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Eigenvalues of linear viscoelastic systems

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ABSTRACT

The calculation of eigenvalues of single- and multiple-degree-of-freedom linear viscoelastic systems is considered. The assumed viscoelastic forces depend on the past history of motion via convolution integrals over exponentially decaying kernel functions. Current methods to solve this type of problem normally use the state-space approach involving additional internal variables. Such approaches often increase the size of the eigenvalue problem to be solved and can become computationally expensive for large systems. Here an approximate non-state-space based approach is proposed for this type of problem. The proposed approximations are based on certain physical assumptions which simplify the underlying characteristic equation to be solved. Closed-form approximate expressions of the complex and real eigenvalues of the system are derived. These approximate expressions are obtained as functions of the elastic eigenvalues only. This enables one to approximately calculate the eigenvalues of complex viscoelastic systems by simple post-processing of the elastic (undamped) eigenvalues. Representative numerical examples are given to verify the accuracy of the derived expressions.

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1. Introduction

The characterization of energy dissipation in complex vibrating structures such as aircrafts and helicopters is of fundamental importance. Noise and vibration are not only uncomfortable to the users of these complex dynamical systems, but also may lead to fatigue, fracture and even failure of such systems. The increasing use of composite structural materials, active control and damage tolerant systems in the aerospace and automotive industries have lead to renewed demand for energy absorbing and high damping materials. Effective applications of such materials in complex engineering dynamical systems require robust and efficient analytical and numerical methods. Due to the superior damping characteristics, the dynamics of viscoelastic materials and structures have received significant attention over the past two decades. This paper is aimed at developing computationally efficient and physically insightful approximate numerical methods for linear viscoelastic systems.

A key feature of viscoelastic systems is the incorporation of the time history of the state-variables in the equation of motion. Several different models are available for viscoelastic systems. We use the Biot model [1] which allows one to incorporate a wide range of functions in the frequency domain by means of summation of simple 'pole residue forms'. Several authors have considered this model due to its simplicity and generality (see for example Refs. [2–4]). The equation of motion of a single-degree-of-freedom (SDOF) linear viscoelastic system can

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be expressed by

$$m_u\ddot{u}(t) + \int_0^t \mathscr{G}(t-\tau)\dot{u}(\tau)\,\mathrm{d}\tau + k_e u(t) = f(t). \tag{1}$$

Here $u(t)$ is the displacement, $f(t)$ is the forcing, m_u is the mass, k_e is the elastic stiffness and $\mathscr{G}(t)$ is the viscoelastic stiffness kernel function. The kernel function $\mathscr{G}(t)$, or functions similar to it, is known by many names such as retardation functions, heredity functions, after-effect functions or relaxation functions in the context of different subjects. Eq. (1) is very general and for any engineering applications some specific form of $\mathscr{G}(t)$ have to be assumed. A wide variety of mathematical expressions could be used for the kernel functions $\mathscr{G}(t)$ as long as the rate of energy dissipation is non-negative. Here we will use a viscoelastic material model for which the kernel function has the form

$$\mathscr{G}(t) = k_{v_0}\delta(t) + \sum_{k=1}^{n} a_k \mathrm{e}^{-b_k t} k_{v_k} \tag{2}$$

or in the Laplace domain

$$G(s) = k_{v_0} + \sum_{k=1}^{n} \frac{a_k}{s+b_k} k_{v_k}. \tag{3}$$

The constants a_k, b_k, k_{v_0} and k_{v_k} are the viscoelastic material parameters and n denotes the number of terms in the series. Some other viscoelastic modeling approaches, such as the GHM (Golla–Hughes–McTavish) approach [5,6] uses the following expression for the kernel function:

$$sG(s) = G^{\infty}\left[1 + \sum_k \alpha_k \frac{s^2 + 2\xi_k\omega_k s}{s^2 + 2\xi_k\omega_k s + \omega_k^2}\right]. \tag{4}$$

For the fractional derivative approach proposed by Bagley and Torvik [7], the kernel function takes the form

$$G(s) = \frac{E_1 s^{\alpha} - E_0 b s^{\beta}}{1 + b s^{\beta}} \quad (0<\alpha,\beta<1). \tag{5}$$

For the ADF (anelastic displacement field) approach [8,9] one has

$$G(s) = 1 + \sum_{k=1}^{n} \frac{\Delta_k s}{s+\beta_k}. \tag{6}$$

Although these models are physically different, it will be seen later that they can be treated in an unified way for the calculation of some of the eigenvalues of the system.

Eq. (1), together with the kernel in Eq. (2), represents an integro-differential equation. Several authors have proposed [2,3,5–11] state-space approach based on the internal variables for this type of equation. The main reasons for seeking an alternative to the state-space approach in this paper include:

- although exact in nature, the state-space approach for viscoelastic systems is computationally very intensive for real-life multiple-degree-of-freedom (MDOF) systems due to the huge number of internal variables;
- the physical insights offered by methods in the original space (e.g., the modal analysis) is lost in a state-space based approach.

Regarding the first point, McTavish [12], Woodhouse [13] and Adhikari [14] proposed approximate methods in the space of the original problem. These methods are applied to frequency dependent damping when the damping is small and they neglect the overdamped modes. A direct time-domain approach to obtain the solution of Eq. (1) was proposed by Adhikari and Wagner [15]. This method is computationally efficient and accurate but does not provide much physical insight. Based on a variational principle, Qian and Hansen [16] derived a substructure synthesis method where the viscoelastic system eigensolution is obtained from the undamped system eigensolution. Daya and Potier-Ferry [17] proposed an asymptotic numerical method for the calculation of natural frequencies and loss-factors of viscoelastic systems.

The calculation of the eigenvalues by solving the nonlinear eigenvalue problem corresponding to the equation of motion (1) is the main topic of the paper. The objective is to derive the approximations in a way so that they can be easily extended to MDOF systems where the computational efficiency can make a real difference. We have derived closed-form approximate expressions of the eigenvalues of the system for three mathematically different cases based on the number of kernel functions. First SDOF systems are considered and then the results are extended to MDOF systems. The approximations utilize Taylor series expansion in the complex domain and are based on certain simplifying physical assumptions. The validity of the assumptions and the accuracy of the results are verified by numerical calculations.

2. Nonlinear eigenvalue problem for viscoelastic systems

The equation of motion (1) can be expressed in the Laplace domain as

$$s^2 m_u \bar{u}(s) + sG(s)\bar{u}(s) + k_e \bar{u}(s) = \bar{f}(s) \tag{7}$$

or

$$d(s)\bar{u}(s) = \bar{f}(s).$$

Here the dynamic stiffness

$$d(s) = s^2 m_u + sG(s) + k_e \tag{8}$$

and $\bar{u}(s), \bar{f}(s)$ and $G(s)$ are, respectively, the Laplace transforms of displacement, forcing and viscoelastic kernel function. The eigenvalues of the system can be obtained by solving the characteristic equation

$$s_j^2 m_u + s_j G(s_j) + k_e = 0$$

or

$$s_j^2 m_u + s_j \left(k_{v_0} + \sum_{k=1}^{n} \frac{a_k}{s_j + b_k} k_{v_k} \right) + k_e = 0, \quad j = 1, \ldots, m, \tag{9}$$

where m is the order of the characteristic polynomial. Following the approach outlined in Refs. [14,18], the dynamic response, that is the solution of Eq. (1), can be expressed in terms of the eigenvalues as

$$u(t) = \sum_{j=1}^{m} \gamma_j \left[\int_0^t \{ e^{s_j(t-\tau)} f(\tau) + \mathscr{G}(\tau) u_0 \} \, d\tau + e^{s_j t} \{ m\dot{u}_0 + s_j m u_0 \} \right], \quad \forall t > 0. \tag{10}$$

Here u_0 and $\dot{u}_0$ are, respectively, the initial displacement and velocity and the constants γ_j can be expressed as

$$\gamma_j = \frac{1}{\left. \dfrac{\partial d(s)}{\partial s} \right|_{s=s_j}}. \tag{11}$$

For systems with only the elastic stiffness term, the order of the characteristic polynomial $m = 2$. For viscoelastic systems in general m is more than two. From Eq. (9) one can observe that in general $m = 2 + n$. This is the key difference between a viscoelastic system and an elastic system where the number of eigenvalues is exactly two. The two conjugated complex eigenvalues correspond to the oscillatory motions of the system and are called *elastic mode* or *vibration mode*. The modes corresponding to the 'additional' n eigenvalues are called *damping modes* or *overdamped modes*. For stable passive systems the damping modes are over-critically damped (i.e., negative real eigenvalues) and not oscillatory in nature. In this paper both the complex-conjugate modes and the damping modes will be derived. In the following sections closed-form approximate expressions of eigenvalues are derived for the elastic modes and damping modes.

3. Complex-conjugate eigenvalues

The main motivation behind the proposed approximations is that the approximate eigenvalues can be 'constructed' from the eigenvalues of the underlying elastic system. The eigenvalues of the underlying damped elastic system can in turn be expressed in terms of the undamped eigenvalues. Combining these together, one can therefore obtain the eigenvalues of viscoelastic systems by simple 'post-processing' of the eigenvalues of the underlying undamped systems only. The eigenvalues (appearing in a complex-conjugatae pair) of the underlying damped elastic system [19] is given by

$$s_0 = -\zeta_n \omega_n \pm i\omega_n \sqrt{1 - \zeta_n^2} \approx -\zeta_n \omega_n \pm i\omega_n, \tag{12}$$

where the undamped natural frequency

$$\omega_n = \sqrt{k_e / m_u} \tag{13}$$

and ζ_n is the viscous damping factor. Viscously damped elastic system can be considered as a special case of Eq. (7) when the function $G(s)$ is replaced by $G(s \to 0)$. Therefore, for the purpose of numerical approximations, we can obtain an equivalent viscous damping coefficient

$$c = \lim_{s \to 0} G(s) = k_{v_0} + \sum_{k=1}^{n} a_k k_{v_k} / b_k. \tag{14}$$

From this expression, the viscous damping factor ζ_n can be obtained as

$$\zeta_n = c/2\sqrt{k_e m_u} = \left(k_{v_0} + \sum_{k=1}^{n} a_k k_{v_k}/b_k\right) \Big/ 2\sqrt{k_e m_u}. \tag{15}$$

For a truly damped elastic system, the solution given by Eq. (12) would have been the exact solution of the characteristic equation (9). Since in general this is not the case, the difference between the elastic solution and the true solution of the characteristic equation (9) is essentially arising due to the 'varying' nature of the function $G(s)$. The approximate solution obtained here are based on keeping this fact in mind.

The central idea is that the actual solution of the characteristic equation (9) can be obtained by expanding the solution in a Taylor series around s_0. The error arising in the resulting solution would then depend on the 'degree of variability' of the function $G(s)$. We assume that the true solution of Eq. (9) can be expressed as

$$s = s_0 + \delta, \tag{16}$$

where δ is a small quantity. Substituting this into the characteristic equation we have

$$(s_0+\delta)^2 m_u + (s_0+\delta)G(s_0+\delta) + k_e = 0. \tag{17}$$

Expanding $G(s_0+\delta)$ in a Taylor series in δ around s_0 one has

$$(s_0+\delta)^2 m_u + (s_0+\delta)\left(G(s_0) + \delta\frac{\partial G(s_0)}{\partial s} + \cdots\right) + k_e = 0$$

or

$$\left(m_u + \frac{\partial G(s_0)}{\partial s}\right)\delta^2 + \left(s_0\left(2m_u + \frac{\partial G(s_0)}{\partial s}\right) + G(s_0)\right)\delta + s_0(s_0 m_u + G(s_0)) + k_e \approx 0. \tag{18}$$

Keeping only the first-order terms in δ we have

$$\delta^{(1)} \approx -\frac{s_0(s_0 m_u + G(s_0)) + k_e}{s_0\left(2m_u + \dfrac{\partial G}{\partial s}(s_0)\right) + G(s_0)}. \tag{19}$$

The superscript 1 in Eq. (19) is used to denote that this is a first-order approximation.

One can improve the accuracy by retaining higher-order terms in δ. Retaining upto second-order terms in δ in the Taylor expansion of Eq. (18) we have

$$\delta^{(2)} = \frac{-B - \sqrt{B^2 - 4AC}}{2A}, \tag{20}$$

where

$$A = m_u + \frac{1}{2}\frac{\partial^2 G(s_0)}{\partial s^2}s_0 + \frac{\partial G(s_0)}{\partial s}, \quad B = 2m_u s_0 + s_0\frac{\partial G(s_0)}{\partial s} + G(s_0) \tag{21}$$

and

$$C = s_0^2 m_u + s_0 G(s_0) + k_e. \tag{22}$$

In the above expressions $\partial G(s_0)/\partial s$ and $\partial^2 G(s_0)/\partial s^2$ are, respectively, the first- and second-order derivative of $G(s)$ evaluated at $s = s_0$. When the kernel function is given by Eq. (3), we have

$$\frac{\partial G(s_0)}{\partial s} = -\sum_{k=1}^{n}\frac{a_k}{(s_0+b_k)^2}k_{v_k},$$

$$\frac{\partial^2 G(s_0)}{\partial s^2} = \sum_{k=1}^{n}\frac{2a_k}{(s_0+b_k)^3}k_{v_k}. \tag{23}$$

Our numerical works show that retaining terms higher than the second-order results considerably complex expressions and the accuracy gained is not very significant. As a result we have not perused this approach in the rest of the paper. Based on our numerical works we recommend the second-order expression in Eq. (20) as it gives an excellent accuracy and additional computation cost is just marginally higher compared to the first-order approximation. The expression of the approximate eigenvalue derived here shows that the complex-conjugate eigenvalues of a general viscoelastic system can be obtained by post-processing of the undamped eigenvalue ω_n and equivalent viscous damping factor ζ_n. This approximation of the complex-conjugate eigenvalues is valid for any kernel function. It could therefore be used with viscoelastic models given by Eqs. (4)–(6). On the contrary, the real eigenvalues are specific to a particular kernel function. In the next section, the approximation to the real eigenvalues is derived for the Biot model.

4. Real eigenvalues

While the complex-conjugate eigenvalues can be expected to be close to the eigenvalues of the equivalent elastic damped system, no such analogy can be made for the real solution as the equivalent elastic system does not have one. This makes the calculation of the real eigenvalues more challenging. The nature of the real eigenvalues is expected to be specific to the corresponding kernel functions. For this reason the methods proposed here are only valid for the Biot model. For the convenience of analytical developments, the following three cases are considered separately:

- single-degree-of-freedom system with single exponential kernel ($n=1$);
- single-degree-of-freedom system with double exponential kernels ($n=2$);
- single-degree-of-freedom system with multiple exponential kernels ($n>2$).

4.1. Systems with single exponential kernel

For this case when $n=1$ the eigenvalue equation can be simplified from Eq. (9) as

$$s^2 m_u + sG(s) + k_e = 0 \quad \text{where } G(s) = k_{\nu_0} + \frac{a_1}{s+b_1} k_{\nu_1}. \tag{24}$$

Eq. (24) is a third-order polynomial in s and it can be solved exactly in closed form. A more detailed study on the properties of the exact solutions have been carried out in Refs. [20–22]. Here we propose an approximate solution so that the approximation method can be extended to more complex cases.

We first multiply the characteristic equation (24) by $(s+b_1)$ and rewrite

$$(s^2 m_u + sk_{\nu_0} + k_e)(b_1 + s) + sa_1 k_{\nu_1} = 0. \tag{25}$$

To obtain an initial guess the 'small viscoelasticity' approximation is introduced so that $sa_1 k_{\nu_1} \approx 0$. Since $(s^2 m_u + sk_{\nu_0} + k_e) \neq 0$ as we are considering the real solution only, the first guess is obtained as

$$b_1 + s_0 = 0 \quad \text{or} \quad s_0 = -b_1. \tag{26}$$

We take the first approximation of the real root as

$$s = s_0 + \Delta = -b_1 + \Delta, \tag{27}$$

where Δ is a small quantity. Eq. (27) indicates that the real eigenvalues are expected to be close to the relaxation parameters. This is one of the most crucial observations made in this paper. Substituting s from Eq. (27) into the characteristic equation one obtains

$$((-b_1+\Delta)^2 m_u + (-b_1+\Delta)k_{\nu_0} + k_e)\Delta + (-b_1+\Delta)a_1 k_{\nu_1} = 0. \tag{28}$$

Expanding this equation results

$$m_u\Delta^3 + (-b_1 m_u + k_{\nu_0})\Delta^2 + (b_1^2 m_u - b_1 k_{\nu_0} + k_e + a_1 k_{\nu_1})\Delta - a_1 b_1 k_{\nu_1} = 0. \tag{29}$$

After neglecting all the terms associated with Δ^n for $n>1$ this equation results

$$\Delta \approx \frac{a_1 b_1 k_{\nu_1}}{b_1^2 m_u - b_1 k_{\nu_0} + k_e + a_1 k_{\nu_1}}. \tag{30}$$

Therefore, the approximate real solution is given by

$$s = s_0 + \Delta \approx -b_1 + \frac{a_1 b_1 k_{\nu_1}}{b_1^2 m_u - b_1 k_{\nu_0} + k_e + a_1 k_{\nu_1}}. \tag{31}$$

The third-order characteristic polynomial in Eq. (29) can be solved exactly. However, recall that the main reason for this approximation is not to solve this specific problem, but to solve MDOF systems with multiple kernels using the insight obtained from this simple problem. In the next section this approximation technique is extended to SDOF systems with two kernels.

4.2. Systems with two exponential kernels

This section is the next logical step towards solving the general multiple kernel case. For this case the kernel function $G(s)$ takes the form

$$G(s) = k_{\nu_0} + \frac{a_1}{s+b_1} k_{\nu_1} + \frac{a_2}{s+b_2} k_{\nu_2}. \tag{32}$$

The characteristic function with this type of kernel function can be expressed as

$$s^2 m_u + s\left(k_{v_0} + \frac{a_1}{s+b_1}k_{v_1} + \frac{a_2}{s+b_2}k_{v_2}\right) + k_e = 0. \tag{33}$$

This a polynomial in s of order four and therefore has four roots. The two complex-conjugate roots can be obtained using the expressions derived before. In this section only the real roots are considered. Multiplying Eq. (33) by $(s+b_1)(s+b_2)$ we have

$$(s^2 m_u + s k_{v_0} + k_e)(s+b_1)(s+b_2) + s(a_1 k_{v_1}(s+b_2) + a_2 k_{v_2}(s+b_1)) = 0. \tag{34}$$

Similar to the previous case, the first solution is approximated as

$$s_1 = -b_1 + \Delta_1. \tag{35}$$

Substituting this in Eq. (34) and neglecting the higher-order terms $\Delta_1^n,\ n>1$ one obtains

$$\Delta_1 \approx \frac{(b_1-b_2)a_1 b_1 k_{v_1}}{(b_2-b_1)(b_1^2 m_u - b_1 k_{v_0} + k_e + a_1 k_{v_1}) - b_1(a_1 k_{v_1} + a_2 k_{v_2})}. \tag{36}$$

Using this, the approximation to the first real solution is given by

$$s_1 \approx -b_1 + \frac{(b_1-b_2)a_1 b_1 k_{v_1}}{(b_2-b_1)(b_1^2 m_u - b_1 k_{v_0} + k_e + a_1 k_{v_1}) - b_1(a_1 k_{v_1} + a_2 k_{v_2})}. \tag{37}$$

The second solution can be obtained from the same expression by swapping the subscripts 1 and 2.

4.3. The general case: systems with multiple exponential kernels

For this case the kernel function $G(s)$ is given by Eq. (3). The characteristic equation with this type of kernel function can be expressed by Eq. (9). This a polynomial in s of order $(n+2)$ and therefore, has $(n+2)$ roots. In this section we derive approximate solutions with the view of generalizing them to MDOF systems.

There are n number of pure real roots corresponding to n terms in the series in Eq. (3). Multiplying the characteristic equation (9) by the product $\prod_j^n (s+b_j)$ we have

$$(s^2 m_u + s k_{v_0} + k_e)\prod_{j=1}^{n}(s+b_j) + s\sum_{l=1}^{n}\left(a_l k_{v_l}\prod_{\substack{j=1\\ j\neq l}}^{n}(b_j+s)\right) = 0. \tag{38}$$

Like the previous case, we use the approximation

$$s_k = -b_k + \Delta_k, \quad k = 1,2,\ldots,n. \tag{39}$$

Substituting this into the characteristic equation results

$$\begin{aligned}&((-b_k+\Delta_k)^2 m_u + (\Delta_k - b_k)k_{v_0} + k_e)\Delta_k \prod_{\substack{j=1\\ j\neq k}}^{n}(\Delta_k + b_j - b_k)\\ &+ (\Delta_k - b_k)\left(a_k k_{v_k}\prod_{\substack{j=1\\ j\neq k}}^{n}(\Delta_k + b_j - b_k)\sum_{l=1}^{n} a_l k_{v_l}\Delta_k \prod_{\substack{j=1\\ j\neq l\\ j\neq k}}^{n}(\Delta_k + b_j - b_k)\right) = 0.\end{aligned} \tag{40}$$

Expanding further and retaining only the first-order terms in Δ_k, after some simplifications we have

$$\Delta_k \approx \frac{b_k a_k k_{v_k} p_1}{[b_k^2 m_u - b_k k_{v_0} + k_e]p_1 + [-b_k(p_2+p_3) + a_k k_{v_k} p_1]} \tag{41}$$

with

$$p_1 = \prod_{\substack{j=1\\ j\neq k}}^{n}(b_j - b_k), \quad p_2 = a_k k_{v_k}\sum_{\substack{j=1\\ j\neq k}}^{n}\prod_{\substack{r=1\\ r\neq j\\ r\neq k}}^{n}(b_r - b_k)$$

Table 1
The exact and approximate eigenvalues of an SDOF system with a single kernel.

b_1	Exact solution (state-space)	Proposed approximate solution	Percentage error
1.5	-1.4500	-1.4517	0.12
Complex-conjugate solution	-0.0250 ± 1.4382i	-0.0250 ± 1.4382i	-0.00 ± 0.00i

and

$$p_3 = \sum_{\substack{j=1 \\ j \neq k}}^{n} a_j k_{v_j} \prod_{\substack{r=1 \\ r \neq j,k}}^{n} (b_r - b_k). \tag{42}$$

It can be verified that the general expression (41) reduces to Eq. (36) derived in the previous subsection when $n = 2$. Eqs. (39), (41) and (42) completely define the n real eigenvalues of an SDOF viscoelastic system with a general kernel function. In the next section the accuracy of these approximate expressions are verified by numerical examples.

5. Numerical examples

To understand the accuracy provided by the approximate expressions derived in the preceding sections, we consider three representative numerical examples covering the three cases. The approximate results obtained from these expressions are compared with the exact solutions obtained from the state-space method. For the calculation of percentage error, the state-space results are considered as benchmarks.

5.1. An SDOF system with a single kernel

A single-degree-of-freedom system with one kernel is studied to investigate the accuracy of the approximate eigenvalues derived in Sections 3 and 4.1. For the numerical calculations we consider $m_u = 1\,\text{kg}$, $k_e = 2\,\text{N/m}$, $k_{v_0} = 0$. It is considered that $a_1 k_{v_1}/b_1 = 2\zeta_n\omega_n$, $b_1 = 1.5$ and the damping factor constant $\zeta_n = 0.1$. The accuracy of the proposed approximation is shown in Table 1. The complex-conjugate eigenvalues turn out to be almost exact while there is a small error in the real eigenvalue.

5.2. An SDOF system with two kernels

We consider a single-degree-of-freedom system with two exponential kernels to investigate the accuracy of the approximate eigenvalues derived in Sections 3 and 4.2. For the numerical calculations we consider $m_u = 1\,\text{kg}$, $k_e = 2\,\text{N/m}$ and $k_{v_0} = 0$. It is assumed that all $a_k k_{v_k}/b_k$ are of the same value so that $a_k k_{v_k}/b_k = 2\zeta_n\omega_n$, $\forall k = 1, 2$, $b_1 = 1$, $b_2 = 5$ and the damping factor constant $\zeta_n = 0.1$. The accuracy of the proposed approximation is shown in Table 2. The complex-conjugate eigenvalues turn out to be almost exact while there are small errors in the two real eigenvalues. Error corresponding to the larger value of b_k is smaller than error corresponding to the smaller value of b_k.

5.3. The general case: an SDOF system with multiple kernels

We consider a single-degree-of-freedom system with eight kernels to investigate the accuracy of the approximate eigenvalues derived in Sections 3 and 4.3. For the numerical calculations we consider $m_u = 1\,\text{kg}$, $k_e = 2\,\text{N/m}$, $k_{v_0} = 0$. It is assumed that all $a_k k_{v_k}/b_k$ are of the same value so that $a_k k_{v_k}/b_k = 2\zeta_n\omega_n$, $\forall k = 1, 2, \ldots, 8$. The damping factor constant $\zeta_n = 0.1$ and the values of b_k for $k = 1, 2, \ldots, 8$ are selected as 1.4973, 1.5231, 1.7454, 1.7657, 1.9317, 1.9442, 1.9558 and 2.0677. The approximate eigenvalues obtained using the proposed method are compared with the results obtained from exact state-space solution in Table 3. The complex-conjugate eigenvalues are obtained very accurately using the proposed approximation. The real eigenvalues are not as accurate as the complex-conjugate eigenvalues. However, recall that the motion corresponding to the real eigenvalues are purely dissipative in nature and therefore do not significantly affect the dynamic response of the system.

6. Multiple-degree-of-freedom system

In this section the results obtained for the SDOF system are extended to general MDOF systems. The equation of motion of an N-degree-of-freedom linear viscoelastic system can be expressed by extending the corresponding SDOF equation (1) as

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \int_0^t \mathscr{G}(t-\tau)\dot{\mathbf{u}}(\tau)\,\mathrm{d}\tau + \mathbf{K}_e\mathbf{u}(t) = \mathbf{f}(t). \tag{43}$$

Table 2
The exact and approximate eigenvalues of an SDOF system with two kernels.

$b_k,\ k=1,2$	Exact solution (state-space)	Proposed approximate solution	Percentage error
1.0	−0.9531	−0.9539	0.0868
5.0	−4.9739	−4.9743	0.0084
Complex-conjugate solution	-0.0365 ± 1.4519i	-0.0365 ± 1.4519i	0.00 ± 0.00i

Table 3
The exact and approximate eigenvalues of an SDOF system with eight kernels.

$b_k,\ k=1,2,\ldots,8$	Exact solution (state-space)	Proposed approximate solution	Percentage error
1.4973	−1.3025	−1.4003	7.5121
1.5231	−1.5105	−1.4825	1.8499
1.7454	−1.6586	−1.6581	0.0291
1.7657	−1.7560	−1.7286	1.5607
1.9317	−1.8668	−1.7315	7.2454
1.9442	−1.9375	−1.8981	2.0334
1.9558	−1.9513	−1.9313	1.0219
2.0677	−2.0511	−2.0302	1.0184
Complex-conjugate solution	-0.1984 ± 1.5920i	-0.1984 ± 1.5915i	0.0335 ± 0.0056i

Here $\mathbf{u}(t) \in \mathbb{R}^N$ is the displacement vector, $\mathbf{f}(t) \in \mathbb{R}^N$ is the forcing vector, $\mathbf{M} \in \mathbb{R}^{N\times N}$ is the mass matrix, $\mathbf{K}_e \in \mathbb{R}^{N\times N}$ is the elastic stiffness matrix and $\mathscr{G}(t)$ is the matrix of viscoelastic kernel functions. In the Laplace domain, the equation of motion can be expressed as

$$\mathbf{D}(s)\bar{\mathbf{u}} = \mathbf{0}, \tag{44}$$

where the dynamic stiffness matrix $\mathbf{D}(s)$ is given by

$$\mathbf{D}(s) = s^2\mathbf{M} + s\mathbf{G}(s) + \mathbf{K}_e \in \mathbb{C}^n \tag{45}$$

with

$$\mathbf{G}(s) = \mathbf{K}_{\nu_0} + \sum_{k=1}^{n} \frac{a_k}{s+b_k}\mathbf{K}_{\nu_k} \in \mathbb{C}^n. \tag{46}$$

Eq. (46) is the extension of the corresponding scalar equation (3) to the matrix case. The eigenvalues of the system s_j can be obtained by solving the characteristic polynomial

$$\det[\mathbf{D}(s_j)] = 0, \quad j = 1,2,\ldots m. \tag{47}$$

We assume that the order of the characteristic polynomial is m so that there are m eigenvalues of the system. In this work we consider that all the eigenvalues are distinct. For undamped elastic systems or elastic systems with viscous damping, the order of the characteristic polynomial $m = 2N$. For viscoelastic systems in general m is more than $2N$, that is $m = 2N + p;\ p \geqslant 0$. If all of the $\mathbf{K}_{\nu_k}$ matrices are of full rank, then $p = nN$. This shows that although the system has N degrees-of-freedom, the number of eigenvalues is more than $2N$. This is a major difference between a viscoelastic system and an elastic system where the number of eigenvalues is exactly $2N$, including any multiplicities. Like the SDOF case, N complex-conjugate pair of eigenvalues correspond to *elastic modes* or *vibration modes*. These modes are related to the N modes of vibration of the structural system. Physically, the assumption of '$2N$ complex-conjugate pairs of eigenvalues' implies that all the elastic modes are oscillatory in nature, that is, they are sub-critically damped. The additional nN eigenvalues correspond to *non-viscous modes* or *overdamped modes*. Here the complex eigenvalues are derived for a matrix with general kernel functions. For the real eigenvalues the Biot model with two cases namely, when $n = 1$ and when $n > 1$ are considered for analytical convenience.

6.1. Complex-conjugate eigenvalues

The aim of this section is to obtain the complex-conjugate eigenvalues using the undamped elastic eigenvalues. The undamped elastic eigenvalue problem of an MDOF system is given by

$$\mathbf{K}_e\mathbf{x}_j = \omega_j^2\mathbf{M}\mathbf{x}_j, \quad j = 1,2,\ldots,N, \tag{48}$$

where ω_j^2 and $\mathbf{x}_j$ are the eigenvalues and mass-normalized eigenvectors of the system. We define the matrices

$$\boldsymbol{\Omega} = \text{diag}[\omega_1, \omega_2, \ldots, \omega_N] \tag{49}$$

and

$$\mathbf{X} = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N]. \tag{50}$$

There are various efficient methods available to obtain the undamped eigensolution (see for example Ref. [23]) and many of them have been integrated within the commercially available general purpose finite element software. Using these, the characteristic equation (47) can be transformed into the modal coordinates as

$$\det[s_j^2 \mathbf{I}_N + s_j \mathbf{G}'(s_j) + \boldsymbol{\Omega}^2] = 0, \quad j = 1, 2, \ldots m, \tag{51}$$

where $\mathbf{I}_N$ is an N-dimensional identity matrix. The viscoelastic kernel function matrix in the modal coordinates $\mathbf{G}'(s)$ is defined as

$$\begin{aligned} \mathbf{G}'(s) &= \mathbf{X}^{\mathrm{T}} \mathbf{G}(s) \mathbf{X} = (\mathbf{X}^{\mathrm{T}} \mathbf{K}_{\nu_0} \mathbf{X}) + \sum_{k=1}^{n} \frac{a_k}{s + b_k} (\mathbf{X}^{\mathrm{T}} \mathbf{K}_{\nu_k} \mathbf{X}) \\ &= \mathbf{K}'_{\nu_0} + \sum_{k=1}^{n} \frac{a_k}{s + b_k} \mathbf{K}'_{\nu_k}, \end{aligned} \tag{52}$$

where the matrices

$$\mathbf{K}'_{\nu_0} = \mathbf{X}^{\mathrm{T}} \mathbf{K}_{\nu_0} \mathbf{X} \quad \text{and} \quad \mathbf{K}'_{\nu_k} = \mathbf{X}^{\mathrm{T}} \mathbf{K}_{\nu_k} \mathbf{X}, \ \ k = 1, 2, \ldots, n. \tag{53}$$

We consider that the system has small non-proportionality so that the off-diagonal entries of the $\mathbf{G}'$ matrix are small compared to the diagonal entries, that is $G'_{kl}(s_j) \leqslant G'_{kk}(s_j), \ \forall s_j, \ k \neq l$. This approximation is often employed in the dynamic analysis of damped systems [24–27]. Considering the j-th set of Eq. (51) one obtains

$$s_j^2 + s_j G'_{jj}(s_j) + \omega_j^2 \approx 0. \tag{54}$$

This equation is similar to Eq. (9) and can be solved in a similar way. The modal damping factor ζ_j can be defined as

$$\zeta_j = \frac{\lim_{s \to 0} G'_{jj}(s)}{2\omega_j} = \left(K'_{\nu_{0_{jj}}} + \sum_{k=1}^{n} \frac{a_k}{b_k} K'_{\nu_{k_{jj}}} \right) \Big/ 2\omega_j, \tag{55}$$

where the matrices $\mathbf{K}'_{\nu_0}$ and $\mathbf{K}'_{\nu_k}$ are defined in Eq. (53). The complete solution can be written as

$$s_j = s_{0_j} + \delta_j, \tag{56}$$

where

$$s_{0_j} = -\zeta_j \omega_j \pm \mathrm{i} \omega_j \sqrt{1 - \zeta_j^2}, \tag{57}$$

$$\delta_j \approx \frac{-B_j - \sqrt{B_j^2 - 4A_j C_j}}{2A_j}, \tag{58}$$

$$A_j = 1 + \frac{1}{2} \frac{\partial^2 G'_{jj}(s_{0_j})}{\partial s^2} s_{0_j} + \frac{\partial G'_{jj}(s_{0_j})}{\partial s}, \tag{59}$$

$$B_j = 2s_{0_j} + s_{0_j} \frac{\partial G'_{jj}(s_{0_j})}{\partial s} + G'_{jj}(s_{0_j}) \tag{60}$$

and

$$C_j = s_{0_j}^2 + s_{0_j} G'_{jj}(s_{0_j}) + \omega_j^2. \tag{61}$$

In the above expressions $\partial G'_{jj}(s_{0_j})/\partial s$ and $\partial^2 G'_{jj}(s_{0_j})/\partial s^2$ are, respectively, the first- and second-order derivative of $G'_{jj}(s)$ evaluated at $s = s_{0_j}$ and can be obtained following Eq. (23). Eqs. (55)–(61) completely define the complex-conjugate eigenvalues of an MDOF viscoelastic system.

6.2. Real eigenvalues

The real eigenvalues are obtained using an approach similar to the SDOF system. After neglecting the off-diagonal terms of the $\mathbf{G}'$ matrix, the governing characteristic equation for every mode can be expressed by Eq. (54). This equation can be solved for the real eigenvalues. For systems with single exponential kernel, there are in total $3N$ number of eigenvalue of

which N are real. Following Eqs. (27)–(31), the real eigenvalues can expressed as

$$s_j = -b_1 + \Delta_j, \quad \forall j = 1, \ldots, N, \tag{62}$$

where

$$\Delta_j \approx \frac{a_1 b_1 K'_{\nu_{1_{jj}}}}{b_1^2 - b_1 K'_{\nu_{0_{jj}}} + \omega_j^2 + a_1 K'_{\nu_{1_{jj}}}}. \tag{63}$$

This equation and Eq. (62) completely define all the N real eigenvalues.

For systems with n kernels, there are in general nN number of purely real eigenvalues. The approximate eigenvalues can be obtained extending Eqs. (39) and (41) as

$$s_{jk} = -b_k + \Delta_{jk}, \quad \forall j = 1, \ldots, N, \ k = 1, \ldots, n, \tag{64}$$

where

$$\Delta_{jk} \approx \frac{b_k a_k K'_{\nu_{k_{jj}}} p_1}{[b_k^2 - b_k K'_{\nu_{0_{jj}}} + \omega_j^2]p_1 + [-b_k(p_2 + p_3) + a_k K'_{\nu_{k_{jj}}} p_1]} \tag{65}$$

with

$$p_1 = \prod_{\substack{l=1 \\ l \neq k}}^{n} (b_l - b_k), \quad p_2 = a_k K'_{\nu_{k_{jj}}} \sum_{\substack{l=1 \\ l \neq k}}^{n} \prod_{\substack{r=1 \\ r \neq l \\ r \neq k}}^{n} (b_r - b_k)$$

and

$$p_3 = \sum_{\substack{m=1 \\ m \neq k}}^{n} a_m K'_{\nu_{m_{jj}}} \prod_{\substack{r=1 \\ r \neq m,k}}^{n} (b_r - b_k). \tag{66}$$

From Eq. (64) note that there are N real solutions around each b_k. Eqs. (64)–(66) completely define *all* the nN number of real eigenvalues.

Table 4
Exact and approximate eigenvalues of the three-DOF system.

Exact solution (state-space)	Proposed approximate solution	Percentage error
Real eigenvalues		
−0.9380	−0.9425	0.4797
−1.2995	−1.3044	0.3771
−1.4507	−1.7413	20.0317
−1.5754	−1.8096	14.8661
−1.7405	−1.5761	9.4456
−1.8095	−1.4507	19.8287
−0.6301	−0.6301	0
−1.1276	−1.1276	0
−1.4505	−1.7096	17.8628
−1.5507	−1.8081	16.5990
−1.7096	−1.5507	9.2946
−1.8081	−1.4505	19.7777
−0.6798	−0.6731	0.9856
−1.1295	−1.1289	0.0531
−1.4505	−1.7085	17.7870
−1.5501	−1.8080	16.6376
−1.7086	−1.5501	9.2766
−1.8080	−1.4505	19.7732
Complex-conjugate eigenvalues		
$-0.4109 \pm 2.6579i$	$-0.4116 \pm 2.6591i$	$0.1704 \pm 0.0451i$
$-0.4359 \pm 2.0939i$	$-0.4359 \pm 2.0939i$	$0 \pm 0i$
$-0.1674 \pm 0.8523i$	$-0.1649 \pm 0.8528i$	$1.4934 \pm 0.0587i$

6.3. Numerical example

We consider a three degree-of-freedom system to illustrate the proposed method and verify the accuracy of the approximate expressions. The mass, stiffness and viscoelastic matrices in the Laplace domain for the problem are considered as

$$\mathbf{M} = \begin{bmatrix} m_u & 0 & 0 \\ 0 & m_u & 0 \\ 0 & 0 & m_u \end{bmatrix}, \quad \mathbf{K}_e = \begin{bmatrix} 2k_u & -k_u & 0 \\ -k_u & 2k_u & -k_u \\ 0 & -k_u & 2k_u \end{bmatrix} \tag{67}$$

and

$$\mathbf{G}(s) = \mathbf{K}_\nu \sum_{k=1}^{6} \frac{a_k}{s+b_k} \quad \text{where } \mathbf{K}_\nu = \begin{bmatrix} 0.30 & -0.15 & -0.05 \\ -0.15 & 0.30 & -0.15 \\ -0.05 & -0.15 & 0.30 \end{bmatrix}. \tag{68}$$

For the numerical value we consider $m_u = 1$ kg, $k_u = 1$ N/m and $a_k = b_k$. It is assumed that $\mathbf{K}_{\nu_0} = \mathbf{0}$, all the $\mathbf{K}_{\nu_k}$ matrices are the same and the values of b_k for $k = 1, 2, \ldots, 6$ are selected as 1.4565, 1.0185, 1.8214, 1.4447, 1.6154, and 1.7919. The approximate eigenvalues obtained using the proposed method are compared with the results obtained from exact state-space solution in Table 4. Like the SDOF examples considered before, the complex-conjugate eigenvalues are obtained very accurately using the proposed approximation. The real eigenvalues are not as accurate as the complex-conjugate eigenvalues. However, recall that the motion corresponding to the real eigenvalues are purely dissipative in nature and therefore may not significantly contribute to the dynamic response of the system around the resonant frequencies.

7. Conclusions

Single and multiple-degree-of-freedom linear viscoelastic systems are considered. The calculation of eigenvalues of viscoelastic systems requires the solution of a nonlinear eigenvalue problem. So far mainly state-space based methods involving additional internal variables have been used for this type of problems. In this paper a new non-state-space approach has been proposed for such nonlinear eigenvalue problems. A key motivation behind this is to reduce the additional computational cost necessary for viscoelastic systems compared to undamped elastic systems. Approximate expressions have been derived for the complex and real eigenvalues with single and multiple exponential kernels. It is assumed that all the eigenvalues are distinct. For the SDOF system it has been assumed that the viscoelastic system can be considered as a small perturbation from the underlying elastic system. For the MDOF system, it has been additionally assumed that the system can be approximately diagonalized using the undamped elastic modes. These approximations allow one to obtain the eigenvalues of viscoelastic systems by simple post-processing of the undamped elastic eigenvalues, which in turn can be obtained using a general purpose finite element software. The complex eigenvalues are close to the eigenvalues of the underlying damped elastic system while the real eigenvalues are close to the relaxation parameters. The expression for the complex eigenvalues is applicable to any general viscoelastic kernel while the expression for the real eigenvalues is specific to the Biot model only. Although the real solution is applicable to the Biot model only, the novel technique developed for this purpose may inspire solutions to other type of viscoelastic models discussed literature.

The accuracy of the proposed approximations were verified numerically against the exact state-space eigenvalues for few example problems. The complex-conjugate eigenvalues turn out to be more accurate compared to the real eigenvalues. This is particularly encouraging because complex eigenvalues dominate the dynamic response of linear systems. The method presented offers a reduction in computational effort because neither the state-space formalisms nor the additional internal variables are employed. This approach might provide further physical insight and computational advantage for MDOF systems as only familiar undamped natural frequencies and damping factors are required to obtain the eigenvalues of the system.

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