u}(x) = \sum_{k=0}^{\infty} \mathbf{D}_k x^k \sum_{k=0}^{\infty} \mathbf{c}_k x^k = \sum_{k=0}^{\infty} x^k \sum_{m=0}^{k} \mathbf{D}_{k-m}\mathbf{c}_m = \sum_{k=0}^{\infty} \overline{\mathbf{d}}_k x^k, \tag{13}$$

where the three vectors $\overline{\mathbf{p}}_k$, $\overline{\mathbf{b}}_k$ and $\overline{\mathbf{d}}_k$ are defined as

$$\overline{\mathbf{p}}_k = \begin{bmatrix} \overline{n}_{1,k} \\ \overline{n}_{2,k} \end{bmatrix} = \sum_{m=0}^{k} (m+2)(m+1)\mathbf{P}_{k-m}\mathbf{c}_{m+2};$$

$$\overline{\mathbf{b}}_k = \begin{bmatrix} \overline{b}_{1,k} \\ \overline{b}_{2,k} \end{bmatrix} = \sum_{m=0}^{k} (m+1)\mathbf{B}_{k-m}\mathbf{c}_{m+1}; \quad \overline{\mathbf{d}}_k = \begin{bmatrix} \overline{d}_{1,k} \\ \overline{d}_{2,k} \end{bmatrix} = \sum_{m=0}^{k} \mathbf{D}_{k-m}\mathbf{c}_m, \tag{14}$$

and the nonlinear term $N\mathbf{u}(x)$ is decomposed as

$$N\mathbf{u}(x) = \sum_{k=0}^{\infty} x^k \mathbf{A}_k(\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_k), \tag{15}$$

where the vector $\mathbf{A}_k(\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_k)$ can be defined as

$$\mathbf{A}_k(\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_k) = \begin{bmatrix} A_{1,k} \\ A_{2,k} \end{bmatrix} = \begin{bmatrix} A_{1,k}(c_{1,0}, c_{1,1}, \ldots, c_{1,k}; c_{2,0}, c_{2,1}, \ldots, c_{2,k}) \\ A_{2,k}(c_{1,0}, c_{1,1}, \ldots, c_{1,k}; c_{2,0}, c_{2,1}, \ldots, c_{2,k}) \end{bmatrix}. \tag{16}$$

The coefficients $A_{1,k}$ and $A_{2,k}$ are known as Adomian polynomials [15–17]. Substitute Eqs. (11)–(15) into the Eq. (7), one can have

$$\begin{aligned} \mathbf{u}(x) = \sum_{k=0}^{\infty} \mathbf{c}_k x^k &= \mathbf{\Phi}(x) + L^{-1}\left(\sum_{k=0}^{\infty} \mathbf{g}_k x^k\right) - L^{-1}\left(\sum_{k=0}^{\infty} \overline{\mathbf{p}}_k x^k\right) - L^{-1}\left(\sum_{k=0}^{\infty} \overline{\mathbf{b}}_k x^k\right) \\ &\quad - L^{-1}\left(\sum_{k=0}^{\infty} \overline{\mathbf{d}}_k x^k\right) - L^{-1}\left(\sum_{k=0}^{\infty} x^k \mathbf{A}_k(\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_k)\right) \\ &= \mathbf{u}(0) + \mathbf{u}'(0)x + \sum_{k=0}^{\infty} \frac{\mathbf{g}_k - \overline{\mathbf{p}}_k - \overline{\mathbf{b}}_k - \overline{\mathbf{d}}_k - \mathbf{A}_k(\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_k)}{(k+1)(k+2)} x^{k+2}. \end{aligned} \tag{17}$$

By collecting the coefficients of like powers of x, the following recurrence relations for $\mathbf{c}_k$ can be obtained:

$$\mathbf{c}_0 = \mathbf{u}(0); \quad \mathbf{c}_1 = \mathbf{u}'(0), \tag{18}$$

$$\mathbf{c}_k = \frac{\mathbf{g}_{k-2} - \overline{\mathbf{p}}_{k-2} - \overline{\mathbf{b}}_{k-2} - \overline{\mathbf{d}}_{k-2} - \mathbf{A}_{k-2}(\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_{k-2})}{k(k-1)}, \quad k = 2, 3, 4, \ldots, \tag{19}$$

where $\mathbf{g}_{k-2}$, $\overline{\mathbf{p}}_{k-2}$, $\overline{\mathbf{b}}_{k-2}$, $\overline{\mathbf{d}}_{k-2}$, and $\mathbf{A}_{k-2}(\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_{k-2})$ can be determined by the Eqs. (10), (14), and (16). The coefficient vectors $\mathbf{c}_k$ ($k \geqslant 2$) of each term in the series (8) can be decided by the recurrence relation (19), and the power series solutions (8) of the system (6) of differential equations in the initial/boundary value problems yield simple recurrence relations for the coefficient vectors $\mathbf{c}_k$. Following Refs. [21–24], the power series solutions (8) converge to $\mathbf{u}(x)$. However, in practice all the coefficient vectors $\mathbf{c}_k$ in the series (8) cannot be determined exactly, and the solutions can only be approximated by a truncated series $\sum_{k=0}^{n-1} \mathbf{c}_k x^k$, where n is the approximate term of the power series solutions.

3. Using the AMDM to analyze the free vibration of uniform Timoshenko beams

Consider a uniform elastic Timoshenko beam of finite length l as shown in Fig. 1, the beam is made of homogeneous and isotropic materials and is constrained with the rotational and translational flexible ends, and with a concentrated mass at the right end, with account taken of the rotatory inertia of the mass, and its eccentricity. The two coupled equations of motion for transverse vibrations of the uniform Timoshenko beam are given by [25–27]

$$\rho A \frac{\partial^2 y}{\partial t^2} - \frac{\partial}{\partial x}\left[\kappa G A \left(\frac{\partial y}{\partial x} - \psi\right)\right] = 0, \tag{20}$$

$$\rho I \frac{\partial^2 \psi}{\partial t^2} - \kappa G A \left(\frac{\partial y}{\partial x} - \psi\right) - \frac{\partial}{\partial x}\left(EI \frac{\partial \psi}{\partial x}\right) = 0, \tag{21}$$

where $y = y(x, t)$ and $\psi = \psi(x, t)$ are the total transverse deflection of the beam and the angle of rotation of the cross-section due to bending of the beam at position x and time t, respectively. E is the Young's modulus of the beam material, G the shear modulus of the beam material, κ the shear correction factor of the beam, A the cross-sectional area of the beam, I the area moment of inertia of the beam, ρ the mass density of the beam material (mass per unit volume). The boundary conditions are given by

$$\kappa G A \left[\frac{\partial y(x,t)}{\partial x} - \psi(x,t)\right] - k_{TL} y(x,t) = 0, \tag{22}$$

$$EI \frac{\partial \psi(x,t)}{\partial x} - k_{RL} \psi(x,t) = 0, \tag{23}$$

at $x = 0$, and

$$\kappa G A \left[\frac{\partial y(x,t)}{\partial x} - \psi(x,t)\right] + k_{TR} y(x,t) + M \frac{\partial^2 y(x,t)}{\partial t^2} + Me \frac{\partial^2 \psi(x,t)}{\partial t^2} = 0, \tag{24}$$

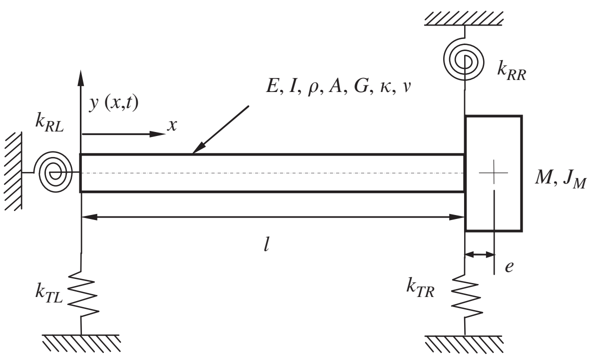

Fig. 1. An uniform Timoshenko beam with elastically restrained ends ($k_{TL}, k_{RL}, k_{TR}, k_{RR}$) and with a tip mass ($M, J_M, e$) at the right end.

$$EI\frac{\partial\psi(x,t)}{\partial x}+k_{RR}\psi(x,t)+(J_M+Me^2)\frac{\partial^2\psi(x,t)}{\partial t^2}+Me\frac{\partial^2 y(x,t)}{\partial t^2}=0, \tag{25}$$

at $x=l$, where k_{TL}, k_{RL} and k_{TR}, k_{RR} are the translational spring constants and the rotational spring constants at the left end and right end of the beam, respectively, and M, J_M, and e are the concentrated mass attached at beam tip, the moment of inertia of the tip mass, the eccentricity which is the distance between the beam tip and the center of the tip mass at the right end of the beam, respectively.

For time harmonic vibration with angular frequency ω, the two coupled governing equations of motion (20) and (21) are given by

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\kappa GA\left(\frac{\mathrm{d}Y(x)}{\mathrm{d}x}-\Psi(x)\right)\right]+\rho A\omega^2 Y(x)=0, \tag{26}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[EI\frac{\mathrm{d}\Psi(x)}{\mathrm{d}x}\right]+\kappa GA\left[\frac{\mathrm{d}Y(x)}{\mathrm{d}x}-\Psi(x)\right]+\rho I\omega^2\Psi(x)=0, \tag{27}$$

where $Y(x)$ is the modal transverse deflection and $\Psi(x)$ is the modal angle of rotation due to bending. The boundary conditions (22)–(25) can be written by

$$\left[\kappa GA\frac{\mathrm{d}Y(x)}{\mathrm{d}x}-k_{TL}Y(x)-\kappa GA\Psi(x)\right]\Bigg|_{x=0}=0, \tag{28}$$

$$\left[EI\frac{\mathrm{d}\Psi(x)}{\mathrm{d}x}-k_{RL}\Psi(x)\right]\Bigg|_{x=0}=0, \tag{29}$$

and

$$\left[\kappa GA\frac{\mathrm{d}Y(x)}{\mathrm{d}x}+(k_{TR}-M\omega^2)Y(x)-(\kappa GA+Me\omega^2)\Psi(x)\right]\Bigg|_{x=l}=0, \tag{30}$$

$$\left[EI\frac{\mathrm{d}\Psi(x)}{\mathrm{d}x}-Me\omega^2Y(x)+[k_{RR}-(J_M+Me^2)\omega^2]\Psi(x)\right]\Bigg|_{x=l}=0. \tag{31}$$

Without loss of generality, introduce the following dimensionless quantities:

$$X=\frac{x}{l};\quad Y(X)=\frac{Y(x)}{l};\quad \Psi(X)=\Psi(x)$$

$$\Omega^2=\frac{\rho A\omega^2 l^4}{EI};\quad \eta=\frac{I}{Al^2};\quad \xi=\frac{\kappa GAl^2}{EI}=\frac{\kappa}{2\eta(1+v)}$$

$$K_{TL}=\frac{k_{TL}l^3}{EI};\quad K_{TR}=\frac{k_{TR}l^3}{EI};\quad K_{RL}=\frac{k_{RL}l}{EI};\quad K_{RR}=\frac{k_{RR}l}{EI}$$

$$\mu=\frac{M}{M_B}=\frac{M}{\rho Al};\quad \delta=\frac{e}{l};\quad \gamma=\sqrt{\frac{J_M}{Ml^2}} \tag{32}$$

then the Eqs. (26) and (27) can be rewritten in the dimensionless form as follows:

$$\frac{\mathrm{d}}{\mathrm{d}X}\left[\xi\left(\frac{\mathrm{d}Y(X)}{\mathrm{d}X}-\Psi(X)\right)\right]+\Omega^2 Y(X)=0, \tag{33}$$

$$\xi\left[\frac{\mathrm{d}Y(X)}{\mathrm{d}X}-\Psi(X)\right]+\frac{\mathrm{d}}{\mathrm{d}X}\left[\frac{\mathrm{d}\Psi(X)}{\mathrm{d}X}\right]+\eta\Omega^2\Psi(X)=0, \tag{34}$$

and the boundary conditions (28)–(31) can be also rewritten in the dimensionless form as follows:

$$\left[\xi\frac{\mathrm{d}Y(x)}{\mathrm{d}x}-K_{TL}Y(X)-\xi\Psi(X)\right]\bigg|_{X=0}=0, \tag{35}$$

$$\left[\frac{\mathrm{d}\Psi(X)}{\mathrm{d}X}-K_{RL}\Psi(X)\right]\bigg|_{X=0}=0, \tag{36}$$

and

$$\left[\xi\frac{\mathrm{d}Y(X)}{\mathrm{d}X}+(K_{TR}-\mu\Omega^2)Y(X)-[\xi+\delta\mu\Omega^2]\Psi(X)\right]\bigg|_{X=1}=0, \tag{37}$$

$$\left\{\frac{\mathrm{d}\Psi(X)}{\mathrm{d}X}-\delta\mu\Omega^2Y(X)+[K_{RR}-\mu(\gamma^2+\delta^2)\Omega^2]\Psi(X)\right\}\bigg|_{X=1}=0. \tag{38}$$

Eqs. (33) and (34) can be written in the matrix form as

$$\begin{bmatrix}\xi & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}Y''(X)\\ \Psi''(X)\end{bmatrix}+\begin{bmatrix}0 & -\xi\\ \xi & 0\end{bmatrix}\begin{bmatrix}Y'(X)\\ \Psi'(X)\end{bmatrix}+\begin{bmatrix}\Omega^2 & 0\\ 0 & -\xi+\eta\Omega^2\end{bmatrix}\begin{bmatrix}Y(X)\\ \Psi(X)\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}. \tag{39}$$

Here primes denote differentiation with respect to X, and furthermore, Eq. (39) can be reduced in the following expression which is similar to Eq. (6):

$$\mathbf{u}''(X)+\mathbf{B}\mathbf{u}'(X)+\mathbf{D}\mathbf{u}(X)=\mathbf{g}(X)=\mathbf{0}, \tag{40}$$

where

$$\mathbf{u}(X)=\begin{bmatrix}Y(X)\\ \Psi(X)\end{bmatrix},\quad \mathbf{u}'(X)=\begin{bmatrix}Y'(X)\\ \Psi'(X)\end{bmatrix},\quad \mathbf{u}''(X)=\begin{bmatrix}Y''(X)\\ \Psi''(X)\end{bmatrix}, \tag{41}$$

and

$$\mathbf{B}=\begin{bmatrix}0 & -1\\ \xi & 0\end{bmatrix},\quad \mathbf{D}=\begin{bmatrix}\dfrac{\Omega^2}{\xi} & 0\\ 0 & -\xi+\eta\Omega^2\end{bmatrix}. \tag{42}$$

From the previous mentions in Eqs. (8), (18), and (19), one can get the dimensionless modal transverse deflection $Y(X)$ and dimensionless modal angle of rotation $\Psi(X)$ by the AMDM. The power series solutions of Eq. (39) can be found as follows:

$$\mathbf{u}(X)=\begin{bmatrix}Y(X)\\ \Psi(X)\end{bmatrix}=\begin{bmatrix}\sum_{k=0}^{\infty}c_{1,k}X^k\\ \sum_{k=0}^{\infty}c_{2,k}X^k\end{bmatrix}=\sum_{k=0}^{\infty}\mathbf{c}_kX^k=\mathbf{\Phi}(X)+\sum_{k=2}^{\infty}\mathbf{c}_kX^k, \tag{43}$$

where

$$\mathbf{\Phi}(X)=\mathbf{c}_0+\mathbf{c}_1X=\mathbf{u}(0)+\mathbf{u}'(0)X, \tag{44}$$

$$\mathbf{c}_0=\mathbf{u}(0)=\begin{bmatrix}Y(0)\\ \Psi(0)\end{bmatrix},\quad \mathbf{c}_1=\mathbf{u}'(0)=\begin{bmatrix}Y'(0)\\ \Psi'(0)\end{bmatrix}, \tag{45}$$

and the recurrence relations for $\mathbf{c}_k$ can be obtained

$$\mathbf{c}_k = \frac{-1}{k(k-1)}[(k-1)\mathbf{B}\mathbf{c}_{k-1} + \mathbf{D}\mathbf{c}_{k-2}], \quad k = 2, 3, 4, \ldots . \tag{46}$$

Therefore, we can find the coefficient vectors $\mathbf{c}_k$ from the recurrent equation (46) and finally we can get the solution vector $\mathbf{u}(X) = [Y(X) \;\; \Psi(X)]^{\mathrm{T}}$ from Eq. (43). The series solution, of course, is $\sum_{k=0}^{\infty} \mathbf{c}_k X^k$. However, in practice all the coefficient vectors $\mathbf{c}_k$ in series solution cannot be determined exactly, and the solutions can only be approximated by a truncated series $\sum_{k=0}^{n-1} \mathbf{c}_k X^k$ with n-term approximation, and one can now form successive approximants

$$\boldsymbol{\varphi}^{[n]}(X) = \begin{bmatrix} Y^{[n]}(X) \\ \Psi^{[n]}(X) \end{bmatrix} = \sum_{k=0}^{n-1} \mathbf{c}_k X^k, \tag{47}$$

as n increases and the boundary conditions are also met. Thus

$$\boldsymbol{\varphi}^{[1]}(X) = \mathbf{c}_0, \quad \boldsymbol{\varphi}^{[2]}(X) = \boldsymbol{\varphi}^{[1]}(X) + \mathbf{c}_1 X, \quad \boldsymbol{\varphi}^{[3]}(X) = \boldsymbol{\varphi}^{[2]}(X) + \mathbf{c}_2 X^2, \ldots, \tag{48}$$

serve as approximate solutions with increasing accuracy as $n \to \infty$, and is also obligated to, of course, satisfy the boundary conditions, that is

$$\mathbf{u}(X) = \lim_{n\to\infty} \boldsymbol{\varphi}^{[n]}(X). \tag{49}$$

The boundary conditions (35)–(38) can be also written in matrix form as

$$\begin{bmatrix} \xi & 0 \\ 0 & 1 \end{bmatrix} \mathbf{c}_1 + \begin{bmatrix} -K_{TL} & -\xi \\ 0 & -K_{RL} \end{bmatrix} \mathbf{c}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{50}$$

and

$$\begin{bmatrix} \xi & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Y'(1) \\ \Psi'(1) \end{bmatrix} + \begin{bmatrix} K_{TR} - \mu\Omega^2 & -\xi - \delta\mu\Omega^2 \\ -\delta\mu\Omega^2 & K_{RR} - \mu(\gamma^2 + \delta^2)\Omega^2 \end{bmatrix} \begin{bmatrix} Y(1) \\ \Psi(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{51}$$

Table 1
The relations between $\mathbf{c}_0 = \mathbf{P}_0(\Omega)\mathbf{a}$ and $\mathbf{c}_1 = \mathbf{P}_1(\Omega)\mathbf{a}$ for the four special cases: clamped, pinned, guided, and free ($\mathbf{a} = [a_1 \;\; a_2]^{\mathrm{T}}$, a_1 and a_2 are arbitrary constants).

$X = 0$	Boundary conditions	Relations	$\mathbf{P}_0(\Omega)$, $\mathbf{P}_1(\Omega)$
Clamped $K_{TL} \to \infty$; $K_{RL} \to \infty$	$Y = 0$ $\Psi = 0$	$\mathbf{c}_0 = \mathbf{0}$; $\mathbf{c}_1 = \mathbf{a}$	$\mathbf{P}_0(\Omega) = \mathbf{0}$ $\mathbf{P}_1(\Omega) = \mathbf{I}$
Pinned $K_{TL} \to \infty$; $K_{RL} = 0$	$Y = 0$ $\Psi' = 0$	$\mathbf{c}_0 = \begin{bmatrix} 0 \\ a_1 \end{bmatrix}$; $\mathbf{c}_1 = \begin{bmatrix} a_2 \\ 0 \end{bmatrix}$	$\mathbf{P}_0(\Omega) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ $\mathbf{P}_1(\Omega) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$
Guided $K_{TL} = 0$; $K_{RL} \to \infty$	$Y' = \Psi$ $\Psi = 0$	$\mathbf{c}_0 = \begin{bmatrix} a_1 \\ 0 \end{bmatrix}$; $\mathbf{c}_1 = \begin{bmatrix} 0 \\ a_2 \end{bmatrix}$	$\mathbf{P}_0(\Omega) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\mathbf{P}_1(\Omega) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
Free $K_{TL} = 0$; $K_{RL} = 0$	$Y' = \Psi$ $\Psi' = 0$	$\mathbf{c}_0 = \mathbf{a}$; $\mathbf{c}_1 = \begin{bmatrix} a_2 \\ 0 \end{bmatrix}$	$\mathbf{P}_0(\Omega) = \mathbf{I}$ $\mathbf{P}_1(\Omega) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

The two coefficient vectors $\mathbf{c}_0$ and $\mathbf{c}_1$ in Eq. (45) can be decided by the BCs of Eq. (50) and the relations between $\mathbf{c}_0$ and $\mathbf{c}_1$ are listed in Table 1 for the four special cases: clamped, pinned, guided, and free. In the general cases, assuming the coefficient vector $\mathbf{c}_0 = \mathbf{P}_0(\Omega)\mathbf{a}$ and $\mathbf{P}_0(\Omega) = \mathbf{I}$, where $\mathbf{I}$ is a 2×2 identity matrix and $\mathbf{a} = [a_1 \;\; a_2]^{\mathrm{T}}$, a_1 and a_2 are arbitrary constants. Then another coefficient vector $\mathbf{c}_1$ can be expressed as the function of $\mathbf{a}$, that is, from Eq. (50), by setting

$$\mathbf{c}_1 = \begin{bmatrix} \dfrac{K_{TL}}{\xi} & 1 \\ 0 & K_{RL} \end{bmatrix}; \quad \mathbf{c}_0 = \begin{bmatrix} \dfrac{K_{TL}}{\xi} & 1 \\ 0 & K_{RL} \end{bmatrix}; \quad \mathbf{P}_0(\Omega)\mathbf{a} = \mathbf{P}_1(\Omega)\mathbf{a}, \tag{52}$$

then the initial term $\mathbf{\Phi}(X)$ in Eq. (44) is the function of $\mathbf{a}$ and from recurrence relations (46), the following relationships are given:

$$\mathbf{c}_k = \frac{-1}{k(k-1)}[(k-1)\mathbf{B}\mathbf{c}_{k-1} + \mathbf{D}\mathbf{c}_{k-2}] = \mathbf{P}_k(\Omega)\mathbf{a}, \quad k = 2, 3, 4, \ldots, \tag{53}$$

where

$$\mathbf{P}_0(\Omega) = \mathbf{I}; \quad \mathbf{P}_1(\Omega) = \begin{bmatrix} \dfrac{K_{TL}}{\xi} & 1 \\ 0 & K_{RL} \end{bmatrix}$$

$$\vdots$$

$$\mathbf{P}_k(\Omega) = \frac{-1}{k(k-1)}[(k-1)\mathbf{B}\mathbf{P}_{k-1}(\Omega) + \mathbf{D}\mathbf{P}_{k-2}(\Omega)], \quad k = 2, 3, 4, \ldots . \tag{54}$$

The 2×2 matrices $\mathbf{P}_k(\Omega)$ $(k = 0, 1, 2, \ldots, n-1)$ are the functions of Ω. Hence one can find that the coefficient vectors $\mathbf{c}_k$ $(k = 1, 2, 3, \ldots, n-1)$ are the functions of $\mathbf{a}$ and Ω. In the meantime, the n-term approximation $\boldsymbol{\varphi}^{[n]}(X)$ of the solution vector $\mathbf{u}(X)$ is also the function of $\mathbf{a}$ and Ω, that is

$$\boldsymbol{\varphi}^{[n]}(X) = \sum_{k=0}^{n-1} \mathbf{c}_k X^k = \sum_{k=0}^{n-1} \mathbf{P}_k(\Omega)\mathbf{a}X^k. \tag{55}$$

By substituting $\boldsymbol{\varphi}^{[n]}(X)$ into Eq. (51), one can obtain

$$\mathbf{F}^{[n]}(\Omega)\mathbf{a} = \mathbf{0}, \tag{56}$$

where

$$\mathbf{F}^{[n]}(\Omega) = \sum_{k=0}^{n-2}(k+1)\begin{bmatrix} \xi & 0 \\ 0 & 1 \end{bmatrix}\mathbf{P}_{k+1}(\Omega) + \sum_{k=0}^{n-1}\begin{bmatrix} K_{TR} - \mu\Omega^2 & -\xi - \delta\mu\Omega^2 \\ -\delta\mu\Omega^2 & K_{RR} - \mu(\gamma^2 + \delta^2)\Omega^2 \end{bmatrix}\mathbf{P}_k(\Omega), \tag{57}$$

where $\mathbf{F}^{[n]}(\Omega)$ is the 2×2 matrix which is decided by the approximate term n and dimensionless natural frequency Ω. For nontrivial solution vectors $\mathbf{a}$ in Eq. (56) one can obtain the frequency equation by the Cramer's rule

$$|\mathbf{F}^{[n]}(\Omega)| = 0, \tag{58}$$

where $|\mathbf{F}^{[n]}(\Omega)|$ is the determinant of $\mathbf{F}^{[n]}(\Omega)$. Hence the ith estimated dimensionless natural frequency $\Omega_i^{[n]}$ corresponding to n can be obtained by Eq. (58) and the approximate term n is decided by the following equation:

$$|\Omega_i^{[n]} - \Omega_i^{[n-1]}| \leqslant \varepsilon, \tag{59}$$

where $\Omega_i^{[n-1]}$ is the ith estimated dimensionless natural frequency corresponding to the approximate term $n-1$, and ε is a preset small value. If Eq. (59) is satisfied, then $\Omega_i^{[n]}$ is the ith dimensionless natural frequency. Substituting $\Omega_i^{[n]}$ into Eq. (47) we have

$$\boldsymbol{\varphi}_i^{[n]}(X) = \begin{bmatrix} Y_i^{[n]}(X) \\ \Psi_i^{[n]}(X) \end{bmatrix} = \sum_{k=0}^{n-1} \mathbf{c}_k^{[i]} X^k, \tag{60}$$

where $\mathbf{c}_k^{[i]}$ is $\mathbf{c}_k$ whose Ω is substituted by $\Omega_i^{[n]}$, and $\boldsymbol{\varphi}_i^{[n]}(X)$ is the ith mode shape function corresponding to the ith dimensionless natural frequency $\Omega_i^{[n]}$. By normalizing Eq. (60), the ith normalized mode shape function is defined as

$$\overline{\boldsymbol{\varphi}}_i^{[n]}(X) = \frac{\boldsymbol{\varphi}_i^{[n]}(X)}{\sqrt{\int_0^1 [\boldsymbol{\varphi}_i^{[n]}(X)]^2 \,\mathrm{d}X}} = \begin{bmatrix} Y_i^{[n]}(X) \Big/ \sqrt{\int_0^1 [Y_i^{[n]}(X)]^2 \,\mathrm{d}X} \\ \Psi_i^{[n]}(X) \Big/ \sqrt{\int_0^1 [\Psi_i^{[n]}(X)]^2 \,\mathrm{d}X} \end{bmatrix} = \begin{bmatrix} \overline{Y}_i^{[n]}(X) \\ \overline{\Psi}_i^{[n]}(X) \end{bmatrix}, \tag{61}$$

where $\overline{\boldsymbol{\varphi}}_i^{[n]}(X)$ is the ith normalized mode shape function of the beam corresponding to the ith dimensionless natural frequency $\Omega_i^{[n]}$.

Hence, by using the method of AMDM, we can easily solve the vibration problem of uniform Timoshenko beams with various boundary conditions. The proposed method is very efficient with the aid of symbolic computation.

4. Verifications and examples

In order to demonstrate the feasibility and the efficiency of AMDM in this paper, the four cases are discussed as follows. By using AMDM, one can obtain the natural frequencies and mode shapes of the beam with various boundary conditions at both ends. The computed results are compared with the analytical and numerical results in the literatures.

4.1. A clamped–free beam

In this case, the system properties are given as $\eta = 0.0004$ and $\xi = 625$. The BCs are $Y(0) = 0,\ \Psi(0) = 0$ and $Y'(1) - \Psi(1) = 0,\ \Psi'(1) = 0$, that is $K_{TL} \to \infty,\ K_{RL} \to \infty$ and $K_{TR} = 0,\ K_{RR} = 0$. From Table 1 one can set $\mathbf{c}_0 = \mathbf{0}$, $\mathbf{c}_1 = \mathbf{a}$ and $\mathbf{P}_1(\Omega) = \mathbf{I}$, by substituting them into Eq. (54), then the matrices $\mathbf{P}_k(\Omega)$ can be obtained

$$\mathbf{P}_2(\Omega) = \begin{bmatrix} 0 & 0.5 \\ -312.5 & 0 \end{bmatrix}$$

$$\mathbf{P}_3(\Omega) = \begin{bmatrix} -104.1667 - 2.6667 \times 10^{-4}\,\Omega^2 & 0 \\ 0 & -6.6667 \times 10^{-5}\,\Omega^2 \end{bmatrix}$$

$$\mathbf{P}_4(\Omega) = \begin{bmatrix} 0 & -8.3333 \times 10^{-5} \\ -2.4253 \times 10^{-12} + 5.2083 \times 10^{-2}\,\Omega^2 & 0 \end{bmatrix}$$

$$\mathbf{P}_5(\Omega) = \begin{bmatrix} -4.8506 \times 10^{-13} + 1.875 \times 10^{-2}\,\Omega^2 + 2.1333 \times 10^{-8}\,\Omega^4 & 0 \\ 0 & 8.3333 \times 10^{-3}\,\Omega^2 + 1.3333 \times 10^{-9}\,\Omega^4 \end{bmatrix}$$

$$\vdots \qquad\qquad \vdots$$

$$\mathbf{P}_k(\Omega) = \frac{-1}{k(k-1)}[(k-1)\mathbf{B}\mathbf{P}_{k-1}(\Omega) + \mathbf{D}\mathbf{P}_{k-2}(\Omega)], \tag{62}$$

Table 2
Convergence results of the ith estimated dimensionless natural frequency $\Omega_i^{[n]}$ for $n = 60$ approximate terms ($\varepsilon = 0.00001$).

n	$\Omega_1^{[n]}$	$\Omega_2^{[n]}$	$\Omega_3^{[n]}$	$\Omega_4^{[n]}$	$\Omega_5^{[n]}$	$\Omega_6^{[n]}$
5	2.13468	57.60496				
6	0.43470	38.92292				
13	3.49974	140.86534				
14	3.49944	116.22340				
17	3.49980	21.48919	36.32149	170.97845		
18	3.49980	21.50294	37.19355	146.19060		
23	3.49980	21.35455				
24	3.49980	21.35465	58.12890	75.58746		
34	3.49980	21.35465	57.47045	106.99807	147.25981	265.02921
35	3.49980	21.35465	57.47046	106.88348		
41	3.49980	21.35465	57.47046	106.92639	166.22669	385.80415
42	3.49980	21.35465	57.47046	106.92637	166.60817	293.72594
50	3.49980	21.35465	57.47046	106.92637	166.65999	233.87972
51	3.49980	21.35465	57.47046	106.92637	166.66000	233.78540
58	3.49980	21.35465	57.47046	106.92637	166.66000	233.84932
59	3.49980	21.35465	57.47046	106.92637	166.66000	233.84932
60	3.49980	21.35465	57.47046	106.92637	166.66000	233.84932

from above, the n-term approximation $\boldsymbol{\varphi}^{[n]}(X)$ in Eq. (55) can be given and substitute it into the BCs at $X = 1$, one can obtain

$$\mathbf{F}^{[n]}(\Omega)\mathbf{a} = \left\{ \sum_{k=1}^{n-1} \begin{bmatrix} k & -1 \\ 0 & k \end{bmatrix} \mathbf{P}_k(\Omega) \right\} \mathbf{a} = \mathbf{0} \tag{63}$$

and the frequency equation (58) becomes

$$\left| \sum_{k=1}^{n-1} \begin{bmatrix} k & -1 \\ 0 & k \end{bmatrix} \mathbf{P}_k(\Omega) \right| = \mathbf{0}. \tag{64}$$

Hence the ith estimated dimensionless natural frequency $\Omega_i^{[n]}$ can be calculated from Eq. (64) by the computational technique and is listed in Table 2 for $n = 60$. From this table, we can obtain any eigenvalue one at a time. The larger the approximate term is, more eigenvalues one can find. From Eq. (59) and Table 2, we have

$$|\Omega_i^{[16]} - \Omega_i^{[15]}| \leqslant \varepsilon = 0.00001. \tag{65}$$

Thus, the first dimensionless natural frequency Ω_1 corresponding to $n = 16$ can be obtained as

$$\Omega_1 = \Omega_1^{[16]} = 3.49980. \tag{66}$$

By substituting $\Omega_1^{[16]}$ into Eq. (55) and normalizing it by Eq. (61), the first mode shape functions is given as

$$\begin{aligned} \bar{Y}_1^{[16]}(X) = {} & 1.53646 \times 10^{-2} X + 3.48967 X^2 - 1.60053 X^3 \\ & - 7.12394 \times 10^{-3} X^4 + 3.52871 \times 10^{-3} X^5 + 1.18737 \times 10^{-1} X^6 \\ & - 2.33403 \times 10^{-2} X^7 - 1.03881 \times 10^{-4} X^8 + 2.22341 \times 10^{-5} X^9 \\ & + 2.88591 \times 10^{-4} X^{10} - 3.61015 \times 10^{-5} X^{11} - 1.60662 \times 10^{-7} X^{12} \\ & + 2.15395 \times 10^{-8} X^{13} + 1.47159 \times 10^{-7} X^{14} - 1.35004 \times 10^{-8} X^{15}, \end{aligned} \tag{67}$$

$$\begin{aligned}\overline{\Psi}_1^{[16]}(X) = {} & 3.26124X - 2.24357X^2 - 2.66305 \times 10^{-3}X^3 \\ & + 4.58010 \times 10^{-3}X^4 + 3.32881 \times 10^{-1}X^5 - 7.63382 \times 10^{-2}X^6 \\ & - 2.32990 \times 10^{-4}X^7 + 6.67868 \times 10^{-5}X^8 + 1.34840 \times 10^{-3}X^9 \\ & - 1.85540 \times 10^{-4}X^{10} - 6.60618 \times 10^{-7}X^{11} + 1.03292 \times 10^{-7}X^{12} \\ & + 9.62567 \times 10^{-7}X^{13} - 9.46108 \times 10^{-8}X^{14} - 3.59283 \times 10^{-10}X^{15}. \end{aligned} \tag{68}$$

By using the given analytical method [1], the first dimensionless natural frequency and mode shape functions can be obtained as

$$\Omega_1 = \Omega_1^{[a]} = 3.4998, \tag{69}$$

$$\begin{aligned}\overline{Y}_1^{[a]}(X) = {} & 0.9971[\cosh(1.8675X) - \cos(1.8740X)] \\ & - 0.73052\ \sinh(1.8675X) + 0.7362\ \sin(1.8740X), \end{aligned} \tag{70}$$

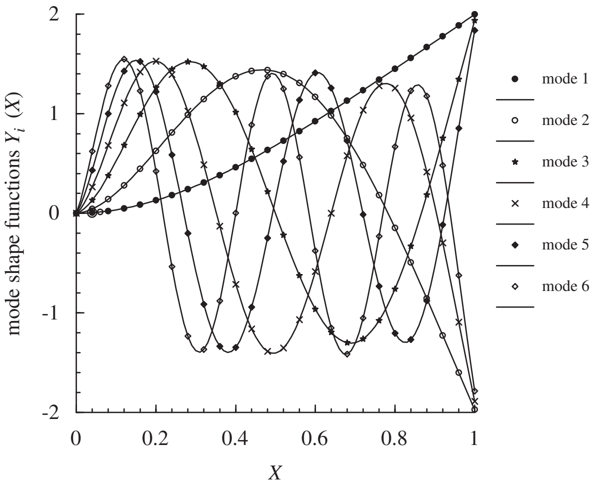

Fig. 2. The first six mode shape functions $Y_1(X) - Y_6(X)$ (—, analytical mode shape function).

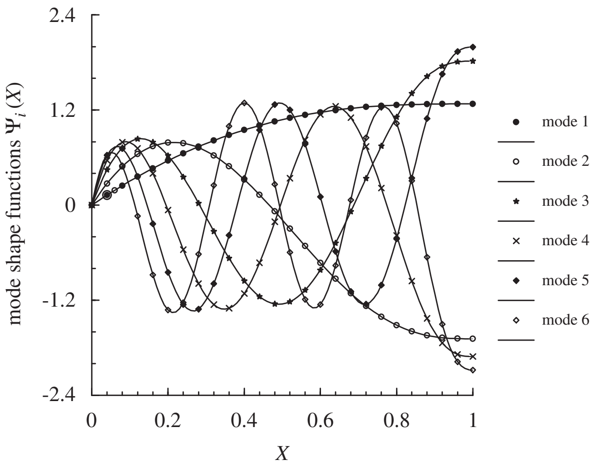

Fig. 3. The first six mode shape functions $\Psi_1(X) - \Psi_6(X)$ (—, analytical mode shape function).

$$\overline{\Psi}_1^{[a]}(X) = -0.6411[\cosh(1.8675X) - \cos(1.8740X)] + 0.8750\sinh(1.8675X) + 0.8683\sin(1.8740X), \quad (71)$$

where $\Omega_1^{[a]}$, $\overline{Y}_1^{[a]}(X)$, and $\overline{\Psi}_1^{[a]}(X)$ are analytical solutions of the first dimensionless natural frequency and mode shape functions, respectively. One can deduce that $\Omega_1 = \Omega_1^{[16]} = \Omega_1^{[a]} = 3.49980$ from Eqs. (66) and (69). Following the same procedure as shown above, the other dimensionless natural frequencies and mode shapes can be obtained. In Table 2, as the approximate term number n increases, the dimensionless natural frequencies $\Omega_1 - \Omega_6$ converge to 3.49980, 21.35465, 57.47046, 106.92637, 166.660, 233.84932, very quickly one by one without missing any frequency. Those complete natural frequencies lead to corresponding mode shapes correctly, which are shown in Figs. 2 and 3.

4.2. A pinned–pinned beam

In this case, the system properties are given as $\eta = 0.01$, $\nu = 0.25$, and $\kappa = 2/3$ (the same parameters as Ref. [12]). The BCs are $Y(0) = 0$, $\Psi'(0) = 0$, and $Y(1) = 0$, $\Psi'(1) = 0$, that is $K_{TL} \to \infty$, $K_{RL} = 0$, $K_{TR} \to \infty$, $K_{RR} = 0$. From Table 1 one can set $\mathbf{c}_0 = \mathbf{P}_0(\Omega)\mathbf{a}$, $\mathbf{c}_1 = \mathbf{P}_1(\Omega)\mathbf{a}$, and substitute them into

Table 3
The first three dimensionless natural frequencies $\Omega_1 - \Omega_3$ of a pinned–pinned beam for $n = 36$ approximate terms ($\eta = 0.01$, $\nu = 0.25$, $\kappa = 2/3$); (I) Karami's results [12], (II) analytical solutions [1].

	Present			(I)	(II)
	$n = 17$	$n = 27$	$n = 36$		
Ω_1	8.21469	8.21469	8.21469	8.21	8.2147
Ω_2	24.95166	24.22810	24.22810	24.23	24.2281
Ω_3	28.72891	41.63873	41.54164	41.54	41.5416

Table 4
The first six dimensionless natural frequencies $\sqrt{\Omega_1} - \sqrt{\Omega_6}$ of a pinned–pinned beam for $n = 65$ approximate terms ($\varepsilon = 0.00001$, $\nu = 0.3$, $\kappa = 5/6$); (I) Lee's results [13].

h/l	Method	$\sqrt{\Omega_1}$	$\sqrt{\Omega_2}$	$\sqrt{\Omega_3}$	$\sqrt{\Omega_4}$	$\sqrt{\Omega_5}$	$\sqrt{\Omega_6}$
0[a]	Present	3.14159	6.28319	9.42478	12.56637	15.70796	18.84956
	(I)	3.14159	6.28319	9.42478	12.5664	15.7080	18.8496
0.002	Present	3.14158	6.28310	9.42449	12.56569	15.70661	18.84842
	(I)	3.14158	6.28310	9.42449	12.5657	15.7066	18.8473
0.005	Present	3.14153	6.28265	9.42298	12.56212	15.69965	18.83532
	(I)	3.14153	6.28265	9.42298	12.5621	15.6997	18.8352
0.01	Present	3.14133	6.28106	9.41761	12.54941	15.67492	18.79264
	(I)	3.14133	6.28106	9.41761	12.5494	15.6749	18.7926
0.02	Present	3.14053	6.27471	9.39631	12.49941	15.57841	18.62823
	(I)	3.14053	6.27471	9.39632	12.4994	15.5784	18.6282
0.05	Present	3.13498	6.23136	9.25537	12.18132	14.99264	17.68103
	(I)	3.13498	6.23136	9.25537	12.1813	14.9926	17.6810
0.1	Present	3.11568	6.09066	8.84052	11.34310	13.61317	15.67904
	(I)	3.11568	6.09066	8.84052	11.3431	13.6132	15.6790
0.2	Present	3.04533	5.67155	7.83952	9.65709	11.22204	12.60221
	(I)	3.04533	5.67155	7.83952	9.65709	11.2220	12.6022

[a]Euler–Bernoulli beam.

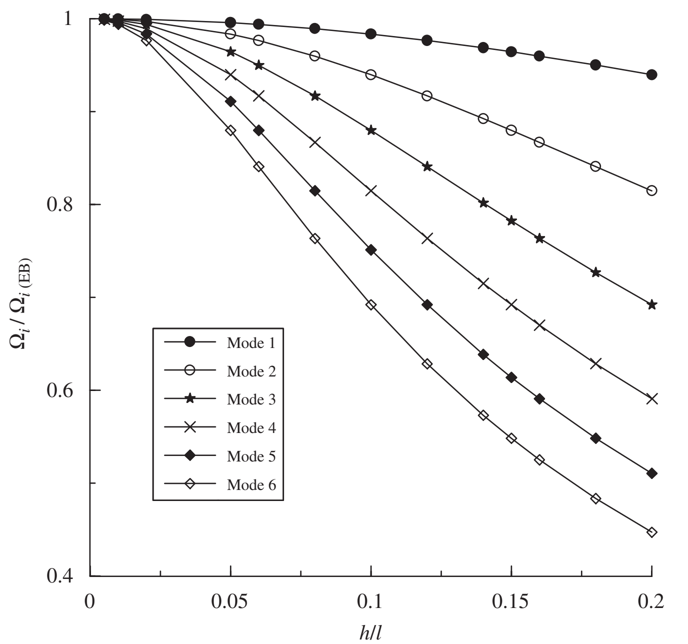

Fig. 4. The first six dimensionless frequency ratio $\Omega_i/\Omega_{i(EB)}$ versus height-to-length h/l ($\Omega_{i(EB)}=$ the ith dimensionless natural frequency of Euler–Bernoulli beam)

Table 5
The first three dimensionless natural frequencies $\Omega_1-\Omega_3$ of a Timoshenko beam with two elastically restrained ends for n approximate terms ($\nu=0.3$, $\kappa=0.85$, $\eta=0.01$, $K_{TL}=K_{TR}=K_{RL}=K_{RR}$, $\varepsilon=0.00001$); (I) Maurizi's results [5].

K_{TL}	n	Ω_1		Ω_2		Ω_3	
		Present	(I)	Present	(I)	Present	(I)
1	13	1.40219	1.40	4.64014	4.64	–	18.92
	17	1.40219	1.40	4.64050	4.64	19.03083	18.92
	26	1.40219	1.40	4.64051	4.64	18.93370	18.92
100	20	10.17325	10.17	18.45166	18.41	30.57089	31.35
	25	10.17325	10.17	18.45097	18.41	31.40052	31.35
	31	10.17325	10.17	18.45097	18.41	31.40181	31.35

Eq. (54), the matrices $\mathbf{P}_k(\Omega)$ can be given as

$$\begin{aligned}
\mathbf{P}_2(\Omega) &= \begin{bmatrix} 0 & 0 \\ 13.3333-0.005\Omega^2 & -13.3333 \end{bmatrix}\\
\mathbf{P}_3(\Omega) &= \begin{bmatrix} 4.4444-1.6667\times10^{-3}\,\Omega^2 & -4.4444-6.25\times10^{-3}\,\Omega^2 \\ 0 & 0 \end{bmatrix}\\
\mathbf{P}_4(\Omega) &= \begin{bmatrix} 0 & 0 \\ -1.1111\times10^{-2}\,\Omega^2+4.1667\times10^{-6}\,\Omega^4 & 5.2778\times10^{-2}\,\Omega^2 \end{bmatrix}.\\
&\vdots \qquad\qquad\qquad \vdots
\end{aligned}\tag{72}$$

Table 6
The first three dimensionless natural frequencies $\Omega_1 - \Omega_3$ of the Timoshenko beam with two elastically restrained ends ($\nu = 0.3$, $\kappa = 0.85$, $\eta = 0.01$, $K_{TL} = K_{TR}$, $K_{RL} = K_{RR}$, $\varepsilon = 0.00001$).

K_{TL}	K_{RL}: 0			10^{-1}			1			10			10^2			∞		
	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3
0	0[a]	0[a]	16.81947[a]	0	1.44289	17.06640	0	4.06816	18.79373	0	7.31416	23.33168	0	8.27422	25.19319	0[b]	8.40466[b]	25.46170[b]
10^{-4}	0.01414	0.02315	16.81949	0.01414	1.44308	17.06642	0.01414	4.06822	18.79374	0.01414	7.31419	23.33169	0.01414	8.27424	25.19320	0.01414	8.40469	25.46171
10^{-3}	0.04472	0.07319	16.81964	0.04472	1.44473	17.06656	0.04472	4.06878	18.79387	0.04472	7.31445	23.33177	0.04472	8.27445	25.19327	0.04472	8.40489	25.46177
10^{-2}	0.14141	0.23145	16.82112	0.14141	1.46119	17.06802	0.14141	4.07431	18.79513	0.14141	7.31701	23.33260	0.14142	8.27653	25.19393	0.14142	8.40691	25.46242
10^{-1}	0.44673	0.73176	16.83596	0.44674	1.61651	17.08256	0.44683	4.12921	18.80773	0.44699	7.34262	23.34086	0.44703	8.29729	25.20060	0.44704	8.42707	25.46886
1	1.39898	2.30933	16.98428	1.39944	2.71515	17.22791	1.40219	4.64050	18.93370	1.40704	7.59299	23.42339	1.40847	8.50132	25.26717	1.40866	8.62540	25.53320
10	4.03294	7.15474	18.45007	4.04497	7.27061	18.66448	4.11880	8.02617	20.17923	4.25498	9.67414	24.23977	4.29676	10.26283	25.92636	4.30239	10.34557	26.17046
10^2	7.40296	18.44190	28.80152	7.49099	18.44218	28.87138	8.09115	18.44411	29.37318	9.57492	18.44897	30.78330	10.17325	18.45097	31.40181	10.26065	18.45126	31.49355
10^3	8.29253	24.68492	42.41438	8.41893	24.73669	42.42545	9.31960	25.11677	42.50446	11.93916	26.29496	42.72766	13.22383	26.88717	42.82986	13.42618	26.98021	42.84542
10^4	8.39333	25.38535	44.21791	8.52450	25.45214	44.25174	9.46373	25.94851	44.49885	12.24686	27.55564	45.24486	13.64335	28.40091	45.60490	13.86531	28.53576	45.66043
∞	8.40466[c]	25.46170[c]	44.40051[c]	8.53638	25.53024	44.43747	9.47999	26.04028	44.70817	12.28190	27.69958	45.53239	13.69133	28.57663	45.93282	13.91558[d]	28.71679[d]	45.99466[d]

[a] Free–free.
[b] Guided–guided.
[c] Pinned–pinned.
[d] Clamped–clamped.

Table 7
The first three dimensionless natural frequencies $\Omega_4 - \Omega_6$ of the Timoshenko beam with two elastically restrained ends ($\nu = 0.3$, $\kappa = 0.85$, $\eta = 0.01$, $K_{TL} = K_{TR}$, $K_{RL} = K_{RR}$, $\varepsilon = 0.00001$).

K_{TL}	K_{RL}																	
	0			10^{-1}			1			10			10^2			∞		
	Ω_4	Ω_5	Ω_6	Ω_4	Ω_5	Ω_6	Ω_4	Ω_5	Ω_6	Ω_4	Ω_5	Ω_6	Ω_4	Ω_5	Ω_6	Ω_4	Ω_5	Ω_6
0	33.96173[a]	51.81109[a]	59.52834[a]	34.20753	52.04943	59.65340	36.04842	53.92774	60.67519	41.69303	60.50755	64.83753	44.07162	63.20991	66.86969	44.40051[b]	63.55317[b]	67.14323[b]
10^{-4}	33.96173	51.81110	59.52834	34.20753	52.04943	59.65340	36.04843	53.92774	60.67519	41.69304	60.50755	64.83753	44.07163	63.20991	66.86969	44.40051	63.55318	67.14323
10^{-3}	33.96178	51.81111	59.52834	34.20758	52.04945	59.65340	36.04847	53.92776	60.67519	41.69308	60.50758	64.83753	44.07167	63.20993	66.86969	44.40055	63.55320	67.14323
10^{-2}	33.96221	51.81127	59.52836	34.20802	52.04961	59.65342	36.04892	53.92795	60.67520	41.69350	60.50785	64.83754	44.07204	63.21020	66.86971	44.40092	63.55347	67.14324
10^{-1}	33.96656	51.81286	59.52853	34.21239	52.05124	59.65358	36.05342	53.92988	60.67532	41.69773	60.51051	64.83762	44.07583	63.21286	66.86985	44.40464	63.55610	67.14340
1	34.01013	51.82882	59.53021	34.25621	52.06758	59.65519	36.09851	53.94918	60.67651	41.74005	60.53713	64.83845	44.11372	63.23941	66.87124	44.44175	63.58244	67.14492
10	34.45430	51.98991	59.54706	34.70249	52.23253	59.67139	36.55481	54.14348	60.68844	42.16253	60.80323	64.84669	44.49063	63.50402	66.88506	44.81088	63.84480	67.16003
10^2	39.24881	53.72316	59.72068	39.48708	54.00051	59.83817	41.23295	56.16961	60.81018	46.12020	63.38747	64.92597	47.94701	66.00907	67.01476	48.18929	66.32131	67.30167
10^3	54.41685	61.36374	63.41088	54.58464	61.42061	63.71984	55.78095	61.95791	66.16527	58.25808	65.44234	74.31955	58.76948	67.72539	76.25848	58.82551	68.06443	76.42448
10^4	56.98091	63.25716	66.78950	57.15304	63.28155	67.07177	58.43904	63.52487	69.35190	61.43154	66.04779	78.12577	61.92519	68.35228	80.34029	61.97161	68.71565	80.49050
∞	57.17719[c]	63.55317[c]	67.14323[c]	57.34894	63.57531	67.42123	58.63963	63.79545	69.67207	61.75076	66.17245	78.51099	62.27041	68.46673	80.82702	62.31837[d]	68.83263[d]	80.98017[d]

[a]Free–free.
[b]Guided–guided.
[c]Pinned–pinned.
[d]Clamped–clamped.

Table 8
The first five dimensionless natural frequencies $\Omega_1 - \Omega_5$ of the cantilever Timoshenko beam with a tip mass at the free end ($\eta = 0.0004$, $\xi = 625$, $\mu = 1.0$, $\gamma^2 = 0.125$, $\delta = 0$, $K_{TL} = K_{RL} \to \infty$, $K_{TR} = K_{RR} = 0$, $\varepsilon = 0.00001$); (I) Posiadala's results [10], (II) Bruch's results [3].

Method	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
Present	1.39820	5.72942	23.63976	58.40659	106.53806
(I)	1.40	5.73	23.64	58.41	106.54
(II)	1.40	5.73	23.64	58.41	106.54

Table 9
The square root of the dimensionless fundamental natural frequency $\sqrt{\Omega_1}$ of the cantilever Timoshenko beam elastically restrained and carrying a tip mass at the free end ($v = 0.25$, $\kappa = 4/3$, $\eta = (h/l)^2/12$, $\delta = 0$, $\gamma = 0$, $K_{TL} = K_{RL} \to \infty$, $K_{RR} = 0$, $\varepsilon = 0.00001$); (I) Karami's results [12], (II) Matsuda's results [8].

μ	h/l	$K_{TR} = 1$			$K_{TR} = 10$		
		Present	(I)	(II)	Present	(I)	(II)
1	0.01	1.34084	1.3408	1.3408	1.79884	1.7988	1.7988
	0.1	1.33930	1.3392	1.3389	1.79779	1.7978	1.7976
	0.2	1.33471	1.3350	1.3330	1.79471	1.7946	1.7940
20	0.01	0.66678	0.6668	0.6668	0.89527	0.8953	0.8953
	0.1	0.66619	0.6662	0.6660	0.89501	0.8950	0.8949
	0.2	0.66443	0.6643	0.6636	0.89426	0.8942	0.8939

Following the same procedure as shown in case 4.1, the n-term approximation $\boldsymbol{\varphi}^{[n]}(X)$ in Eq. (55) can be given and substitute it into the BCs at $X = 1$, one can obtain the frequency equation

$$|\mathbf{F}^{[n]}(\Omega)| = \left| \mathbf{P}_0(\Omega) + \sum_{k=1}^{n-1} \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \mathbf{P}_k(\Omega) \right| = 0. \tag{73}$$

Hence the ith dimensionless natural frequency $\Omega_i^{[n]}$ can be given by Eqs. (73) and (59), the first three dimensionless natural frequencies are listed in Table 3 for the approximate terms $n = 36$. From this table, the calculated results compared with the Ref. [12] are in close agreement.

Another example in this case is also shown as follows. The square roots of the first six dimensionless natural frequencies $\sqrt{\Omega_1} - \sqrt{\Omega_6}$ in the pinned-pinned beam are listed in Table 4. The results presented here are for $\eta = (h/l)^2/12$, $v = 0.3$ and $\kappa = 5/6$, where h is the height of a rectangular cross-section beam (the same parameters as Ref. [13]). From this table, the calculated results compared with Ref. [13] are in close agreement. Besides, the first six dimensionless natural frequency ratios $\Omega_1/\Omega_{1(EB)} - \Omega_6/\Omega_{6(EB)}$ versus the height-to-length ratio h/l are shown in Fig. 4 (where Ω_{EB} is the dimensionless natural frequency of Euler–Bernoulli beam). It is evident that as h/l increases, the natural frequency decreases. The effect of shear deformation and rotary inertia is negligible when h/l is very small and the influence of shear deformation and rotary inertia on the natural frequency of the beam is more pronounced for higher modes.

4.3. A beam with two elastically restrained ends

In this case, the system properties are given as $\eta = 0.01$, $v = 0.3$, $\kappa = 0.85$, and $K_{TL} = K_{TR} = K_{RL} = K_{RR}$ (the same parameters as Ref. [5]). From the previous analysis one can set $\mathbf{c}_0 = \mathbf{P}_0(\Omega)\mathbf{a}$, $\mathbf{P}_0(\Omega) = \mathbf{I}$ and substitute them into Eq. (54), the matrices $\mathbf{P}_k(\Omega)$ can be obtained, and then substitute into Eqs. (58) and (59), the dimensionless natural frequencies can be found. The first three dimensionless natural frequencies $\Omega_1 - \Omega_3$ are listed in Table 5. From this table, the calculated results compared with Ref. [5] are also in close agreement. In Tables 6 and 7, the first six dimensionless natural frequencies $\Omega_1 - \Omega_6$ are given for the Timoshenko beam with two elastically restrained ends ($v = 0.3$, $\kappa = 0.85$, $\eta = 0.01$, $K_{TL} = K_{TR}$, $K_{RL} = K_{RR}$, $\varepsilon = 0.00001$), From the two tables one can find that the larger the spring parameters are, the larger the natural frequencies are, and

Table 10
The first five dimensionless natural frequencies $\Omega_1 - \Omega_5$ of the cantilever Timoshenko beam with a tip mass at the free end ($v = 1/3$, $\kappa = 2/3$, $\delta = 0$, $\gamma = 0$, $K_{TL} = K_{RL} \to \infty$, $K_{TR} = K_{RR} = 0$, $\varepsilon = 0.00001$).

η	$\mu = 0$					$\mu = 0.5$					$\mu = 1$				
	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
[a]	3.51602	22.03449	61.69721	120.90192	199.85953	2.01630	16.90142	51.70092	106.05798	180.12328	1.55730	16.25009	50.89584	105.19828	179.23202
0.0001	3.51194	21.85817	60.54259	116.83029	189.47317	2.01477	16.79574	50.86691	102.85019	171.50250	1.55622	16.15071	50.07968	102.02601	170.66864
0.0004	3.49980	21.35465	57.47046	106.92637	166.66000	2.01021	16.49090	48.61094	94.88795	152.13779	1.55301	15.86381	47.86999	94.14585	151.41994
0.0009	3.47990	20.59032	53.32878	95.20149	142.88874	2.00269	16.01956	45.48888	85.20173	131.43706	1.54770	15.41955	44.80754	84.54938	130.82721
0.0016	3.45267	19.64966	48.88913	84.11329	122.63316	1.99230	15.42565	42.04642	75.82404	113.48501	1.54037	14.85866	41.42534	75.25012	112.95912
0.0025	3.41872	18.61356	44.62358	74.48835	106.28420	1.97921	14.75433	38.65321	67.55370	98.88115	1.53110	14.22330	38.08637	67.04405	98.41966
0.0036	3.37873	17.54697	40.74471	66.36232	93.11252	1.96358	14.04513	35.50163	60.51869	87.14094	1.52001	13.55060	34.98095	60.06215	86.73055
0.0049	3.33347	16.49563	37.30974	59.50859	82.27785	1.94562	13.32875	32.66466	54.59354	77.63847	1.50723	12.86964	32.18233	54.18297	77.26968
0.0064	3.28370	15.48834	34.30052	53.65162	72.97121	1.92554	12.62687	30.14985	49.59152	69.84802	1.49290	12.20114	29.69918	49.22354	69.51097
0.0081	3.23022	14.54100	31.66935	48.52978	64.26923	1.90358	11.95346	27.93438	45.33291	63.36722	1.47716	11.55862	27.50994	45.00827	63.00422
0.01	3.17377	13.66065	29.36143	43.91022	55.98285	1.87996	11.31659	25.98440	41.65876	53.80067	1.46019	10.94999	25.58187	41.38465	53.37008

[a]Euler–Bernoulli beam.

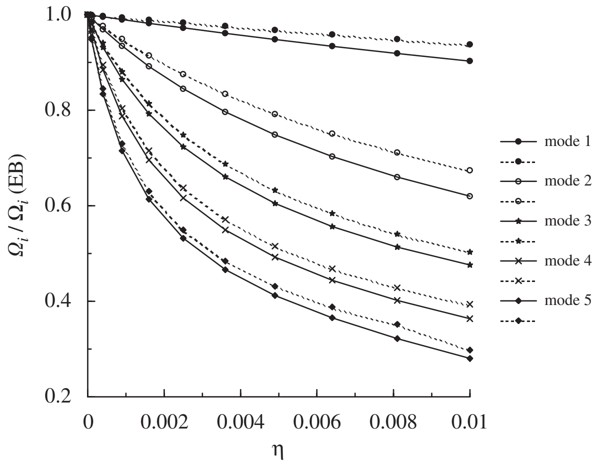

Fig. 5. The first five dimensionless frequency ratio $\Omega_i/\Omega_{i(EB)}$ versus the parameter η for $v = 1/3$, $\kappa = 2/3$, $K_{TL} = K_{RL} \to \infty$, $K_{TR} = K_{RR} = 0$, $\delta = \gamma = 0$ (—, $\mu = 0$; - - - -, $\mu = 1$).

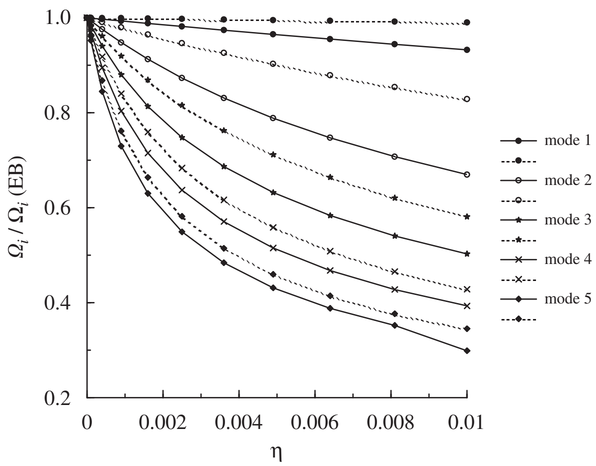

Fig. 6. The first five dimensionless frequency ratio $\Omega_i/\Omega_{i(EB)}$ versus the parameter η for $v = 1/3$, $\kappa = 2/3$, $K_{TL} = K_{RL} \to \infty$, $K_{TR} = K_{RR} = 0$, $\mu = 0.5$, $\delta = 0$ (—, $\gamma = 0$; - - - -, $\gamma = 1$).

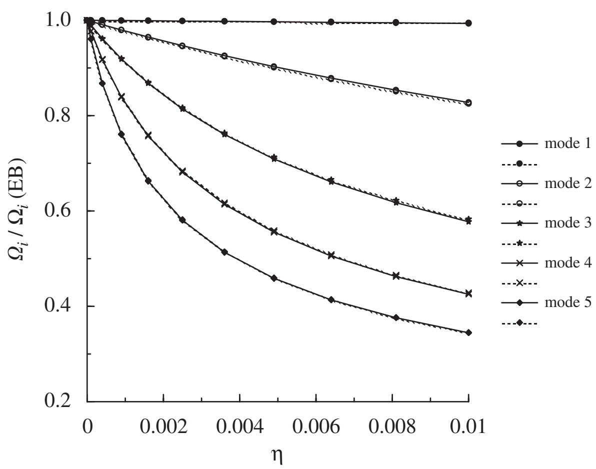

Fig. 7. The first five dimensionless frequency ratio $\Omega_i/\Omega_{i(EB)}$ versus the parameter η for $v = 1/3$, $\kappa = 2/3$, $K_{TL} = K_{RL} \to \infty$,$K_{TR} = K_{RR} = 0$, $\mu = \gamma = 1$ (—, $\delta = 0$; - - - -, $\delta = 1$).

the translational spring parameters K_{TL}, K_{TR} have greater influence on the natural frequencies than the rotational spring parameters K_{RL}, K_{RR}.

4.4. A clamped beam with a tip mass at the right end

In this case, the first five dimensionless natural frequencies $\Omega_1 - \Omega_5$ of the cantilever Timoshenko beam with a tip mass at the free end ($\eta = 0.0004$, $\xi = 625$, $\mu = 1.0$, $\gamma^2 = 0.125$, $\delta = 0$, $K_{TL} = K_{RL} \to \infty$, $K_{TR} = K_{RR} = 0$, $\varepsilon = 0.00001$) are listed in Table 8. From this table, the calculated results compared with Refs. [3,10] are also in close agreement. Table 9 lists the square root $\sqrt{\Omega_1}$ of the dimensionless fundamental natural frequency of the cantilever Timoshenko beam elastically restrained and carrying a tip mass at the free end ($\nu = 0.25$, $\kappa = 4/3$, $\eta = (h/l)^2/12$, $\delta = 0$, $\gamma = 0$, $K_{RR} = 0$, $K_{TL} = K_{RL} \to \infty$, $\varepsilon = 0.00001$), From this table, the calculated results compared with Refs. [8,12] are also in close agreement. In Table 10, the first five dimensionless natural frequencies $\Omega_1 - \Omega_5$ of the cantilever Timoshenko beam with a tip mass at the free end ($\nu = 1/3$, $\kappa = 2/3$, $\delta = 0$, $\gamma = 0$, $K_{TL} = K_{RL} \to \infty$, $K_{TR} = K_{RR} = 0$) are listed. One can find that the dimensionless natural frequencies decrease when the tip mass μ increases and the parameter η has greater influence on the natural frequencies than μ. Finally, In Figs. 5 and 6, It can be observed that the natural frequency ratios decrease when the parameter η increases for fixed μ (or γ), but the natural frequency ratios increase when μ (or γ) increases for fixed η. In Fig. 7, the influence of δ on the natural frequency ratio seems very small.

5. Conclusion

The two coupled governing differential equations with constant coefficients for the free vibrations of uniform Timoshenko beams with a tip mass and elastically ends constraints have been reduced into two recursive algebraic equations. By the method proposed in this study, the closed form series solutions of the free vibrations of uniform beams with various boundary conditions can be obtained. This paper presents an effective method to solve vibration problems of uniform Timoshenko beams with a tip mass and elastically ends constraints. By using the proposed method, any ith natural frequency and mode shape function can be obtained one at a time. The larger the approximate term n is giving, more natural frequency can be found at the same time. The computed results are compared closely with the results obtained by using other analytical and numerical methods. This study provides a unified and systematic procedure which is seemingly simpler and more straightforward than the other methods.

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