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Smooth Karhunen–Loève decomposition to analyze randomly vibrating systems

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ABSTRACT

This paper deals with the application of smooth Karhunen–Loève decomposition (SKLD) procedure applied to random fields. The SKLD is obtained by solving a generalized eigenproblem defined by combining the covariance matrix of the random field with that of the associated time-derivative random field. The main properties of the SKLD procedure thus obtained are described and compared with the classical Karhunen–-Loève decomposition. The SKLD is then applied to the responses of randomly excited vibrating systems in order to perform modal analysis. The resulting SKLD characteristics are discussed, in the case of linear vibrating systems subjected to white noise excitation, in terms of normal modes.

1. Introduction

A new multivariable data-analysis method called the smooth orthogonal decomposition (SOD) method was recently proposed by Chelidze and Zhou [1]. SOD is defined, not appropriately since it is not a proper orthogonal decomposition (POD), as the POD resulting from a maximization problem combined with scalar time series measurements subjected to a minimization constraint involving the associated time derivative of the time series. The definition given is for discrete-time and relies on the definition of a smoothing operator, the constraint used in Ref. [1]. The SOD method can be used to extract normal modes and natural frequencies from multi-degree-of-freedom vibration systems. Free and forced sinusoidal responses have been studied in Ref. [1] and randomly excited systems have been analyzed in Ref. [2]. From a different perspective, it is also discussed in Refs. [3,4].

In this paper, smooth orthogonal decomposition is formulated in terms of a smooth Karhunen–Loève (KL) decomposition (SKLD) for analyzing (continuous-time) random fields. The SKLD is performed by solving a generalized eigenproblem defined in terms of the covariance matrix of the random field and that of its time-derivative. The main properties of the SKLD are described and compared with the classical Karhunen–Loève decomposition. The SKLD is applied to the responses of randomly excited vibrating systems in order to perform modal analysis. The resulting SKLD characteristics are discussed, in the case of vibrating linear systems subjected to white-noise excitation, in terms of the normal modes.

2. Smooth Karhunen–Loève decomposition

Let $\{\mathbf{U}(t), t \in \mathbb{R}\}$ be a $\mathbb{R}^n$-valued random process indexed by $\mathbb{R}$. We assume $\{\mathbf{U}(t), t \in \mathbb{R}\}$ to be a zero-mean second-order stationary process that admits a time derivative process $\{\dot{\mathbf{U}}(t), t \in \mathbb{R}\}$ which is also a second-order stationary process. We

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take $\mathbf{R}_{\mathbf{U}} = \mathbb{E}(\mathbf{U}(t)^T\mathbf{U}(t))$ and $\mathbf{R}_{\dot{\mathbf{U}}} = \mathbb{E}(\dot{\mathbf{U}}(t)^T\dot{\mathbf{U}}(t))$ to denote the covariance matrices of $\{\mathbf{U}(t),\ t \in \mathbb{R}\}$ and $\{\dot{\mathbf{U}}(t), t \in \mathbb{R}\}$, respectively. These matrices, by the hypothesis of stationary process, are time-invariant. We assume they are symmetric positive-definite matrices.

2.1. Decomposition principle

Based on the Karhunen–Loève theory [5–11], $\{\mathbf{U}(t), t \in \mathbb{R}\}$ and $\{\dot{\mathbf{U}}(t), t \in \mathbb{R}\}$ can be decomposed into

$$\mathbf{U}(t,\theta) = \sum_{k=1}^{n} \xi_k(t,\theta)\boldsymbol{\psi}_k \tag{1a}$$

$$\dot{\mathbf{U}}(t,\theta) = \sum_{k=1}^{n} \eta_k(t,\theta)\boldsymbol{\Sigma}_k \tag{1b}$$

where the n-vectors $\boldsymbol{\psi}_k$ (respectively, $\boldsymbol{\Sigma}_k$) solve the eigenproblem

$$\mathbf{R}_{\mathbf{U}}\boldsymbol{\psi}_k = \lambda_k\boldsymbol{\psi}_k \quad (\text{respectively, } \mathbf{R}_{\dot{\mathbf{U}}}\boldsymbol{\Sigma}_k = \mu_k\boldsymbol{\Sigma}_k) \tag{2}$$

and the random processes $\{\xi_k(t), t \in \mathbb{R}\}$ (respectively, $\{\eta_k(t), t \in \mathbb{R}\}$) are given by $\xi_k(t,\theta) = \mathbf{U}(t,\theta)^T\boldsymbol{\psi}_k$ (respectively, $\eta_k(t,\theta) = \mathbf{U}(t,\theta)^T\boldsymbol{\Sigma}_k$).

Note that the variable θ has been used as a reminder that, $\{\mathbf{U}(t), t \in \mathbb{R}\}$ being a random process for fix t, $\mathbf{U}(t)$ is a random variable defined on the space of random events and $\mathbf{U}(t,\theta)$ denotes one realization of the random variable; θ will be omitted from now on.

The main properties of the KLDs (1) are

- $(\boldsymbol{\psi}_k)_k$ and $(\boldsymbol{\Sigma}_k)_k$ are orthogonal basis of $\mathbb{R}^n$,
- $\mathbb{E}(\xi_{k_1}(t)\xi_{k_2}(t)) = \lambda_{k_1}\delta_{k_1,k_2}$ and $\mathbb{E}(\eta_{k_1}(t)\eta_{k_2}(t)) = \nu_{k_1}\delta_{k_1,k_2}$, where δ_{k_1,k_2} is the Kronecker delta,
- the decompositions (Eq. (1)) satisfy the optimality relations

$$\mathbb{E}\left(\left\|\mathbf{U}(t) - \sum_{k=1}^{p}\xi_k(t)\boldsymbol{\psi}_k\right\|^2\right) = \sum_{k=1}^{p}\lambda_k \leqslant \mathbb{E}\left(\left\|\mathbf{U}(t) - \sum_{k=1}^{p}\tilde{\xi}_k(t)\mathbf{B}_k\right\|^2\right) \tag{3}$$

$$\mathbb{E}\left(\left\|\dot{\mathbf{U}}(t) - \sum_{k=1}^{p}\eta_k(t)\boldsymbol{\Sigma}_k\right\|^2\right) = \sum_{k=1}^{p}\mu_k \leqslant \mathbb{E}\left(\left\|\dot{\mathbf{U}}(t) - \sum_{k=1}^{p}\tilde{\eta}_k(t)\mathbf{B}_k\right\|^2\right) \tag{4}$$

for any integer $p \leqslant n$ and any arbitrary orthogonal basis $(\mathbf{B}_k)_k$ of $\mathbb{R}^n$, (see Ref. [10]).

The eigenvalues (λ_k and μ_k) are called the Karhunen–Loève values (KLVs), the eigenvectors ($\boldsymbol{\psi}_k$ and $\boldsymbol{\Sigma}_k$) are called the Karhunen–Loève modes (KLMs), and the scalar random processes ($\{\xi_k(t)\}$ and $\{\eta_k(t)\}$) are called the Karhunen–Loève components (KLCs).

As described in Ref. [12], the KLMs, $\boldsymbol{\psi}_k$, of $\{\mathbf{U}(t), t \in \mathbb{R}\}$ generally differ from the KLMs, $\boldsymbol{\Sigma}_k$, of $\{\dot{\mathbf{U}}(t), t \in \mathbb{R}\}$ and hence the KLD of $\{\dot{\mathbf{U}}(t),\ t \in \mathbb{R}\}$ generally differs from the time derivative of Eq. (1a)

$$\dot{\mathbf{U}}(t) = \sum_{k=1}^{n}\dot{\xi}_k(t)\boldsymbol{\psi}_k \tag{5}$$

A necessary condition for a vector to satisfy the two eigenproblems (2) is that the vector must solve the generalized eigenproblem

$$\mathbf{R}_{\mathbf{U}}\boldsymbol{\Gamma}_k = \nu_k\mathbf{R}_{\dot{\mathbf{U}}}\boldsymbol{\Gamma}_k \tag{6}$$

The generalized eigenproblem (6) is a *statistical* version (for *continuous-time* random process), of the generalized eigenvalue problem introduced in Ref. [1] to characterize the SOD, what constitutes a major difference. In definition (6) only the covariance matrices $\mathbf{R}_{\mathbf{U}}$ and $\mathbf{R}_{\dot{\mathbf{U}}}$ are used, no other operator is necessary. The idea comes from Ref. [12]. The results are, of course, similar to the ones presented in Refs. [1–3], but now, since one relies on the covariance matrices, one has a powerful computation tool, not available before.

The smooth Karhunen–Loève decomposition of the random field will then be given by

$$\mathbf{U}(t) = \sum_{k=1}^{n}\zeta_k(t)\boldsymbol{\Gamma}_k \tag{7}$$

where the vector $\boldsymbol{\Gamma}_k$ solve the generalized eigenproblem (6).

In this new definition, the following notation is used: the eigenvalues ν_k are called the smooth Karhunen-Loève values (SKLVs), the eigenvectors $\mathbf{\Gamma}_k$ are called the smooth Karhunen–Loève modes (SKLMs), and the scalar random processes $\{\zeta_k(t)\}$ are called the smooth Karhunen–Loève components (SKLCs).

In line with the approach proposed in Ref. [1], the SKLD can also be defined from a constrained variational problem in which the SKLMs are described as

$$\max_{\mathbf{\Gamma}\in\mathbb{R}^n} \frac{\mathbb{E}(\mathbf{U}(t)^T\mathbf{\Gamma})^2}{\mathbb{E}(\dot{\mathbf{U}}(t)^T\mathbf{\Gamma})^2} = \max_{\mathbf{\Gamma}\in\mathbb{R}^n} \frac{\mathbf{\Gamma}^T\mathbf{R}_{\mathbf{U}}\mathbf{\Gamma}}{\mathbf{\Gamma}^T\mathbf{R}_{\dot{\mathbf{U}}}\mathbf{\Gamma}} \tag{8}$$

Observe we can maximize in the unit ball, $B(0,1)\in\mathbb{R}^n$, instead of $\mathbb{R}^n$, with the same result, [13]. The objective function used here differs significantly from that used to define the KLD (see Ref. [14]). Here the denominator of the objective function takes the covariance matrix of the time-derivative process $\{\dot{\mathbf{U}}(t), t\in\mathbb{R}\}$ into account (which justifies the name *smooth* KLD). The solutions to Eq. (8) are given by the eigenproblem (6).

2.2. Some properties of the SKLD

2.2.1. SKLV, SKLM, and SKLC properties

Since the matrices $\mathbf{R}_{\mathbf{U}}$ and $\mathbf{R}_{\dot{\mathbf{U}}}$ are symmetric positive-definite matrices, all the SKLVs (eigenvalues) ν_k are strictly positive and the set of vectors $\mathbf{\Gamma}_k$ (the SKLMs) constitutes a basis which is orthogonal with respect to both $\mathbf{R}_{\mathbf{U}}$ and $\mathbf{R}_{\dot{\mathbf{U}}}$. Note that the SKLM are unique to a scaling constant.

The SKLCs of $\{\mathbf{U}(t), t\in\mathbb{R}\}$ are given by

$$\zeta_k(t) = \frac{\mathbf{\Gamma}_k^T\mathbf{R}_{\mathbf{U}}\mathbf{U}(t)}{\mathbf{\Gamma}_k^T\mathbf{R}_{\mathbf{U}}\mathbf{\Gamma}_k} = \frac{\mathbf{\Gamma}_k^T\mathbf{R}_{\dot{\mathbf{U}}}\mathbf{U}(t)}{\mathbf{\Gamma}_k^T\mathbf{R}_{\dot{\mathbf{U}}}\mathbf{\Gamma}_k} \tag{9}$$

Note that the scalar processes $\{\zeta_k(t), t\in\mathbb{R}\}$ can be defined from either $\mathbf{R}_{\mathbf{U}}$ or $\mathbf{R}_{\dot{\mathbf{U}}}$.

As expected, the scalar processes $\{\zeta_k(t),\ t\in\mathbb{R}\}$ are correlated

$$\mathbb{E}(\zeta_k(t)\zeta_l(t)) = \frac{\mathbf{\Gamma}_k^T\mathbf{R}_{\mathbf{U}}\mathbf{R}_{\mathbf{U}}\mathbf{R}_{\mathbf{U}}\mathbf{\Gamma}_l}{\mathbf{\Gamma}_k^T\mathbf{R}_{\mathbf{U}}\mathbf{\Gamma}_k\mathbf{\Gamma}_l^T\mathbf{R}_{\mathbf{U}}\mathbf{\Gamma}_l} = \frac{\mathbf{\Gamma}_k^T\mathbf{R}_{\dot{\mathbf{U}}}\mathbf{R}_{\dot{\mathbf{U}}}\mathbf{R}_{\dot{\mathbf{U}}}\mathbf{\Gamma}_l}{\mathbf{\Gamma}_k^T\mathbf{R}_{\dot{\mathbf{U}}}\mathbf{\Gamma}_k\mathbf{\Gamma}_l^T\mathbf{R}_{\dot{\mathbf{U}}}\mathbf{\Gamma}_l} \tag{10}$$

The SKLVs do not depend on the energy distribution, and the SKLD does not satisfy the optimality relationship (3).

2.2.2. Linear transformation of the SKLD

Let $\{\mathbf{V}(t), t\in\mathbb{R}\}$ be a $\mathbb{R}^n$-valued random process defined as

$$\mathbf{V}(t) = \mathbf{A}\mathbf{U}(t) \tag{11}$$

where $\mathbf{A}$ is square invertible matrix.

From the relationship

$$\mathbf{R}_{\mathbf{V}} = \mathbf{A}\mathbf{R}_{\mathbf{U}}\mathbf{A}^T \quad \text{and} \quad \mathbf{R}_{\dot{\mathbf{V}}} = \mathbf{A}\mathbf{R}_{\dot{\mathbf{U}}}\mathbf{A}^T$$

it can be shown that the SKLVs of $\{\mathbf{V}(t),\ t\in\mathbb{R}\}$ coincide with those of $\{\mathbf{U}(t), t\in\mathbb{R}\}$ and the sets of the SKLMs comply with the relationship

$$\mathbf{\Gamma}_k(V) = \mathbf{A}^{-T}\mathbf{\Gamma}_k(U) \tag{12}$$

where $\mathbf{\Gamma}_k(U)$ (respectively, $\mathbf{\Gamma}_k(V)$) denotes a SKLM of $\{\mathbf{U}(t), t\in\mathbb{R}\}$ (respectively, $\{\mathbf{V}(t), t\in\mathbb{R}\}$). Lastly, in line with (9), the SKLCs are invariant with respect to the linear change of variables if and only if $\mathbf{A}\mathbf{A}^T = \mathbf{I}$.

2.3. SKLD in practice

To draw up the SKLD, the covariance matrices of $\{\mathbf{U}(t), t\in\mathbb{R}\}$ and $\{\dot{\mathbf{U}}(t), t\in\mathbb{R}\}$ are required. These matrices are known as the characteristics of the random processes but they can also be approximated by sampling $\{\mathbf{U}(t),\ t\in\mathbb{R}\}$ and $\{\dot{\mathbf{U}}(t), t\in\mathbb{R}\}$ or by simply sampling one of the processes $\{\mathbf{U}(t), t\in\mathbb{R}\}$ or $\{\dot{\mathbf{U}}(t), t\in\mathbb{R}\}$. The procedure presented in Ref. [2] corresponds to the second case.

3. Relations between SKLD, KLD, and linear normal decomposition

Consider a discrete mechanical system with d degrees of freedom. Let $\mathbf{U}(t)$ be the displacement vector, $\mathbf{U}(t)$ is assumed to obey the equations of motion

$$\mathbf{M}\ddot{\mathbf{U}}(t) + \mathbf{C}\dot{\mathbf{U}}(t) + \mathbf{K}\mathbf{U}(t) = \mathbf{F}(t) \tag{13}$$

where $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are $d \times d$ symmetric square matrices and the excitation vector, $\{\mathbf{F}(t), t \in \mathbb{R}\}$, is a random-vector process.

The linear normal modes (LNM) are classically defined in terms of the free responses of the associated undamped system as follows:

$$\mathbf{K}\boldsymbol{\Phi}_k = \mathbf{M}\boldsymbol{\Phi}_k\boldsymbol{\Omega}_k^2$$

where $\boldsymbol{\Phi} = [\boldsymbol{\Phi}_1 \cdots \boldsymbol{\Phi}_i \cdots \boldsymbol{\Phi}_d]$ denotes the modal matrix with the normalization condition $\boldsymbol{\Phi}^T\mathbf{M}\boldsymbol{\Phi} = \mathbf{I}$, which means that $\boldsymbol{\Phi}^T\mathbf{K}\boldsymbol{\Phi} = \boldsymbol{\Omega}^2 = \mathrm{diag}(\omega_i^2)$; ω_i^2 and $\boldsymbol{\Phi}_i$ denote the squared natural frequencies and the associated normal mode vectors.

Using the normal mode vectors as the basis for representation of displacement field, the modal-displacement vector $\mathbf{Q}(t)$ defined by

$$\mathbf{U}(t) = \boldsymbol{\Phi}\mathbf{Q}(t) = \sum_{i=1}^{d} \boldsymbol{\Phi}_i Q_i(t) \tag{14}$$

satisfies the following second-order differential equation

$$\ddot{\mathbf{Q}}(t) + \boldsymbol{\Theta}\dot{\mathbf{Q}}(t) + \boldsymbol{\Omega}^2\mathbf{Q}(t) = \boldsymbol{\Phi}^T\mathbf{F}(t) \tag{15}$$

where $\boldsymbol{\Theta} = \boldsymbol{\Phi}^T\mathbf{C}\boldsymbol{\Phi}$.

3.1. Theoretical results

Here we focus on the steady-state response of Eq. (13) or (15) when the damping is proportional (i.e., $\boldsymbol{\Phi}^T\mathbf{C}\boldsymbol{\Phi} = \mathrm{diag}(2\tau_i\omega_i)$ is also diagonal) and the excitation is a zero-mean white-noise random excitation (i.e., $\mathbf{R}_F(\tau) = \mathbb{E}(\mathbf{F}(t+\tau)\mathbf{F}^T(t)) = \mathbf{S}_F\delta(\tau)$, where the intensity $\mathbf{S}_F$ is a symmetric constant matrix).

If the matrix $\boldsymbol{\Phi}^T\mathbf{S}_F\boldsymbol{\Phi}$ is diagonal (i.e., if the modal-excitation terms $\boldsymbol{\Phi}_i^T\mathbf{F}(t)$ in Eq. (15) are uncorrelated) then, as established in Ref. [12], the covariance matrices $\mathbf{R}_{\mathbf{Q}}$ and $\mathbf{R}_{\dot{\mathbf{Q}}}$ of the stationary responses $\{\mathbf{Q}(t), t \in \mathbb{R}\}$ and $\{\dot{\mathbf{Q}}(t), t \in \mathbb{R}\}$ of Eq. (15) will be diagonal. Hence the SKLMs associated with the process $\{\mathbf{Q}(t), t \in \mathbb{R}\}$ will be equal to the vector of the canonical basis of $\mathbb{R}^d$ and the SKLVs will be given by the diagonal terms of the diagonal matrix $\mathbf{R}_{\dot{\mathbf{Q}}}^{-1}\mathbf{R}_{\mathbf{Q}} = (\boldsymbol{\Omega}^2)^{-1}$. Now using the linear relation Eq. (14), it can easily be established that the SKLVs of $\{\mathbf{U}(t), t \in \mathbb{R}\}$ coincide with the SKLVs of $\{\mathbf{Q}(t), t \in \mathbb{R}\}$ and that the SKLMs of $\{\mathbf{U}(t), t \in \mathbb{R}\}$ are given by $\boldsymbol{\Phi}^{-T}$. This relationship is determined up to a multiplicative constant.

It is worth noting that, as indicated in Ref. [1], no assumptions about the mass matrix $\mathbf{M}$ are required to relate the LNMs to the SKLMs. However, if $\mathbf{M} = m\mathbf{I}$, where $\mathbf{I}$ denotes the identity matrix, it is clear that the SKLMs coincide with the KLMs, which, of course, coincide with the LNMs.

To summarize, one of the main advantages of the SKLD is that it gives the natural frequencies inverting the SKLVs (a characteristic which cannot be easily obtained from the KLD) and the normal modal vectors inverting the transpose of the SKLM matrix.

3.2. Example

The example used in Ref. [2] to illustrate the efficiency of the SOD for performing modal analysis on randomly excited systems can be analyzed with the tool presented here. All the units are in the SI system, and we do not show them.

The system consists of a finite chain of eight mass points linked together by a series of identical linear springs and dampers. The first mass is attached to a fixed point by a linear spring and a linear damper, and the last mass is free. All the masses are equal to 1 except the first one, which is equal to m ($m>0$), and the stiffness coefficients of the strings are all equal to 1. The corresponding equation of motion has the form (13), where

$$\mathbf{M} = \begin{pmatrix} m & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{K} = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & & 0 & 0 \\ 0 & -1 & 2 & & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 1 \end{pmatrix} \tag{16}$$

The damping matrix is taken to be $\mathbf{C} = 2\tau_1\omega_1\mathbf{M}$ with $\tau_1 > 0$, which ensures that the damping is proportional and sets the damping ratio in the first linear mode.

The system is first assumed to be excited by a standard vector-valued white-noise process with matrix intensity $\mathbf{S}_F = \frac{1}{2}\mathbf{K} + \mathbf{C}$. This choice ensures that, for all values of m, $\boldsymbol{\Phi}^T\mathbf{S}_F\boldsymbol{\Phi}$ will always be diagonal. If $m = 1$, both the KLD and the SKLD procedures give access to the LNMs. However, if $m \neq 1$, the KLMs can differ from the LNMs, whereas the SKLD always give the LNMs and the corresponding frequencies. These results are illustrated in Fig. 1. The modal assurance criterion (MAC) [15] was used to compare the LNMs and the LNM values predicted by the KLD method and the SKLD method.

The MAC between two vectors U and V is defined by

$$MAC(U,V)=\frac{|U^TV|^2}{(U^TU)(V^TV)} \tag{17}$$

With $\tau_1=0.01$ and ten values of the mass m, Fig. 1a,b gives the maximum MAC values obtained in each LNM with

$$\max_{k=1,\ldots,8} MAC(\Phi_i,\Phi_k^{pred}) \tag{18}$$

where Φ_k^{pred} denotes the LNMs predicted using the KLD method (upper script KLM) and the SKLD method (upper script SKLM). The covariance matrices $\mathbf{R_U}$ and $\mathbf{R_{\dot U}}$ of the stationary responses were obtained by solving the associated Lyapounov's equations (see Ref. [10]). The efficiency of the KLD method for approximating the LNM is highly sensitive to the extent to which the mass matrix differs from the identity matrix. The greater this difference is, the more the KLMs differ from the LNMs. A similar pattern occurs at high damping levels.

It is now assumed that the system is excited by a white-noise process applied to the first mass, i.e.

$$\mathbf{F}(t)=(10\cdots0)^Tf(t)=\mathbf{P}f(t) \tag{19}$$

where $\{f(t),t\in\mathbb{R}\}$ is a white-noise process with intensity $S_0(>0)$. Now $\mathbf{S}_F=S_0\mathbf{P}\mathbf{P}^T$ and hence $\mathbf{\Phi}^T\mathbf{S}_F\mathbf{\Phi}=(\Phi_{1i}\Phi_{1j})$ is not a diagonal matrix. Based on the results established in the previous section, the SKLD do not predict the modal parameters accurately.

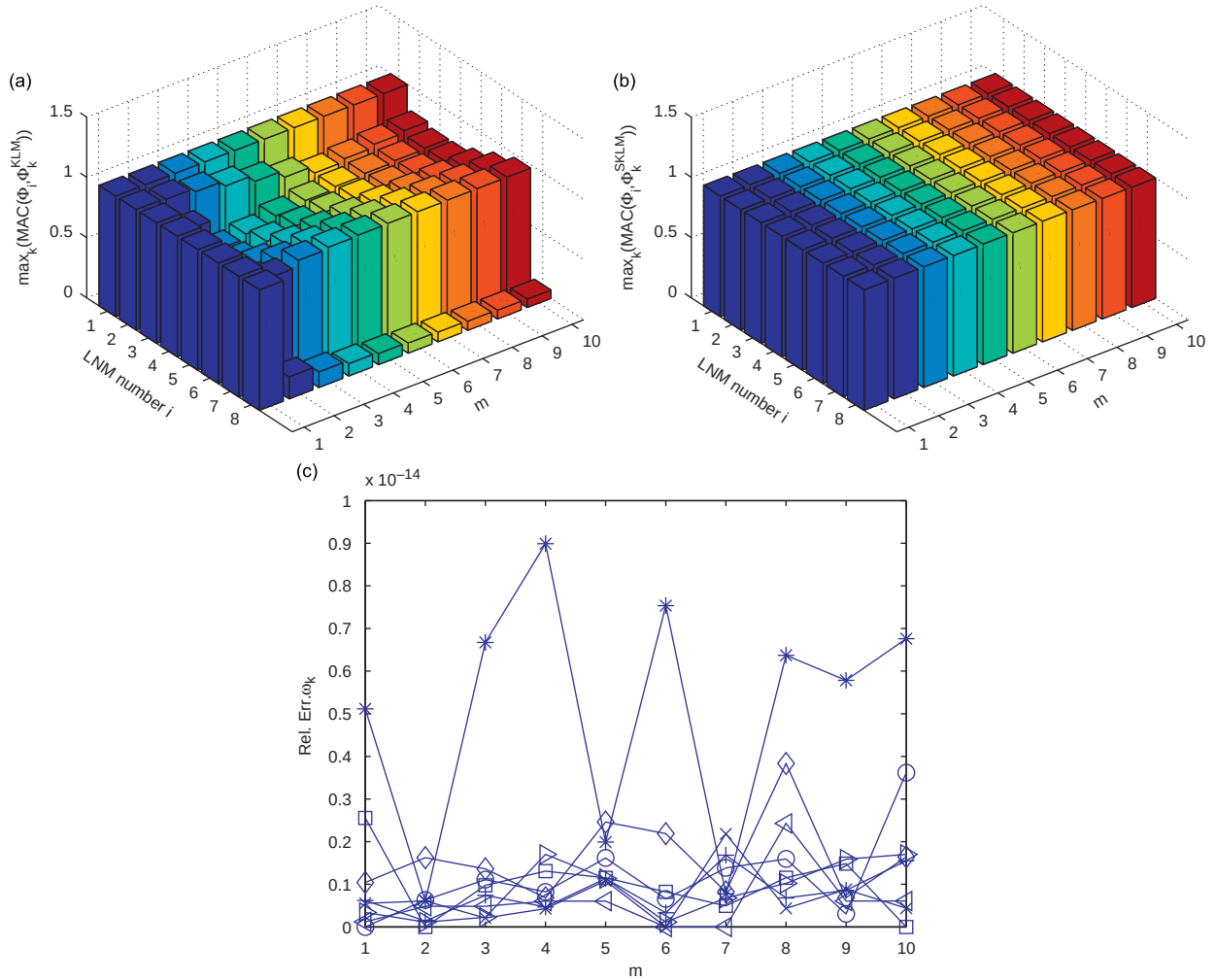

Fig. 1. The maximum MAC values (18) obtained between the LNM and the predicted values (KLD approach (a) and SKLD approach (b)) on the system with $\tau_1=0.01$ and various mass values m excited by a vector valued non-correlated white-noise excitation. The relative error between the natural frequencies and the corresponding values obtained with SKLD (c) (first mode: $*$, second mode: $\circ$, third mode: $+$, fourth mode: $\times$, fifth mode: $\square$, sixth mode: $\diamond$, seventh mode: $\triangleleft$, eighth mode: $\triangleright$).

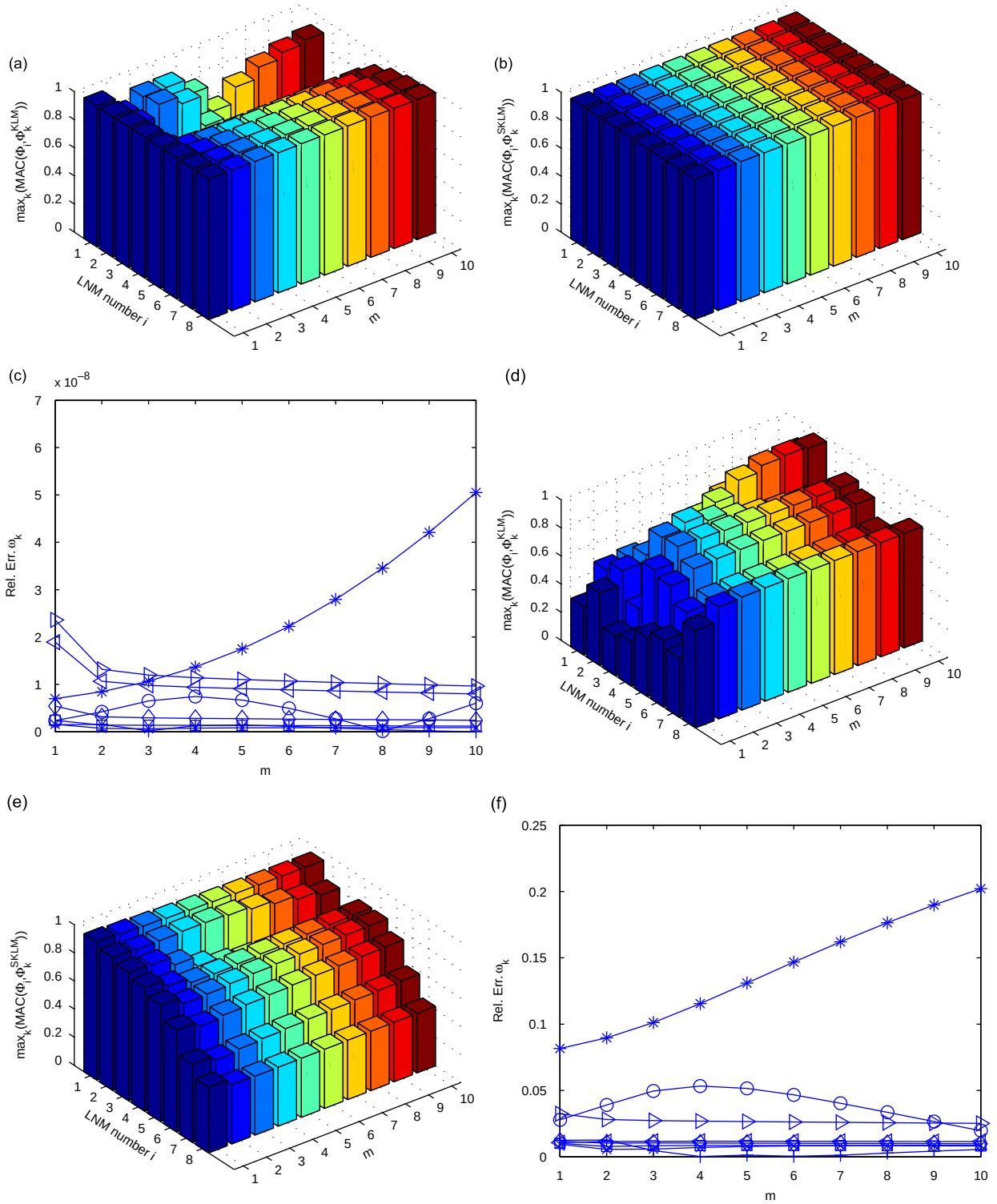

Fig. 2. The maximum MAC values (18) obtained between the LNM and the predicted values one (KLD approach (a) and (d), SKLD approach (b) and (e)) on the system with $\tau_1 = 0.01$ (a–c) and $\tau_1 = 0.8$ (d–f) and various mass values m excited by a white-noise process applied to the first mass. The relative error between the natural frequencies and the corresponding values predicted by the SKLD (c, f) (first mode: $*$, second mode: $\circ$, third mode: $+$, fourth mode: $\times$, fifth mode: $\square$, sixth mode: $\diamond$, seventh mode: $\triangleleft$, eighth mode: $\triangleright$).

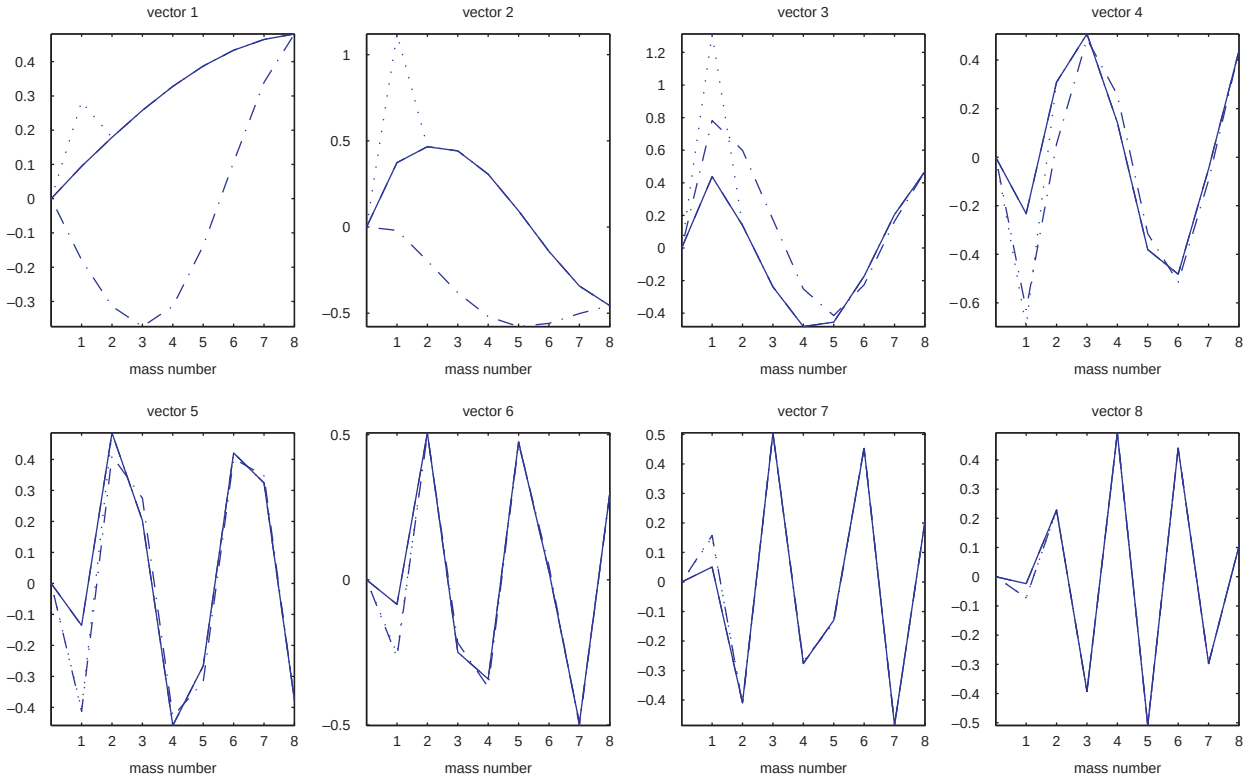

Fig. 3. The LNMs (solid line), the KLMs (dash dot line), the SKLMs (dotted line) and the modes approximated from the SKLMs (dashed line) in the system with $\tau_1 = 0.01$, and $m = 3$, excited by a white-noise process applied to the first mass.

With $\tau_1 = 0.01$ and for ten values of the mass m, the maximum MAC (18) values are shown in Fig. 2a,b. It is clear that the fact that $\mathbf{\Phi}^T\mathbf{S}_F\mathbf{\Phi}$ is not a diagonal matrix does not affect the quality of the LNM predictions obtained with the SKLM. However, at higher damping levels, ($\tau_1 = 0.8$), the SKLD shows a loss of efficiency (Fig. 2d,e).

For modal analysis purpose, the SKLD can also be used to assess the natural frequencies. The efficiency of the procedure is also illustrated in Fig. 2, where the relative errors between the natural frequencies and their predicted values are plotted (Fig. 2c,f). At small damping level, the relative error is numerically acceptable ($\approx 10^{-8}$) in comparison with the ideal case ($\approx 10^{-14}$ see Fig. 1c). At large damping level, the error increases reaching 20% in the first mode (see Fig. 1f). This theoretical finding is in agreement with the numerical study presented in Ref. [2].

Lastly, it is worth comparing in the case of this example all the vectors introduced in the previous discussion: the LNMs, the KLMs, the SKLMs, and the LNM approximated by the SKLD. The case where $\tau_1 = 0.01$ and $m = 3$ is presented in Fig. 3. As was to be expected (see Fig. 2 at $m = 3$), the LNM values predicted by the SKLD are similar to the LNM values, and only the first four KLMs differ significantly from the LNMs. One rather surprising finding obtained is the behavior of the SKLMs which are otherwise similar to the LNM but differ considerably at the first mass, where there is mass inhomogeneity ($m = 3$). We also changed the place of the inhomogeneity and the position of the forcing, always with similar results. So, the result shown is representative.

4. Conclusion

There are several discussions in the literature about the interpretation of POD, of its relation with normal modes, and applications of these concepts to problems [7,9,16–20]. In this study, the smooth orthogonal decomposition method introduced by Ref. [1] is formulated in terms of a smooth Karhunen–Loève decomposition to analyze continuous-time random fields. The SKLD is obtained solving a generalized eigenproblem defined by combining the covariance matrix of the random field with that of the associated time-derivative random field. This definition avoids the use of any smoothing operator and is valid for continuous time. Besides the theoretical simplicity, the definition has several computation advantages, as, for example, the use of Lyapounov's equations to compute the covariances (see Ref. [10]). The SKLD does not have the best decomposition properties of the Karhunen–Loève decomposition. In the context of modal analysis (without excitation data), the SKLD has several advantages with respect to KLD. If the modal forcing components are not correlated and if the damping is proportional, it is possible to estimate, without condition on the mass distribution, the resonance frequencies and normal modes directly. This is true independently of the damping level. Note that if the modal forcing

components are correlated, the efficient of the SKLD method to estimate resonance frequencies and normal modes rapidly decreases when the damping level increases.

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