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A general solution procedure for the forced vibrations of a continuous system with cubic nonlinearities: Primary resonance case

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ABSTRACT

Nonlinear vibrations of a general model of continuous system is considered. The model consists of arbitrary linear and cubic operators. The equation of motion is solved by the method of multiple scales (a perturbation method). The primary resonances of external excitation is analysed. The amplitude and phase modulation equations are presented. Approximate analytical solution is derived. Steady-state solutions and their stability are discussed. Finally, the solution algorithm is applied to two different engineering problems. One of the application is the transverse vibration of an axially moving Euler–Bernoulli beam and the other is a viscoelastic beam.

1. Introduction

Many different nonlinear models addressing vibrations of continuous systems appear in the literature. In most cases, the nature of nonlinearities does not permit exact solutions, hence approximate analytical solutions such as perturbation methods are sought for understanding the physics of the problem. Although the nonlinearities differ much, they possess some common features. For example, in the context of quadratic and cubic nonlinearities, many different forms may be encountered.

In an attempt to understand the effects of arbitrary quadratic and cubic nonlinearities on the solutions, an operator notation was developed by Pakdemirli [1]. The motivation behind the study was to compare the direct-perturbation methods with discretization-perturbation methods by employing a fairly general equation with quadratic and cubic nonlinearities. The discussion was for single mode approximations of free vibrations. Later the analysis was generalized to infinite number of modes for forced vibrations [2]. The advantages of direct-perturbation methods were discussed in detail. Comparison of both methods for a parametically excited linear system expressed by arbitrary linear operators were also done [3]. It is concluded that finite mode truncations of both methods yield different results with direct-perturbation method yielding more precise solutions. Forced vibrations of an arbitrary cubic nonlinear system was employed to compare results of different versions of method of multiple scales [4]. For systems modelled with more than one partial differential equation, a solution procedure was developed [5]. One-to-one internal resonances were considered. Using the model of Ref. [5], arbitrary internal resonance cases were further discussed [6]. For a general cubic nonlinear system, three-to-one internal resonances were considered by Pakdemirli and Özkaya [7]. For the case of arbitrary quadratic nonlinearities,

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two-to-one internal resonances were analysed in a general sense [8]. The new operator notation developed in the previous studies [1–8] were used by other researchers also (see Refs. [9,10] for example).

In this work, additional linear and cubic operators with time derivatives are incorporated into the model. In all previous studies with cubic nonlinearities, only spatial operators were employed. An additional linear and cubic operator containing spatial as well as time derivatives enables to analyse a more general class of continuous systems such as gyroscopic systems which are encountered in axially moving media or pipes conveying fluids. First a general solution procedure is developed for arbitrary operators. Method of multiple scales, a perturbation technique is used in the analysis. Amplitude and phase modulation equations, steady-state solutions and their stability are discussed in a general sense. The formalism is applied to two different problems: (1) nonlinear vibrations of stretched axially moving Euler–Bernoulli beam and (2) nonlinear vibrations of axially moving viscoelastic beam. Applications have very different nonlinearities, one of integro-differential type and the other of differential type. Nevertless, both models possess the common feature of being cubic nonlinear which enables the algorithm developed to be applied directly to these equations. The application problems are discussed in detail. Stability analysis is presented and frequency-response curves are drawn to depict the effects of various parameters on the vibrations of the system. Apart from the examples considered, the general solution procedure developed can be applied to a wide range of physical problems.

2. Equation of motion

The dimensionless model considered is as follows:

$$\ddot{w} + \mathbf{L}_1(w) + \mathbf{L}_2(\dot{w}) + \varepsilon\mu\dot{w} = \varepsilon F \cos \Omega t + \varepsilon\{\mathbf{C}_1(w,w,w) + \mathbf{C}_2(\dot{w},w,w)\} \tag{1}$$

$$\mathbf{B}_1(w) = 0 \quad x = 0, \quad \mathbf{B}_2(w) = 0 \quad x = 1 \tag{2}$$

In this general partial differential equation, the dependent variable $w(x,t)$ represents deflection, independent variables x and t are the spatial and time variables, respectively. Note that there may be more than one spatial variable and a 3-D problem in spatial variables x, y and z has not been excluded from the analysis. $\mathbf{L}_1$ and $\mathbf{L}_2$ are linear differential and/or integral operators. $\mathbf{C}_1$ and $\mathbf{C}_2$ are cubic nonlinear operators. μ represents damping coefficient. F and Ω represents external excitation amplitude and external excitation frequency, respectively. $\mathbf{B}_1$ and $\mathbf{B}_2$ are linear operators of boundary conditions. The representation of boundary conditions should be expressed in a modified form for a 3-D problem. ε is a small dimensionless physical parameter. Dot denotes differentiation with respect to time. To capture the effects of gyroscopic systems, additional linear and cubic operators containing time derivatives are included to the model.

Note that model (1) is a fairly general model and any vibration problem that can be cast into the formalism of Eq. (1) can be solved approximately through the algorithm developed in the following analysis. A restriction of the boundary value problem comes from the boundary conditions, i.e. they are assumed to be linear. If the specific problem contains nonlinear boundary conditions, the general solution algorithm cannot be directly applied to it. This case needs further analysis since the solvability condition brings more terms for nonlinear boundary conditions which are hard to express in a general way. Although the boundary conditions given here represent a 1-D problem such as normalized length, in fact the solution algorithm is more general than that and can be successfully applied to 2- or 3-D problems in spatial variables. Note that both equation of motion and boundary conditions should be expressed in a non-dimensional form for applications.

The cubic operator may not be symmetric with respect to the inner variable and possesses the property of being multilinear as follows:

$$\begin{aligned}\mathbf{C}(c_1w_1 + c_2w_2, c_3w_3 + c_4w_4, c_5w_5 + c_6w_6) = {} & c_1c_3c_5\,\mathbf{C}(w_1,w_3,w_5) + c_1c_3c_6\,\mathbf{C}(w_1,w_3,w_6) \\ & + c_1c_4c_5\mathbf{C}(w_1,w_4,w_5) + c_1c_4c_6\,\mathbf{C}(w_1,w_4,w_6) \\ & + c_2c_3c_5\,\mathbf{C}(w_2,w_3,w_5) + c_2c_3c_6\,\mathbf{C}(w_2,w_3,w_6) \\ & + c_2c_4c_5\,\mathbf{C}(w_2,w_4,w_5) + c_2c_4c_6\,\mathbf{C}(w_2,w_4,w_6)\end{aligned} \tag{3}$$

The above property will be used extensively in the perturbation calculations and the order of terms is extremely important. The cubic operators may be of differential type or integral type or a mixture of both. Sample cubic nonlinearities which are encountered frequently in continuous systems are

$$\mathbf{C}(w,w,w) = w^3, \quad \mathbf{C}(w,w,w) = w'' \int_D w'^2 \, \mathrm{d}x, \quad \mathbf{C}(w,w,w) = w''w'^2$$

$$\mathbf{C}(\dot{w},w,w) = \dot{w}'w'w'', \quad C(\dot{w},w,w) = \dot{w}''w'^2 \tag{4}$$

In general, if the nonlinearity is not completely symmetric such as Eq. (4), a change of arguments will produce different results.

$$\begin{aligned}\mathbf{C}(w_1,w_2,w_3) \neq \mathbf{C}(w_1,w_3,w_2) \neq \mathbf{C}(w_2,w_1,w_3) \\ \neq \mathbf{C}(w_2,w_3,w_1) \neq \mathbf{C}(w_3,w_1,w_2) \neq \mathbf{C}(w_3,w_2,w_1)\end{aligned}$$

$$\mathbf{C}(\dot{w}_1,w_2,w_3) \neq \mathbf{C}(\dot{w}_1,w_3,w_2) \tag{5}$$

Note that although initially the arguments w are the same as expressed in Eq. (4), when explicit solutions are replaced with w in the subsequent analysis, each term might be different and the different terms are expressed as w_1, w_2 and w_3 in Eq. (3) and (5).

3. Perturbation solution

The method of multiple scales [11] is applied directly to the model to find the general solution of Eq. (1). The following expansion for $w(x,t)$ is assumed:

$$w(x,T_0,T_1;\varepsilon)=w_0(x,T_0,T_1)+\varepsilon w_1(x,T_0,T_1)+\cdots \tag{6}$$

where $T_0=t$ is the usual fast time scale and $T_1=\varepsilon t$ is the slow time scale. Time derivatives are expressed in terms of fast and slow time scales as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}=D_0+\varepsilon D_1+\cdots \tag{7}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}=D_0^2+2\varepsilon D_0 D_1+\cdots \tag{8}$$

Where $D_k=\partial/\partial T_k$. Inserting Eqs. (6)–(8) into Eqs. (1) and (2) and separating at each order of ε, one obtains

$$O(\varepsilon^0)$$

$$D_0^2 w_0+\mathbf{L}_1(w_0)+\mathbf{L}_2(D_0 w_0)=0 \tag{9}$$

$$\mathbf{B}_1(w_0)=0 \quad \text{at} \quad x=0, \quad \mathbf{B}_2(w_0) \quad \text{at} \quad x=1 \tag{10}$$

$$O(\varepsilon^1)$$

$$\begin{aligned} D_0^2 w_1+\mathbf{L}_1(w_1)+\mathbf{L}_2(D_0 w_1)= & -2D_0 D_1 w_0-\mu D_0 w_0-\mathbf{L}_2(D_1 w_0)+\mathbf{C}_1(w_0,w_0,w_0)\\ & +\mathbf{C}_2(D_0 w_0,w_0,w_0)+F\cos \Omega T_0 \end{aligned} \tag{11}$$

$$\mathbf{B}_1(w_1)=0 \quad \text{at} \quad x=0, \quad \mathbf{B}_2(w_1)=0 \quad \text{at} \quad x=1 \tag{12}$$

At $O(\varepsilon^0)$, the solution is

$$w_0(x,T_0,T_1)=A_n(T_1)\mathrm{e}^{\mathrm{i}\omega_n T_0}\,Y_n(x)+\bar{A}_n(T_1)\mathrm{e}^{-\mathrm{i}\omega_n T_0}\,\bar{Y}_n(x) \tag{13}$$

where A_n and $\bar{A}_n$ are complex amplitudes and their conjugates, respectively. $Y_n(x)$ satisfy the following equations and boundary conditions:

$$\mathbf{L}_1(Y_n)-\omega_n^2 Y_n+\mathrm{i}\omega_n\mathbf{L}_2(Y_n)=0, \quad n=1,2,\ldots \tag{14}$$

$$\mathbf{B}_1(Y_n)=0 \quad \text{at} \quad x=0, \quad \mathbf{B}_2(Y_n)=0 \quad \text{at} \quad x=1 \tag{15}$$

Due to the dissipative term at the zeroth order, $Y_n(x)$ may not be real and the complex conjugate of the function is incorporated in the zeroth-order solution (13). The above equation and boundary conditions constitute an eigenvalue–eigenfunction problem. ω_n are the eigenvalues and $Y_n(x)$ are the eigenfunctions of the system, respectively. For continuous system, it is clear that there are infinite number of eigenvalues and corresponding eigenfunctions. In this work one mode approximation is assumed with no internal resonances. Substituting Eq. (13) to the right hand side of Eq. (11), one has

$$\begin{aligned} D_0^2 w_1+\mathbf{L}_1(w_1)+\mathbf{L}_2(D_0 w_1)= & \{(-2\mathrm{i}\omega_n D_1 A_n-\mathrm{i}\mu\omega_n A_n)Y_n-D_1 A_n\mathbf{L}_2(Y_n)\}\mathrm{e}^{\mathrm{i}\omega_n T_0}\\ & +(A_n^3\,\mathrm{e}^{3\mathrm{i}\omega_n T_0}\,\mathbf{C}_1(Y_n,Y_n,Y_n)+A_n^2\bar{A}_n\,\mathrm{e}^{\mathrm{i}\omega_n T_0}\,\mathbf{C}_1(Y_n,Y_n,\bar{Y}_n)\\ & +A_n^2\bar{A}_n\,\mathrm{e}^{-\mathrm{i}\omega_n T_0}\,\mathbf{C}_1(\bar{Y}_n,Y_n,Y_n)+A_n^2\bar{A}_n\,\mathrm{e}^{\mathrm{i}\omega_n T_0}\,\mathbf{C}_1(Y_n,\bar{Y}_n,Y_n))\\ & +\mathrm{i}\omega_n(A_n^3\,\mathrm{e}^{3\mathrm{i}\omega_n T_0}\,\mathbf{C}_2(Y_n,Y_n,Y_n)+A_n^2\bar{A}_n\,\mathrm{e}^{\mathrm{i}\omega_n T_0}\,\mathbf{C}_2(Y_n,Y_n,\bar{Y}_n)\\ & -A_n^2\bar{A}_n\,\mathrm{e}^{-\mathrm{i}\omega_n T_0}\,\mathbf{C}_2(\bar{Y}_n,Y_n,Y_n)+A_n^2\bar{A}_n\,\mathrm{e}^{\mathrm{i}\omega_n T_0}\,\mathbf{C}_2(Y_n,\bar{Y}_n,Y_n))\\ & +\tfrac{1}{2}F\,\mathrm{e}^{\mathrm{i}\Omega T_0}+cc \end{aligned} \tag{16}$$

$$\mathbf{B}_1(w_1)=0 \quad \text{at} \quad x=0, \quad \mathbf{B}_2(w_1)=0 \quad \text{at} \quad x=1 \tag{17}$$

where cc stands for complex conjugates of the preceding terms. At order ε, assuming primary resonances, the nearness of external excitation frequency to one of the natural frequencies is expressed as follows:

$$\Omega=\omega_n+\varepsilon\sigma \tag{18}$$

where σ is a detuning parameter of order 1. Substituting Eq. (18) into Eq. (16), after re-arranging yields

$$\begin{aligned} D_0^2 w_1 + \mathbf{L}_1(w_1) + \mathbf{L}_2(D_0 w_1) = & -\mathrm{i}\omega_n(2D_1A_n + \mu A_n)\mathrm{e}^{\mathrm{i}\omega_n T_0} Y_n \\ & - D_1 A_n \,\mathrm{e}^{\mathrm{i}\omega_n T_0}\, \mathbf{L}_2(Y_n) + A_n^2 \bar{A}_n \mathrm{e}^{\mathrm{i}\omega_n T_0}\{C_1(Y_n, Y_n, \bar{Y}_n) \\ & + \mathbf{C}_1(Y_n, \bar{Y}_n, Y_n) + \mathbf{C}_1(\bar{Y}_n, Y_n, Y_n) + \mathrm{i}\omega_n[\mathbf{C}_2(Y_n, Y_n, \bar{Y}_n) \\ & + \mathbf{C}_2(Y_n, \bar{Y}_n, Y_n) - \mathbf{C}_2(\bar{Y}_n, Y_n, Y_n)]\} + \tfrac{1}{2}F\,\mathrm{e}^{\mathrm{i}\omega_n T_0}\,\mathrm{e}^{\mathrm{i}\sigma T_1} \\ & + cc + N.S.T \end{aligned} \tag{19}$$

where *N.S.T* stands for non-secular terms. One assumes a solution for w_1 of the form

$$w_1(x, T_0, T_1) = \phi_1(x, T_1)\mathrm{e}^{\mathrm{i}\omega_n T_0} + cc + W(x, T_0, T_1) \tag{20}$$

where $W(x, T_0, T_1)$ represent solution associated with non-secular terms. Substituting Eq. (20) into Eq. (19) yields

$$\begin{aligned} \mathbf{L}_1(\phi_1) - \omega_n^2\phi_1 + \mathrm{i}\omega_n \mathbf{L}_2(\phi_1) = & -\mathrm{i}\omega_n(2D_1A_n + \mu A_n)Y_n - D_1A_n\mathbf{L}_2(Y_n) \\ & + A_n^2\bar{A}_n\{\mathbf{C}_1(Y_n, Y_n, \bar{Y}_n) + \mathbf{C}_1(Y_n, \bar{Y}_n, Y_n) + \mathbf{C}_1(\bar{Y}_n, Y_n, Y_n) \\ & + \mathrm{i}\omega_n[\mathbf{C}_2(Y_n, Y_n, \bar{Y}_n) + \mathbf{C}_2(Y_n, \bar{Y}_n, Y_n) - \mathbf{C}_2(\bar{Y}_n, Y_n, Y_n)]\} \\ & + \tfrac{1}{2}F\,\mathrm{e}^{\mathrm{i}\sigma T_1} \end{aligned} \tag{21}$$

$$\mathbf{B}_1(\phi_1) = 0 \quad \text{at } x = 0, \quad \mathbf{B}_2(\phi_1) = 0 \quad \text{at } x = 1 \tag{22}$$

Since the homogenous part of Eq. (21) have non-trivial solutions, non-homogenous Eq. (21) have a solution only if a solvability condition is satisfied [11]. For the present model the solvability condition is

$$D_1A_n + \mu k_1 A_n - k_2 A_n^2 \bar{A}_n - \tfrac{1}{2}f\,\mathrm{e}^{\mathrm{i}\sigma T_1} = 0 \tag{23}$$

The coefficients are as follows:

$$k_1 = \frac{\mathrm{i}\omega_n \int_0^1 Y_n\bar{Y}_n\,\mathrm{d}x}{2\mathrm{i}\omega_n \int_0^1 Y_n\bar{Y}_n\,\mathrm{d}x + \int_0^1 \mathbf{L}_2(Y_n)\bar{Y}_n\,\mathrm{d}x} \tag{24}$$

$$\begin{aligned} k_2 = & \left\{\int_0^1 \bar{Y}_n\{\mathbf{C}_1(Y_n, Y_n, \bar{Y}_n) + \mathbf{C}_1(Y_n, \bar{Y}_n, Y_n) + \mathbf{C}_1(\bar{Y}_n, Y_n, Y_n) + \mathrm{i}\omega_n[\mathbf{C}_2(Y_n, Y_n, \bar{Y}_n)\right. \\ & \left. + \mathbf{C}_2(Y_n, \bar{Y}_n, Y_n) - \mathbf{C}_2(\bar{Y}_n, Y_n, Y_n)]\}\mathrm{d}x\right\} \Big/ \left\{2\mathrm{i}\omega_n \int_0^1 Y_n\bar{Y}_n\,\mathrm{d}x + \int_0^1 \mathbf{L}_2(Y_n)\bar{Y}_n\,\mathrm{d}x\right\} \end{aligned} \tag{25}$$

$$f = \frac{\int_0^1 F\bar{Y}_n\,\mathrm{d}x}{2\mathrm{i}\omega_n \int_0^1 Y_n\bar{Y}_n\,\mathrm{d}x + \int_0^1 \mathbf{L}_2(Y_n)\bar{Y}_n\,\mathrm{d}x} \tag{26}$$

The coefficients have real and imaginary parts

$$k_1 = k_{1R} + \mathrm{i}k_{1I}, \quad k_2 = k_{2R} + \mathrm{i}k_{2I}, \quad f = f_R + \mathrm{i}f_I \tag{27}$$

Representing the complex amplitudes in polar form

$$A_n = \tfrac{1}{2}a_n(T_1)\mathrm{e}^{\mathrm{i}\beta_n(T_1)} \tag{28}$$

substituting into Eq. (23), separating into real and imaginary parts, one finally has the amplitude and phase modulation equations

$$a_n' = -\mu k_{1R}a_n + \tfrac{1}{4}k_{2R}a_n^3 + f_R\cos\gamma_n - f_I\sin\gamma_n \tag{29}$$

$$a_n\gamma_n' = a_n\sigma + \mu k_{1I}a_n - \tfrac{1}{4}k_{2I}a_n^3 - f_R\sin\gamma_n - f_I\cos\gamma_n \tag{30}$$

where

$$\gamma_n = \sigma T_1 - \beta_n \tag{31}$$

For steady-state solutions

$$a_n' = \gamma_n' = 0 \tag{32}$$

Substituting Eq. (32) into Eqs. (29) and (30), one has

$$\mu k_{1R}a_n - \tfrac{1}{4}k_{2R}a_n^3 - f_R\cos\gamma_n + f_I\sin\gamma_n = 0 \tag{33}$$

$$\sigma a_n + \mu k_{1I} a_n - \tfrac{1}{4} k_{2I} a_n^3 - f_R \sin\gamma_n - f_I \cos\gamma_n = 0 \tag{34}$$

Elimination of γ_n between the equations yield

$$\sigma = -\mu k_{1I} + \frac{1}{4} k_{2I} a_n^2 \mp \sqrt{\frac{f_R^2 + f_I^2}{a_n^2} - \left(\mu k_{1R} - \frac{1}{4} k_{2R} a_n^2\right)^2} \tag{35}$$

For analysing the stability of the system, Eqs. (29) and (30) are rewritten as follows:

$$a_n' = -\mu k_{1R} a_n + \tfrac{1}{4} k_{2R} a_n^3 + f_R \cos\gamma_n - f_I \sin\gamma_n = F_1(a_n, \gamma_n) \tag{36}$$

$$\gamma' = \sigma + \mu k_{1I} - \frac{1}{4} k_{2I} a_n^2 - \frac{1}{a_n} f_R \sin\gamma_n - \frac{1}{a_n} f_I \cos\gamma_n = F_2(a_n, \gamma_n) \tag{37}$$

To determine the stability of fixed points, the Jacobian matrix is constructed

$$\begin{bmatrix} \frac{\partial F_1}{\partial a_n} & \frac{\partial F_1}{\partial \gamma_n} \\ \frac{\partial F_2}{\partial a_n} & \frac{\partial F_2}{\partial \gamma_n} \end{bmatrix}_{\substack{a_n = a_{n0} \\ \gamma_n = \gamma_{n0}}} \tag{38}$$

Eigenvalues of the Jacobian matrix should not have positive real parts for stability. The approximate solution of the system is

$$w(x, t; \varepsilon) = a_n \cos(\Omega t - \gamma_n) Y_{nR} - a_n \sin(\Omega t - \gamma_n) Y_{nI} + O(\varepsilon) \tag{39}$$

where Y_n can be decomposed into its real and imaginary parts

$$Y_n = Y_{nR} + \mathrm{i} Y_{nI} \tag{40}$$

a_n and γ_n in the approximate solution are governed by Eqs. (36) and (37). Hence, for the general problem, an approximate solution algorithm is developed. The algorithm will be applied to two specific problems in the next section. Note that the approximate solution developed can trivially be extended to three spatial dimensions by expressing the eigenfunctions in 3-D. A restriction of the solution is the absence of internal resonances and natural frequencies should be checked to avoid such resonances before direct implication of the algorithm.

4. Applications

In this section, the general solution algorithm will be applied to two specific vibration problems. Both problems are from axially moving continua. See Refs. [12–20] for some example studies on axially moving continua vibrations. One of the problems contains integro-differential type nonlinearity and the other differential type nonlinearity. Although the nonlinearities are much different in nature, their common feature of being cubic nonlinearity makes them suitable applications for the general model considered.

4.1. Axially moving Euler–Bernoulli beam

For an axially moving Euler–Bernoulli beam, following Ref. [12], the kinetic energy is

$$T = \frac{\rho A}{2} \int_0^L \{[\dot{u} + v_0(1 + u')]^2 + (\dot{w} + v_0 w')^2\}\, \mathrm{d}x \tag{41}$$

where ρA is the mass per unit length of the beam. $u(x, t)$ and $w(x, t)$ are the longitudinal and transverse displacements of the beam, dot denotes differentiation with respect to time and prime denotes differentiation with respect to the spatial variable. v_0 is the axial transport velocity which is constant. The first term in the parenthesis is the longitudinal and the second term is the transverse velocity components as measured by a stationary observer. The potential energy is

$$U = \int_0^L \left(P e_{xx} + \frac{1}{2} E A e_{xx}^2 + \frac{1}{2} E I w''^2\right) \mathrm{d}x \tag{42}$$

where the nonlinear strain is

$$e_{xx} = u' + \tfrac{1}{2} w'^2 \tag{43}$$

where P is the axial tension force, EI the flexural rigidity and EA the axial stiffness. Invoking the Hamilton's principle

$$\delta \int_{t_1}^{t_2} (T-U)\mathrm{d}t = 0 \tag{44}$$

and introducing the dimensionless quantities

$$u^* = \frac{u}{L}, \quad w^* = \frac{w}{L}, \quad x^* = \frac{x}{L}, \quad t^* = t\sqrt{P/\rho AL^2} \tag{45}$$

with the new dimensionless parameters

$$v_0^* = v_0/\sqrt{P/\rho A}, \quad v_\ell = \sqrt{EA/P}, \quad v_f = \sqrt{EI/PL^2} \tag{46}$$

yield finally the coupled equations

$$\ddot{u} + 2v_0\dot{u}' + v_0^2 u'' - v_\ell^2(u' + \tfrac{1}{2}w'^2)' = 0 \tag{47}$$

$$\ddot{w} + 2v_0\dot{w}' + v_0^2 w'' - \{[1 + v_\ell^2(u' + \tfrac{1}{2}w'^2)]w'\}' + v_f^2 w^{IV} = 0 \tag{48}$$

Asterisk notation is dropped for convenience. v_ℓ is the longitudinal and v_f the flexural stiffness parameter. The boundary conditions for the problem are

$$u(0,t) = u(1,t) = 0, \quad w(0,t) = w(1,t) = w''(0,t) = w''(1,t) = 0 \tag{49}$$

The explicit appearance of u in the transverse equation of motion is suppressed by approximating the dynamic tension component as a function of time alone. Over a technologically useful range of parameter values, longitudinal disturbances propagate significantly faster than the transverse ones. On the time scales of the lower transverse modes, tension variations propagate almost instantaneously as the influence of longitudinal inertia is small. With this in mind, Eq. (47) can be integrated with the axial strain and longitudinal displacement yielding

$$e_{xx} = \frac{1}{2}\int_0^1 w'^2\,\mathrm{d}x \tag{50}$$

$$u(x,t) = \frac{x}{2}\int_0^1 w'^2 - \frac{1}{2}\int_0^x w'^2\,\mathrm{d}x \tag{51}$$

The final Eq. (48) reads

$$\ddot{w} + (v_0^2 - 1)w'' + 2v_0\dot{w}' + v_f^2 w^{IV} = \frac{1}{2}v_\ell^2 w''\int_0^1 w'^2\,\mathrm{d}x \tag{52}$$

Adding a viscous damping and harmonic excitation to the equation with introducing a book-keeping small parameter ε to re-order the relative quantities of the terms yields

$$\ddot{w} + (v_0^2 - 1)w'' + 2v_0\dot{w}' + v_f^2 w^{IV} + \varepsilon\mu\dot{w} = \varepsilon F\cos\Omega t + \frac{1}{2}\varepsilon v_\ell^2 w''\int_0^1 w'^2\,\mathrm{d}x \tag{53}$$

$$w(0,t) = w(1,t) = w''(0,t) = w''(1,t) = 0 \tag{54}$$

For some example studies on axially moving Euler Bernoulli beams see [12–15]. For this special problem, the operators are defined to be as follows:

$$\mathbf{L}_1(w) = (v_0^2 - 1)w'' + v_f^2 w^{IV} \tag{55}$$

$$\mathbf{L}_2(\dot{w}) = 2v_0\dot{w}' \tag{56}$$

$$\mathbf{C}_1(w,w,w) = \frac{1}{2}v_\ell^2 w''\int_0^1 w'^2\,\mathrm{d}x \tag{57}$$

$$\mathbf{C}_2(\dot{w},w,w) = 0 \tag{58}$$

The associated eigenfunction–eigenvalue problem given in Eqs. (14) and (15) reduces to

$$v_f^2 Y_n^{IV} + (v_0^2 - 1)Y_n'' + 2v_0\mathrm{i}\omega_n Y_n' - \omega_n^2 Y_n = 0 \tag{59}$$

$$Y_n(0) = Y_n(1) = Y_n''(0) = Y_n''(1) = 0 \tag{60}$$

The solution is

$$Y_n(x) = C_{1n}\,e^{i\beta_{1n}x} + C_{2n}\,e^{i\beta_{2n}x} + C_{3n}\,e^{i\beta_{3n}x} + C_{4n}\,e^{i\beta_{4n}x} \tag{61}$$

where β_{in} satisfy the dispersive relation

$$v_f^2\beta_{in}^4 + (1 - v_0^2)\beta_{in}^2 - 2v_0\omega_n\beta_{in} - \omega_n^2 = 0, \quad i = 1,2,3,4\ldots, \quad n = 1,2,\ldots \tag{62}$$

Applying the simply supported boundary conditions yield

$$C_{1n} + C_{2n} + C_{3n} + C_{4n} = 0 \tag{63}$$

$$C_{1n}\beta_{1n}^2 + C_{2n}\beta_{2n}^2 + C_{3n}\beta_{3n}^2 + C_{4n}\beta_{4n}^2 = 0 \tag{64}$$

$$C_{1n}\,e^{i\beta_{1n}} + C_{2n}\,e^{i\beta_{2n}} + C_{3n}\,e^{i\beta_{3n}} + C_{4n}\,e^{i\beta_{4n}} = 0 \tag{65}$$

$$C_{1n}\beta_{1n}^2\,e^{i\beta_{1n}} + C_{2n}\beta_{2n}^2\,e^{i\beta_{2n}} + C_{3n}\beta_{3n}^2\,e^{i\beta_{3n}} + C_{4n}\beta_{4n}^2\,e^{i\beta_{4n}} = 0 \tag{66}$$

The support condition is found by nontrivial solutions of Eqs. (63)–(66)

$$\begin{aligned}&(e^{i(\beta_{1n}+\beta_{2n})} + e^{i(\beta_{3n}+\beta_{4n})})(\beta_{1n}^2 - \beta_{2n}^2)(\beta_{3n}^2 - \beta_{4n}^2)\\ &\quad + (e^{i(\beta_{1n}+\beta_{3n})} + e^{i(\beta_{2n}+\beta_{4n})})(\beta_{2n}^2 - \beta_{4n}^2)(\beta_{3n}^2 - \beta_{1n}^2)\\ &\quad + (e^{i(\beta_{2n}+\beta_{3n})} + e^{i(\beta_{1n}+\beta_{4n})})(\beta_{1n}^2 - \beta_{4n}^2)(\beta_{2n}^2 - \beta_{3n}^2) = 0\end{aligned} \tag{67}$$

ω_n and β_{in} can be numerically calculated by using the dispersive relation and the support condition. Coefficients are obtained by elimination from Eqs. (63)–(66) . Finally the mode shapes are [13]

$$\begin{aligned}Y_n(x) = c_1\Bigg\{&e^{i\beta_{1n}x} - \frac{(\beta_{4n}^2 - \beta_{1n}^2)(e^{i\beta_{3n}} - e^{i\beta_{1n}})}{(\beta_{4n}^2 - \beta_{2n}^2)(e^{i\beta_{3n}} - e^{i\beta_{2n}})}\,e^{i\beta_{2n}x} - \frac{(\beta_{4n}^2 - \beta_{1n}^2)(e^{i\beta_{2n}} - e^{i\beta_{1n}})}{(\beta_{4n}^2 - \beta_{3n}^2)(e^{i\beta_{2n}} - e^{i\beta_{3n}})}\,e^{i\beta_{3n}x}\\ &+ \left[-1 + \frac{(\beta_{4n}^2 - \beta_{1n}^2)(e^{i\beta_{3n}} - e^{i\beta_{1n}})}{(\beta_{4n}^2 - \beta_{2n}^2)(e^{i\beta_{3n}} - e^{i\beta_{2n}})} + \frac{(\beta_{4n}^2 - \beta_{1n}^2)(e^{i\beta_{2n}} - e^{i\beta_{1n}})}{(\beta_{4n}^2 - \beta_{3n}^2)(e^{i\beta_{2n}} - e^{i\beta_{3n}})}\right] e^{i\beta_{4n}(x)}\Bigg\}\end{aligned} \tag{68}$$

In Fig. 1, fundamental frequency versus axial transport velocity graphics are shown for different flexural stiffness values. Natural frequencies decrease with increasing flexural stiffness values. Frequency-response graphics are shown in Figs. 2–4. All curves show the hardening effect of nonlinear system.

The next step is to calculate the coefficients in the amplitude and phase modulation equations. Substituting the operators (55)–(58) into Eqs.(24)–(26) yields

$$k_1 = \frac{i\omega_n \int_0^1 Y_n\bar{Y}_n\,dx}{2i\omega_n \int_0^1 Y_n\bar{Y}_n\,dx + 2v_0\int_0^1 Y_n'\bar{Y}_n\,dx} \tag{69}$$

$$k_2 = \left\{\frac{1}{2}v_\ell^2 \int_0^1 \bar{Y}_n\left\{2Y_n''\int_0^1 Y_n'\bar{Y}_n'\,dx + \bar{Y}_n''\int_0^1 Y_n'^2\,dx\right\}dx\right\}\Bigg/\left\{2i\omega_n\int_0^1 Y_n\bar{Y}_n\,dx + 2v_0\int_0^1 Y_n'\bar{Y}_n\,dx\right\} \tag{70}$$

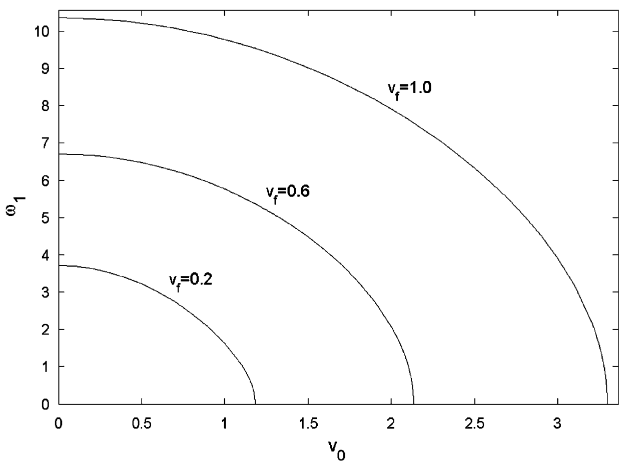

Fig. 1. Fundamental frequencies versus axial transport velocities for various flexural stiffness values.

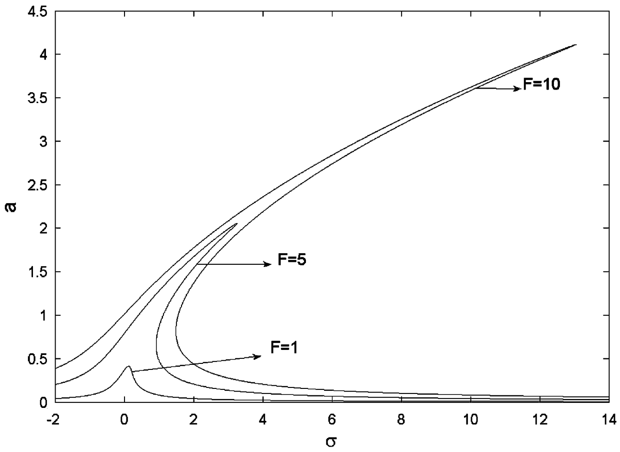

Fig. 2. Frequence-response curves for various external excitation values ($\upsilon_f = 0.2,\ \upsilon_\ell = 0.2,\ \mu = 0.5,\ \upsilon_0 = 0.8$).

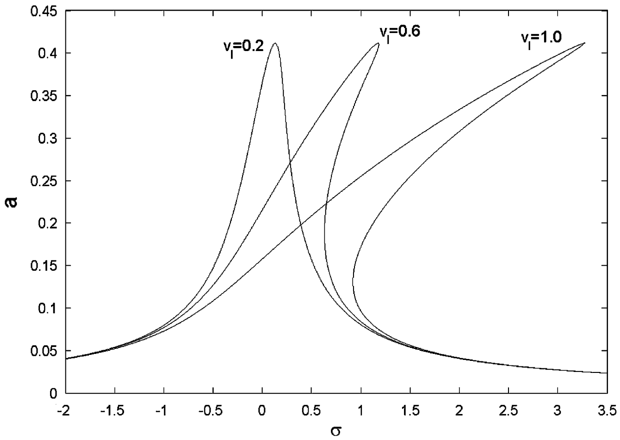

Fig. 3. Frequence-response curves for various longitudinal stiffness values ($\upsilon_f = 0.2,\ \mu = 0.5,\ \upsilon_0 = 0.8,\ F = 1$).

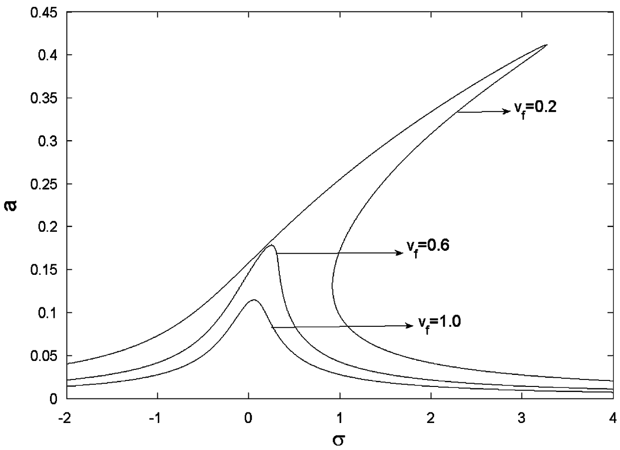

Fig. 4. Frequence-response curves for various flexural stiffness values ($\upsilon_\ell = 1.0,\ \mu = 0.5,\ \upsilon_0 = 0.8,\ F = 1$).

$$f = \frac{\int_0^1 F\bar{Y}_n\,\mathrm{d}x}{2\mathrm{i}\omega_n \int_0^1 Y_n\bar{Y}_n\,\mathrm{d}x + 2\upsilon_0 \int_0^1 Y_n'\bar{Y}_n\,\mathrm{d}x} \tag{71}$$

Substitution of $Y_n(x)$ further into these equations yields the numerical values for the coefficients.

In Fig. 2, frequency response curves are drawn for specific values of external excitation amplitude $F = 1$, 5 and 10 with $v_f = 0.2$, $v_\ell = 0.2$, $v_0 = 0.8$, $\mu = 0.5$, $\omega_1 = 2.42739$. The coefficients of amplitude and phase modulation equations for $F = 1$ are

$$k_1 = 0.3962, \quad k_2 = 3.0865\mathrm{i}, \quad f = -0.0640 + 0.0504\mathrm{i} \tag{72}$$

Curves show that nonlinear effects increase by increasing external excitation.

Fig. 3 shows the effects of longitudinal stiffness values on the frequency response curves. Increasing longitudinal stiffness bends the curves more to the right, increasing the multi-valued region responsible for the well known jump phenomenon.

Fig. 4 shows the influence of flexural stiffness on the frequency response curves. The maximum amplitudes decrease for increasing flexural stiffness. Note that the solutions presented are valid in the subcritical velocity regimes.

4.2. Axially moving viscoelastic beam

The second model represents nonlinear vibrations of an axially moving viscoelastic beam. Following Ref. [17], Newton's second law of motion for a uniform axially moving beam is

$$\rho A(\ddot{w} + 2v_0\dot{w}' + v_0^2 w'') = [(P + A\sigma)w']' - M'' \tag{73}$$

where $\sigma(x,t)$ is the axial disturbed stress and $M(x,t)$ is the bending moment. Other notation is the same as in the previous example. Viscoelastic material obeys the Kelvin model

$$\sigma = Ee_{xx} + \eta\dot{e}_{xx} \tag{74}$$

with the strain

$$e_{xx} = \tfrac{1}{2}w'^2 \tag{75}$$

which is used to account for geometric nonlinearity due to small but finite stretching of the beam. For a slender beam, the linear moment–curvature relationship can be used

$$M = EIw'' + \eta I\dot{w}'' \tag{76}$$

Substituting Eqs. (74)–(76) into Eq. (73) yields the equation of transverse motion of an axially moving viscoelastic beam

$$\rho A(\ddot{w} + 2v_0\dot{w}' + v_0^2 w'') - Pw'' + EIw^{IV} + \eta I\dot{w}^{IV} = \tfrac{3}{2}Ew'^2 w'' + 2\eta w'\dot{w}'w'' + \eta w'^2\dot{w}'' \tag{77}$$

Introducing similar non-dimensionless parameters as in the previous example, adding a harmonic external excitation and reordering nonlinear and harmonic excitation terms with a book keeping small parameter yields finally

$$\ddot{w} + (v_0^2 - 1)w'' + 2v_0\dot{w}' + v_f^2 w^{IV} + \alpha\dot{w}^{IV}\varepsilon\mu\dot{w} = \varepsilon F\cos\Omega t + \varepsilon\{\tfrac{3}{2}v_\ell^2 w''w'^2 + 2\alpha k\dot{w}'w'w'' + \alpha k\dot{w}''w'^2\} \tag{78}$$

with boundary conditions

$$w(0,t) = w(1,t) = w''(0,t) = w''(1,t) = 0 \tag{79}$$

where

$$v_\ell = \sqrt{\frac{EA}{P}}, \quad v_f = \sqrt{\frac{EI}{PL^2}}, \quad \alpha = \frac{I\eta}{L^3\sqrt{\rho AP}}, \quad k = \frac{A\eta}{L\sqrt{\rho AP}} \tag{80}$$

η represents viscosity, α and k are dimensionless parameters related to the viscosity. For similar studies on viscoelastic axially moving beams, the reader is referred to Refs. [17–20]. Note that if the spatial variation of tension is rather small, one can use the averaged value of the disturbed tension

$$\frac{1}{L}\int_0^L A\sigma\,\mathrm{d}x$$

to replace the exact value of $A\sigma$ in Eq. (73). Selecting this choice and neglecting viscoelastic terms yield exactly the same integro-differential model discussed in the previous example [17].

For this specific problem, the general operators are defined as

$$\mathbf{L}_1(w) = (v_0^2 - 1)w'' + v_f^2 w^{IV} \tag{81}$$

$$\mathbf{L}_2(\dot{w}) = 2v_0\dot{w}' + \alpha\dot{w}^{IV} \tag{82}$$

$$\mathbf{C}_1(w,w,w) = \tfrac{3}{2}v_\ell^2 w''w'^2 \tag{83}$$

$$\mathbf{C}_2(\dot{w},w,w) = 2\alpha k\dot{w}'w'w'' + \alpha k\dot{w}''w'^2 \tag{84}$$

The associated eigenvalue–eigenfunction problem as given in Eqs. (14) and (15) reduces to

$$(v_f^2 + \mathrm{i}\omega_n\alpha)Y_n^{IV} + (v_0^2 - 1)Y_n'' + 2v_0\mathrm{i}\omega_n Y_n' - \omega_n^2 Y_n = 0 \tag{85}$$

$$Y_n(0) = Y_n(1) = Y_n''(0) = Y_n''(1) = 0 \tag{86}$$

for which a solution is assumed of the below form

$$Y_n(x) = C_{1n}\,\mathrm{e}^{\mathrm{i}\beta_{1n}x} + C_{2n}\,\mathrm{e}^{\mathrm{i}\beta_{2n}x} + C_{3n}\,\mathrm{e}^{\mathrm{i}\beta_{3n}x} + C_{4n}\,\mathrm{e}^{\mathrm{i}\beta_{4n}x} \tag{87}$$

where β_{in} satisfies the dispersive relation

$$(v_f^2 + \mathrm{i}\omega_n\alpha)\beta_{in}^4 + (1 - v_0^2)\beta_{in}^2 - 2v_0\omega_n\beta_{in} - \omega_n^2 = 0, \quad i = 1,2,3,4\ldots, \; n = 1,2,\ldots \tag{88}$$

After applying the simply supported boundary conditions, the following equations will be found for the coefficients

$$C_{1n} + C_{2n} + C_{3n} + C_{4n} = 0 \tag{89}$$

$$C_{1n}\beta_{1n}^2 + C_{2n}\beta_{2n}^2 + C_{3n}\beta_{3n}^2 + C_{4n}\beta_{4n}^2 = 0 \tag{90}$$

$$C_{1n}\,\mathrm{e}^{\mathrm{i}\beta_{1n}} + C_{2n}\,\mathrm{e}^{\mathrm{i}\beta_{2n}} + C_{3n}\,\mathrm{e}^{\mathrm{i}\beta_{3n}} + C_{4n}\,\mathrm{e}^{\mathrm{i}\beta_{4n}} = 0 \tag{91}$$

$$C_{1n}\beta_{1n}^2\,\mathrm{e}^{\mathrm{i}\beta_{1n}} + C_{2n}\beta_{2n}^2\,\mathrm{e}^{\mathrm{i}\beta_{2n}} + C_{3n}\beta_{3n}^2\,\mathrm{e}^{\mathrm{i}\beta_{3n}} + C_{4n}\beta_{4n}^2\,\mathrm{e}^{\mathrm{i}\beta_{4n}} = 0 \tag{92}$$

The support condition is also found by nontrivial solution condition of Eqs. (89)–(92)

$$\begin{aligned}&(\mathrm{e}^{\mathrm{i}(\beta_{1n}+\beta_{2n})} + \mathrm{e}^{\mathrm{i}(\beta_{3n}+\beta_{4n})})(\beta_{1n}^2 - \beta_{2n}^2)(\beta_{3n}^2 - \beta_{4n}^2)\\ &\quad + (\mathrm{e}^{\mathrm{i}(\beta_{1n}+\beta_{3n})} + \mathrm{e}^{\mathrm{i}(\beta_{2n}+\beta_{4n})})(\beta_{2n}^2 - \beta_{4n}^2)(\beta_{3n}^2 - \beta_{1n}^2)\\ &\quad + (\mathrm{e}^{\mathrm{i}(\beta_{2n}+\beta_{3n})} + \mathrm{e}^{\mathrm{i}(\beta_{1n}+\beta_{4n})})(\beta_{1n}^2 - \beta_{4n}^2)(\beta_{2n}^2 - \beta_{3n}^2) = 0\end{aligned} \tag{93}$$

Numerical values of ω_n and β_{in} can be numerically calculated by using the dispersive relation and the support condition in a similar way. Coefficients can be obtained by elimination of Eqs. (89)–(92). Finally the mode shapes are found to be

$$\begin{aligned}Y_n(x) = c_1\Bigg\{&\mathrm{e}^{\mathrm{i}\beta_{1n}x} - \frac{(\beta_{4n}^2 - \beta_{1n}^2)(\mathrm{e}^{\mathrm{i}\beta_{3n}} - \mathrm{e}^{\mathrm{i}\beta_{1n}})}{(\beta_{4n}^2 - \beta_{2n}^2)(\mathrm{e}^{\mathrm{i}\beta_{3n}} - \mathrm{e}^{\mathrm{i}\beta_{2n}})}\,\mathrm{e}^{\mathrm{i}\beta_{2n}x} - \frac{(\beta_{4n}^2 - \beta_{1n}^2)(\mathrm{e}^{\mathrm{i}\beta_{2n}} - \mathrm{e}^{\mathrm{i}\beta_{1n}})}{(\beta_{4n}^2 - \beta_{3n}^2)(\mathrm{e}^{\mathrm{i}\beta_{2n}} - \mathrm{e}^{\mathrm{i}\beta_{3n}})}\,\mathrm{e}^{\mathrm{i}\beta_{3n}x}\\ &+ \left[-1 + \frac{(\beta_{4n}^2 - \beta_{1n}^2)(\mathrm{e}^{\mathrm{i}\beta_{3n}} - \mathrm{e}^{\mathrm{i}\beta_{1n}})}{(\beta_{4n}^2 - \beta_{2n}^2)(\mathrm{e}^{\mathrm{i}\beta_{3n}} - \mathrm{e}^{\mathrm{i}\beta_{2n}})} + \frac{(\beta_{4n}^2 - \beta_{1n}^2)(\mathrm{e}^{\mathrm{i}\beta_{2n}} - \mathrm{e}^{\mathrm{i}\beta_{1n}})}{(\beta_{4n}^2 - \beta_{3n}^2)(\mathrm{e}^{\mathrm{i}\beta_{2n}} - \mathrm{e}^{\mathrm{i}\beta_{3n}})}\right]\mathrm{e}^{\mathrm{i}\beta_{4n}x}\Bigg\}\end{aligned} \tag{94}$$

The coefficients of amplitude and phase modulation equations are found by substituting the operators (81)–(84) into Eqs. (24)–(26)

$$k_1 = \frac{\mathrm{i}\omega_n \int_0^1 Y_n\bar{Y}_n\,\mathrm{d}x}{2\mathrm{i}\omega_n \int_0^1 Y_n\bar{Y}_n\,\mathrm{d}x + 2v_0\int_0^1 Y_n'\bar{Y}_n\,\mathrm{d}x + \alpha\int_0^1 Y_n^{IV}\bar{Y}_n\,\mathrm{d}x} \tag{95}$$

$$\begin{aligned}k_2 = &\int_0^1 \bar{Y}_n\left\{3v_\ell^2 Y_n''Y_n'\bar{Y}_n' + \frac{3}{2}v_\ell^2\bar{Y}_n''Y_n'^2 + \alpha ki\omega_n Y_n'^2\bar{Y}_n''2\alpha kiw_n Y_n''Y_n'\bar{Y}_n'\right\}\mathrm{d}x\\ &/\left\{2i\omega_n\int_0^1 Y_n\bar{Y}_n\,\mathrm{d}x + 2v_0\int_0^1 Y_n'\bar{Y}_n\,\mathrm{d}x + \alpha\int_0^1 Y_n^{IV}\bar{Y}_n\,\mathrm{d}x\right\}\end{aligned} \tag{96}$$

$$f = \frac{\int_0^1 F\bar{Y}_n\,\mathrm{d}x}{2\mathrm{i}\omega_n \int_0^1 Y_n\bar{Y}_n\,\mathrm{d}x + 2v_0\int_0^1 Y_n'\bar{Y}_n\,\mathrm{d}x + \alpha\int_0^1 Y_n^{IV}\bar{Y}_n\,\mathrm{d}x} \tag{97}$$

By substituting $Y_n(x)$ further into these equations yields the numerical values for the coefficients.

Figs. 5 and 6 show the fundamental frequency and mean velocity relation of axially moving viscoelastic beam for different flexural stiffness by taking $\alpha = 0.001$ and 0.05. For an increase in α values, fundamental frequencies decrease for each flexural stiffness. Variation of frequencies with velocity as given in Figs. 5 and 6 has not been reported before. Frequency-response curves are shown in Figs. 7–10.

In Fig. 7, nonlinear effect of k parameter is shown for specific values of $v_f = 1.0$, $v_\ell = 0.2$, $v_0 = 0.5$, $\mu = 0.2$, $\omega_1 = 10.20142$, $F = 1$, $\alpha = 0.01$.

The coefficients for $k = 1$ are

$$k_1 = 0.4926 + 0.0238\mathrm{i}, \quad k_2 = -0.9965 + 24.8335\mathrm{i}, \quad f = -0.0284 + 0.0027\mathrm{i} \tag{98}$$

Curves show that maximum amplitudes decrease by increasing k values.

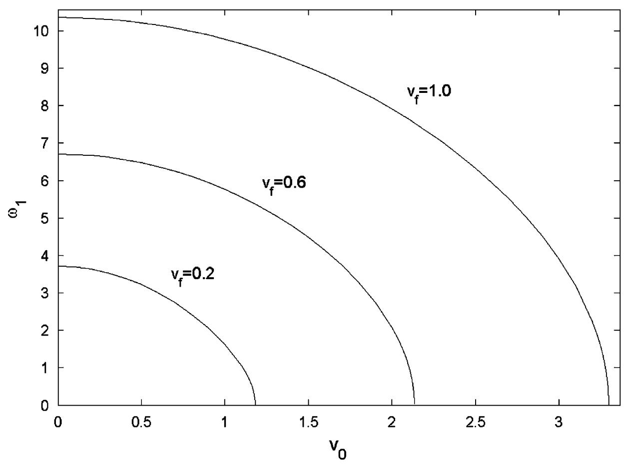

Fig. 5. Fundamental frequencies versus axial transport velocities for various flexural stiffness values ($\alpha = 0.001$).

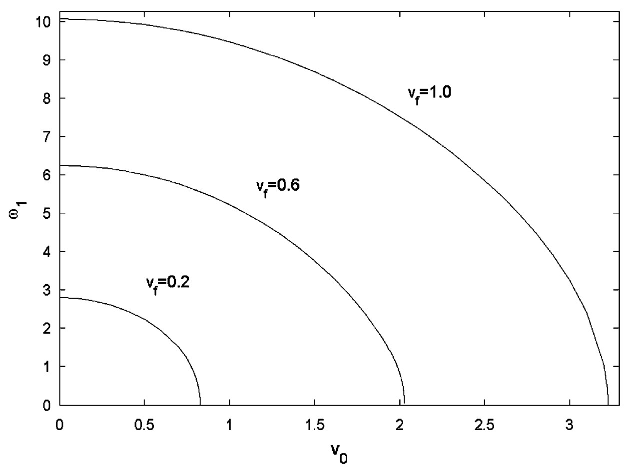

Fig. 6. Fundamental frequencies versus axial transport velocities for various flexural stiffness values ($\alpha = 0.05$).

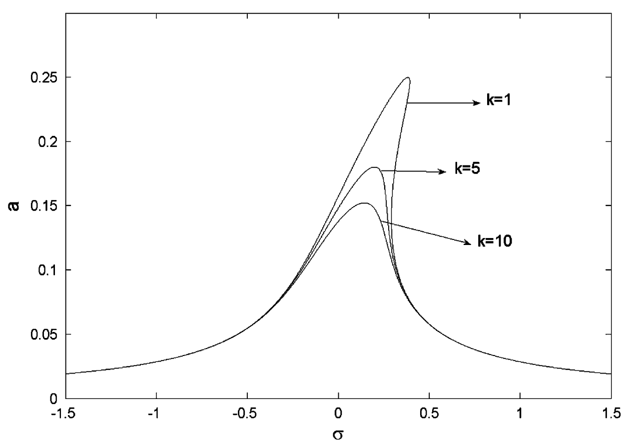

Fig. 7. Frequence-response curves for various k parameter values ($v_f = 1.0,\ v_\ell = 0.2,\ \mu = 0.2,\ v_0 = 0.5,\ \alpha = 0.01,\ F = 1$).

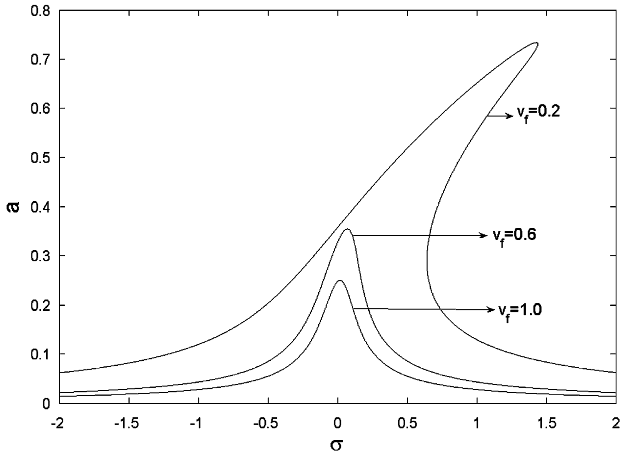

Fig. 8. Frequence-response curves for various flexural stiffness values ($v_\ell = 0.2$, $\mu = 0.2$, $v_0 = 1.1$, $\alpha = 0.001$, $k = 5$, $F = 1$).

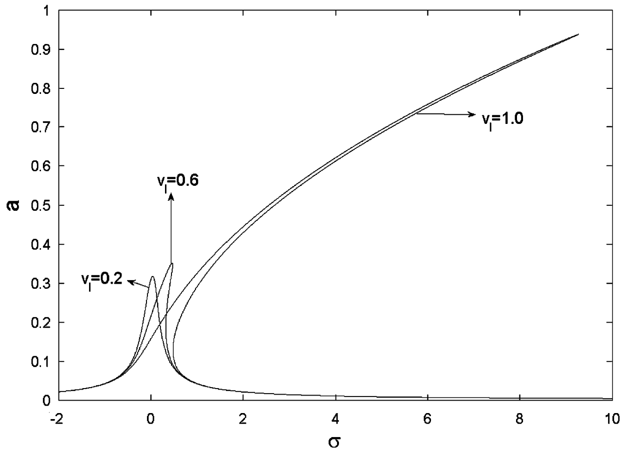

Fig. 9. Frequence-response curves for various longitudinal stiffness values ($v_f = 0.6$, $\mu = 0.2$, $v_0 = 0.5$, $\alpha = 0.01$, $k = 1$, $F = 1$).

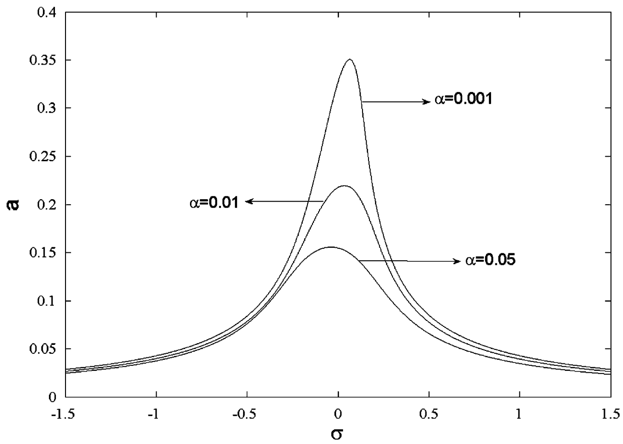

Fig. 10. Frequence-response curves for various α parameter values ($v_f = 0.6$, $v_\ell = 0.2$, $\mu = 0.2$, $v_0 = 1.0$, $k = 5$, $F = 1$).

Fig. 8 shows the effects of flexural stiffness on the frequency response curves. Maximum amplitudes decrease by increasing flexural stiffness. Curves bend more to the right by decreasing flexural stiffness.

Fig. 9 shows the influence of longitudinal stiffness on the frequency response curves. The maximum amplitudes and multi-valued regions responsible for jump phenomena increase for increasing longitudinal stiffness.

Fig. 10 shows the influence of α parameter. The maximum amplitudes decrease for increasing α parameter. As in the previous section, results in this section are also valid in the subcritical velocity regimes.

5. Concluding remarks

A general solution procedure is developed for vibrations of continuous systems with cubic nonlinearities. The arbitrary linear and cubic operators with spatial and time derivatives allow to generalize a wide range class of problems including gyroscopic systems. The approximate solutions, the amplitude and phase modulation equations are derived in terms of the operators. Method of multiple scales is employed in the analysis. The formalism developed is applied to two different problems namely the axially moving Euler–Bernoulli beam and the axially moving viscoelastic beam. Natural frequencies and frequency response curves are presented and variations of the curves with respect to the dimensionless parameters are discussed. The algorithm developed may be applied to many more problems in nonlinear vibrations of continuous systems.

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