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Verification of Lyapunov functions for the analysis of stochastic Liénard equations

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ABSTRACT

Versions of stochastic Liénard equations perturbed by both additive and multiplicative white noise are considered. We discuss existence, uniqueness, continuity, boundedness and moment stability of solutions with the help of several Lyapunov-type functions. The Lyapunov functions are explicitly found to control uniform moment boundedness and stability. A new matching condition on the interaction of resistance and restoring force plays an essential role to guarantee stability (uniform boundedness) of p-th moments by validation of those Lyapunov functions.

1. Introduction

Many authors dealt with systems of ordinary stochastic differential equations, e.g. see Arnold [1,2], Khasminskiǐ [3] and Mao [4]. For an overview on the theory of stochastic differential equations, see also Freidlin and Wentzell [5], Friedman [6], Gard [7], Gikhman and Skorochod [8], Karatzas and Shreve [9], Krylov [10], Mao [11], Oksendal [12], Protter [13] or Revuz and Yor [14]. One of the most commonly met dynamic equations in mechanical and electrical engineering is that of Liénard-type, also known as *Liénard oscillators*, which can be used to model vibrations of physically relevant systems. A detailed study of this general class of equations perturbed by Markovian-type of noise is still missing, although some discussion can be found in Khasminskiǐ's book [3] on existence of a stationary solution process in the case of additive noise (cf. example 2.10 at pp. 92–93, which is due to an idea of M.B. Nevel'son), an asymptotic analysis of singular perturbations as done by Narita [15–17], and a study of stationary solutions under Markovian switching as carried out by Xi and Zhao [18]. Nonrandom Liénard equations were extensively studied in the works of Burton [19–23].

The most known representative of Liénard equations is that of Van der Pol equations (i.e. our model (1) with damping term $f(x) = \varepsilon(x^2 - 1)$ and restoring force $g(x) = x$ as mentioned below), which are used to describe the behavior of simple electronic circuits. Phenomena such as thermal noise forces us to take random Markovian perturbations of Liénard equations into account. The simplest forms of such perturbations are additive and multiplicative white noise.

The aim of our paper is to investigate existence, uniqueness, continuity, boundedness and stability of strong solutions of Itô-type stochastic differential equations

$$\ddot{x} + f(x)\dot{x} + g(x) = \sigma_0\xi_0 + \sigma_1 x\xi_1 + \sigma_2\dot{x}\xi_2 + \sigma_3\sqrt{|f(x)|}\dot{x}\xi_3 \tag{1}$$

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perturbed by independent white noises $\xi_j(t)$ with Gaussian representation

$$W^j(t) = \int_0^t \xi_j(s)\,ds \in \mathcal{N}(0,t), \quad j = 0,1,2,3. \tag{2}$$

For this purpose, we construct Lyapunov-type functions V and verify its use by showing $\mathcal{L}V \leqslant 0$ or $\mathcal{L}V \leqslant cV$ for the infinitesimal generator $\mathcal{L}$ of Eq. (1) with appropriate constant c. The idea of using Lyapunov functions for the analysis of stochastic equations is due to Khasminskiĭ [3] to a large extent. However, in that book [3] and later works, it has not been specified how to construct such functions for nonlinear stochastic dynamics. In fact, each class of nonlinear equations requires its own detailed study to find and verify those functionals controlling the underlying dynamics. We are aiming with this paper to fill that gap and present a verification of a series of appropriate Lyapunov functions V for the entire class of Eqs. (1). Moreover, the explicit knowledge of and construction of V can also be used to control the approximation error of numerical methods applied to Eq. (1). For details, see the axiomatic approach in Schurz [24] and related aspects [25–28,30,29].

In passing, we note that several other attempts have been followed to treat the subclass of stochastic Van der Pol equations. For example, among many other results, Imkeller and Schmalfuss [31] proved the existence of global attractors. Bonzani [32] obtains an analytical approximate solution to a generalized stochastic Van der Pol equation by the use of G.A. Adomian's decomposition method. Schenk-Hoppé [33] discusses possible bifurcation scenarios. Moreover, Tel [34] finds an expansion of the stationary distribution in powers of a bifurcation parameter for such equations.

Eq. (1) is also called *stochastic Liénard equation* or *Liénard oscillator*. For its analysis, we transform Eq. (1) into the equivalent system

$$\dot{x} = y, \quad \dot{y} = -f(x)y - g(x) + \sigma_0\xi_0 + \sigma_1 x\xi_1 + \sigma_2 y\xi_2 + \sigma_3\sqrt{|f(x)|}\,y\xi_3. \tag{3}$$

Hence, we may rewrite Eq. (3) in the Itô differential form

$$dX(t) = a(X(t))\,dt + \sum_{j=0}^{3} b^j(X(t))\,dW^j(t) \tag{4}$$

for solution process $X = (x,y) \in \mathbb{R}^2$ where

$$a_1(x,y) = y, \quad a_2(x,y) = -f(x)y - g(x), \quad b_1^j(x,y) = 0 \quad (j = 0,1,2,3), \tag{5}$$

$$b_2^0(x,y) = \sigma_0, \quad b_2^1(x,y) = \sigma_1 x, \quad b_2^2(x,y) = \sigma_2 y, \quad b_2^3(x,y) = \sigma_3\sqrt{|f(x)|}\,y. \tag{6}$$

The paper is organized as follows. Section 2 reports on existence and uniqueness of continuous Markovian solutions. Section 3 discusses uniform moment boundedness on infinite time-intervals and asymptotic moment stability along Lyapunov-type functions. Section 4 provides some special discussion on the use of a Lyapunov functional modified from Burton's book [23] for stability of its moments, in probability and in almost sure sense. We also make use of certain versions of stochastic LaSalle-type principles to conclude stability and on the structure of related limit sets.

2. Solvability of random Liénard equations

Define

$$\mathcal{F} = \sigma(W^j(t) : t \geqslant 0,\ j = 0,1,2,3) \tag{7}$$

as naturally generated σ-algebra generated by underlying Wiener processes W^j. Let $C^0_{\mathrm{locLip}}(\mathbb{R}^1)$ be the class of all local Lipschitz continuous functions $f : \mathbb{R}^1 \to \mathbb{R}^1$ and $V : \mathbb{R}^2 \to \mathbb{R}^1$ defined by

$$V(x,y) = \frac{1}{2}y^2 + \frac{1}{2}x^2 + \int_0^x g(z)\,dz + K_V + \frac{1}{2}\sigma_0^2 \tag{8}$$

for all $x, y \in \mathbb{R}^1$, where $K_V \geqslant 0$ is a real constant such that $V \geqslant 0$ on $\mathbb{R}^2$.

Theorem 1 (*Existence and uniqueness*). *Assume that*

(i) x_0, y_0 *are independent of naturally generated* σ-*algebra* $\mathcal{F}$,
(ii) $f, g \in C^0_{\mathrm{locLip}}(\mathbb{R}^1)$,
(iii) $\forall x \in \mathbb{R}^1 : \int_0^x g(z)\,dz \geqslant -K_V$ *and* $\lim_{|x|\to+\infty}(x^2 + 2\int_0^x g(z)\,dz) = +\infty$,
(iv) $\mathbb{E}[V(x_0,y_0)] < +\infty$ *where* V *is defined by Eq.* (8), *and*
(v) $\exists K_e\, \forall x \in \mathbb{R}^1 : 1 + \sigma_2^2 + \sigma_3^2|f(x)| - 2f(x) \leqslant K_e < +\infty$.

Then, there is a unique, continuous, Markovian solution (x,y) *of Eq.* (3) *satisfying*

$$\frac{\sigma_0^2}{2} \leqslant \sup_{0 \leqslant t \leqslant T} \mathbb{E}[V(x(t),y(t))] \leqslant \mathbb{E}[V(x_0,y_0)]\exp(\max\{K_e, \sigma_1^2 + 1\}T). \tag{9}$$

Proof. From standard textbooks on stochastic differential equations, we know about the existence of local unique, continuous and Markovian solutions up to the first exit time τ_r from any ball $B_r = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant r^2\}$ since (i), (iv) are satisfied and local Lipschitz coefficients are guaranteed by (ii) too. It remains to show that the local solution can be extended to be a global one by verifying that solutions cannot reach the boundary of $\mathbb{R}^2$. First, note that the function V defined by Eq. (8) is positive-definite and radially unbounded due to assumptions (iii). So it may serve as a Lyapunov function to prove existence and uniqueness of global solutions. We shall borrow some ideas of Khasminskiǐ [3] (see Theorem 4.1, pp. 84–85). For this purpose, define stopping time $\tau_r = \inf\{t \geqslant 0 : (x(t), y(t)) \notin B_r\}$. Set $\tau_r(t) = \min(t, \tau_r)$ for $t \geqslant 0$. Recall the infinitesimal generator of the Markov process related to Eq. (3) is given by the second-order linear partial differential operator

$$\mathscr{L} = y\frac{\partial}{\partial x} - (f(x)y + g(x))\frac{\partial}{\partial y} + \frac{1}{2}(\sigma_0^2 + \sigma_1^2 x^2 + \sigma_2^2 y^2 + \sigma_3^2 |f(x)| y^2)\frac{\partial^2}{\partial y^2} \tag{10}$$

mapping from C^2 to C^0. Recall Dynkin's formula for stopped Markov processes

$$\mathbb{E}[V((x(\tau_r(t)), y(\tau_r(t)))] = \mathbb{E}[V((x(0), y(0)))] + \mathbb{E}\int_0^{\tau_r(t)} \mathscr{L}V((x(\tau_r(s)), y(\tau_r(s))))\,\mathrm{d}s. \tag{11}$$

Now, calculate $\mathscr{L}V(x,y)$. So we arrive at

$$\begin{aligned}\mathscr{L}V(x,y) &= yx - f(x)y^2 + \frac{1}{2}(\sigma_0^2 + \sigma_1^2 x^2 + \sigma_2^2 y^2 + \sigma_3^2 |f(x)| y^2)\\ &\leqslant \frac{\sigma_0^2}{2} + \frac{\sigma_1^2 + 1}{2}x^2 + \frac{1 + \sigma_2^2 + \sigma_3^2|f(x)| - 2f(x)}{2}y^2 \leqslant \max\{\sigma_1^2 + 1, K_e\}V(x,y)\end{aligned} \tag{12}$$

for all $x, y \in \mathbb{R}^1$. Thus, plugging this estimate into Dynkin's formula Eq. (11) leads to the uniform estimation

$$\begin{aligned}\mathbb{E}[V((x(\tau_r(t)), y(\tau_r(t)))] &\leqslant \mathbb{E}[V((x(0), y(0)))] + \max\{\sigma_1^2 + 1, K_e\}\int_0^t \mathbb{E}[V(x(\tau_r(s)), y(\tau_r(s)))]\,\mathrm{d}s\\ &\leqslant (\mathbb{E}[V((x(0), y(0)))])\exp(\max\{\sigma_1^2 + 1, K_e\}T) < +\infty\end{aligned} \tag{13}$$

for $0 \leqslant t \leqslant T$ (while using Gronwall–Bellman inequality). On the other hand, we have

$$\begin{aligned}r^2\mathbb{P}(\{\exists s : 0 \leqslant s < t, (x(s), y(s)) \notin B_r\}) &= r^2\mathbb{E}[I_{\{\tau_r < t\}}] \qquad (14)\\ &\leqslant 2\mathbb{E}[V((x(\tau_r(t)), y(\tau_r(t))))I_{\{\tau_r < t\}}]\\ &\leqslant 2\mathbb{E}[V((x(\tau_r(t)), y(\tau_r(t))))(I_{\{\tau_r < t\}} + I_{\{\tau_r \geqslant t\}})]\\ &= 2\mathbb{E}[V((x(\tau_r(t)), y(\tau_r(t)))_{1 \leqslant n \leqslant N})], \qquad (15)\end{aligned}$$

where I_S denotes the indicator function of subscribed set S. Consequently, for all $0 \leqslant t \leqslant T$, conclude that

$$\mathbb{P}(\{\tau_r < t\}) = \mathbb{P}(\{\exists s : 0 \leqslant s < t, (x(s), y(s)) \notin B_r\}) \leqslant \frac{(2\mathbb{E}[V((x(0), y(0)))])\exp(\max\{\sigma_1^2 + 1, K_e\}T)}{r^2}. \tag{16}$$

Taking the limit $r \to +\infty$ yields that $\mathbb{P}(\{\tau < T\}) = 0$ where τ is the first exit time of process $\{(x(t), y(t)) : t \geqslant 0\}$ from the open set $\mathbb{R}^2$. Hence, the local solution can never explode at finite terminal times T and the unique continuation to a global solution must exist. It remains to check Eq. (9). Apply again Dynkin's formula Eq. (11) (while dropping the exit time τ_r formalism since we know that $\tau_r(t) = t$ (a.s.)) to get to

$$\mathbb{E}[V((x(t), y(t)))] \leqslant (\mathbb{E}[V((x(0), y(0)))])\exp(\max\{\sigma_1^2 + 1, K_e\}t). \tag{17}$$

Now, take the supremum at t. Note that $V(x,y) \geqslant \sigma_0^2/2$ for all $x, y \in \mathbb{R}^1$. Hence, Eq. (9) is obvious. Thus, we have confirmed the conclusion of Theorem 1. □

Remark 2. One may derive more precise uniform bounds on the moments of solutions (x,y) of Eq. (3). This heavily depends on the choice of Lyapunov functional V and the specific parameter choice. Such a study is left to the reader.

Remark 3. Under the assumptions of Theorem 1 with $g(x) \geqslant 0$ we also have uniform boundedness of second moments (and hence first ones too) of displacement x and velocity $y = \dot{x}$ of random Liénard oscillator defined by Eq. (1), i.e.

$$\sup_{0 \leqslant t \leqslant T} \mathbb{E}[x(t)]^2 < +\infty, \quad \sup_{0 \leqslant t \leqslant T} \mathbb{E}[y(t)]^2 < +\infty. \tag{18}$$

As a byproduct, when $g(x) \geqslant 0$, we gain that

$$-\infty < -K_V \leqslant \mathbb{E}\left[\int_0^{x_0} g(z)\,\mathrm{d}z\right] \leqslant \sup_{0 \leqslant t \leqslant T} \mathbb{E}\left[\int_0^{x(t)} g(z)\,\mathrm{d}z\right] < +\infty. \tag{19}$$

Note that trivially $x^2 + y^2 \leqslant 2V(x,y)$ due to condition (iii) in Theorem 1.

Remark 4. One may also derive estimates of moments along certain Lyapunov functionals from below while exploiting similar comparison techniques as in proof above. Such an analysis is omitted here.

3. Moment boundedness and stability analysis

For $x, y \in \mathbb{R}^1$, define $V_s : \mathbb{R}^2 \to \mathbb{R}^1$ by

$$V_s(x,y) = \frac{1}{2}y^2 + \int_0^x g(z)\,\mathrm{d}z. \tag{20}$$

Theorem 5 (*Uniform boundedness of 1st moments*). *Assume that*

(i) x_0, y_0 *are independent of naturally generated* σ*-algebra* $\mathscr{F}$,
(ii) $f, g \in C^0_{\mathrm{locLip}}(\mathbb{R}^1)$,
(iii) $\forall x \in \mathbb{R}^1 : \int_0^x g(z)\,\mathrm{d}z \geqslant 0$,
(iv) $\mathbb{E}[V_s(x_0, y_0)] < +\infty$ *where* V_s *is defined by Eq.* (20),
(v) $\sigma_0 = 0 = \sigma_1$ *and* $\forall x \in \mathbb{R}^1 : \sigma_2^2 + \sigma_3^2|f(x)| - 2f(x) \leqslant 0$.

Then, for Eq. (3), *the following moments are uniformly bounded with respect to time* t:

$$0 \leqslant \sup_{0 \leqslant t < +\infty} \mathbb{E}[y(t)]^2 \leqslant \mathbb{E}[y_0]^2 < +\infty, \tag{21}$$

$$0 \leqslant \sup_{0 \leqslant t < +\infty} \mathbb{E}\left[\int_0^{x(t)} g(z)\,\mathrm{d}z\right] \leqslant \mathbb{E}\left[\int_0^{x_0} g(z)\,\mathrm{d}z\right] < +\infty. \tag{22}$$

Proof. Recall the form of the infinitesimal generator $\mathscr{L}$ as given by Eq. (10). Calculate $\mathscr{L}V_s$ and arrive at

$$\mathscr{L}V_s(x,y) = (\sigma_2^2 + \sigma_3^2|f(x)| - 2f(x))\frac{y^2}{2} \leqslant 0 \tag{23}$$

for all $x, y \in \mathbb{R}^1$. Dynkin's formula Eq. (11) with $\tau_r(t) = t$ yields that

$$0 \leqslant \mathbb{E}[V_s((x(t), y(t)))] \leqslant \mathbb{E}[V_s((x(0), y(0)))] = \mathbb{E}[V_s(x_0, y_0)]. \tag{24}$$

Therefore, in view of $V_s(x,y) \geqslant y^2/2$ and $V_s(x,y) \geqslant \int_0^x g(z)\,\mathrm{d}z$ for all $(x,y) \in \mathbb{R}^2$, we establish the estimates (21) and (22) very easily. □

Remark 6. Note that the Gronwall–Bellman Lemma can be applied to negative integrand kernels as well if the differential inequalities hold uniformly for all initial times and terminal times. This was shown indirectly in Ref. [4] and directly in Ref. [25].

Define $V_p : \mathbb{R}^2 \to \mathbb{R}^1$ by

$$V_p(x,y) = \left(y^2 + 2\int_0^x g(z)\,\mathrm{d}z\right)^{p/2} \tag{25}$$

for $x, y \in \mathbb{R}^1$, where $p \geqslant 2$ is a real constant.

Theorem 7 (*Uniform boundedness of pth moments*). *Assume that*

(i) x_0, y_0 *are independent of naturally generated* σ*-algebra* $\mathscr{F}$,
(ii) $f, g \in C^0_{\mathrm{locLip}}(\mathbb{R}^1)$,
(iii) $\forall x \in \mathbb{R}^1 : \int_0^x g(z)\,\mathrm{d}z \geqslant 0$,
(iv) $\mathbb{E}[V_p(x_0, y_0)] < +\infty$ *where* V_p *is defined by Eq.* (25),
(v) $\sigma_0 = 0$ *and* $\forall x \in \mathbb{R}^1$:

$$(\sigma_2^2(p-1) + (p-1)\sigma_3^2|f(x)| - 2f(x))\frac{y^2}{2} + \frac{(p-1)\sigma_1^2}{2}x^2 \leqslant K_p\left(y^2 + 2\int_0^x g(z)\,\mathrm{d}z\right).$$

Then, for system Eq. (3), *the moments of* V_p *are bounded on* $[0, T]$, *i.e.*

$$\sup_{0 \leqslant t \leqslant T} \mathbb{E}[V_p(x(t), y(t))] \leqslant \mathbb{E}[V_p(x_0, y_0)] \exp(p[K_p]_+ T) < +\infty, \tag{26}$$

where $[\cdot]_+$ *represents the positive part of the inscribed expression.*

Proof. Recall $\mathscr{L}$ given by (10) and calculate

$$\mathscr{L}V_p(x,y) = p[V_p(x,y)]^{(p-2)/p}\left(-f(x)y^2 + \frac{\sigma_1^2x^2+\sigma_2^2y^2+\sigma_3^2|f(x)|y^2}{2}\left(1+\frac{(p-2)y^2}{x^2+y^2}\right)\right) \leqslant pK_pV_p(x,y) \tag{27}$$

for all $x,y\in\mathbb{R}^1$. Therefore, Dynkin's formula Eq. (11) with $\tau_r(t)=t$ and Gronwall–Bellman inequality provide us the conclusion of Theorem 7. □

Define $V_g:\mathbb{R}^2\to\mathbb{R}^1$ by

$$V_g(x,y) = \frac{1}{2}y^2 + ayg(x) + \int_0^x g(z)\,\mathrm{d}z + K_V \tag{28}$$

for $x,y\in\mathbb{R}^1$, where a and K_V are real constants such that V_g is positive.

Theorem 8 (*Instability and stability*). *Assume that*

(i) x_0, y_0 *are independent of naturally generated* σ*-algebra* $\mathscr{F}$,
(ii) $f,g\in C^0_{\mathrm{locLip}}(\mathbb{R}^1)$ *and g is differentiable on* $\mathbb{R}^1$ (μ*-a.e.*),
(iii) *a and* K_V *in* (28) *are such that there is a constant* $\alpha\geqslant 0$ *satisfying*

$$\forall x,y\in\mathbb{R}^1: \quad \alpha(x^2+y^2)\leqslant V_g(x,y), \tag{29}$$

(iv) $\mathbb{E}[V_g(x_0,y_0)]<+\infty$ *where* V_g *is defined by Eq.* (28),
(v) $\sigma_0=0=\sigma_1$ *and* $\exists K_\alpha, K_\beta\in\mathbb{R}^1$ (*constants*)

$$\forall x\in\mathbb{R}^1: K_\alpha\leqslant 2ag'(x)+\sigma_2^2+\sigma_3^2|f(x)|-2f(x)\leqslant K_\beta. \tag{30}$$

Then, for Eq. (3), *the condition* $K_\alpha\geqslant 0$ *and* $\alpha>0$ *implies asymptotic instability of moments along* V_g, *i.e. for adapted initial values satisfying* $\mathbb{E}[x_0^2+y_0^2]>0$, *we have*

$$\liminf_{t\to+\infty}\mathbb{E}[V_g(x(t),y(t))]\geqslant\mathbb{E}[V_g(x_0,y_0)]\geqslant\alpha\mathbb{E}[x_0^2+y_0^2]>0, \tag{31}$$

and $K_\beta\leqslant 0$ *implies stability of moments along* V_g, *i.e.*

$$\sup_{0\leqslant t<+\infty}\mathbb{E}[V_g(x(t),y(t))]=\mathbb{E}[V_g(x_0,y_0)]<+\infty. \tag{32}$$

If $K_\beta<0$ *then we have even almost sure asymptotic stability of y-component and*

$$\forall t\geqslant 0: \quad \alpha\mathbb{E}[y^2(t)]\leqslant\mathbb{E}[V_g(x_0,y_0)]\exp\left(\frac{K_\beta}{2\alpha}t\right). \tag{33}$$

Moreover, the condition $K_\beta<0$ *and* $\alpha>0$ *guarantees asymptotic mean square stability of the y-component of system Eq.* (3).

Proof. Recall $\mathscr{L}$ given by Eq. (10). Calculate and estimate

$$K_\alpha\frac{y^2}{2}\leqslant\mathscr{L}V_g(x,y)=(2ag'(x)+\sigma_2^2+\sigma_3^2|f(x)|-2f(x))\frac{y^2}{2}\leqslant K_\beta\frac{y^2}{2} \tag{34}$$

for all $x,y\in\mathbb{R}^1$. Dynkin's formula Eq. (11) with $\tau_r(t)=t$ and Gronwall–Bellman inequality yield that, if $K_\alpha\geqslant 0$, we have

$$\liminf_{t\to+\infty}\mathbb{E}[V_g((x(t),y(t)))]\geqslant\mathbb{E}[V_g(x_0,y_0)]\geqslant\alpha\mathbb{E}[x_0^2+y_0^2]. \tag{35}$$

Therefore, we can establish asymptotic instability Eq. (31) while requiring that $\alpha>0$ and $K_\alpha\geqslant 0$. On the other hand, we arrive at

$$\mathscr{L}V_g(x,y)=(2ag'(x)+\sigma_2^2+\sigma_3^2|f(x)|-2f(x))\frac{y^2}{2}\leqslant K_\beta\frac{y^2}{2} \tag{36}$$

for all $x,y\in\mathbb{R}^1$. A straightforward application of Dynkin's formula Eq. (11) with $\tau_r(t)=t$ while using Eq. (36) leads to the estimate Eq. (32) of uniform boundedness of moments along V_g whenever $K_\beta\leqslant 0$. More precisely, we obtain the estimates

$$0\leqslant\alpha\mathbb{E}[x^2(t)+y^2(t)]\leqslant\mathbb{E}[V_g(x(t),y(t))]\leqslant\mathbb{E}[V_g(x_0,y_0)], \tag{37}$$

$$0\leqslant\alpha\mathbb{E}[y^2(t)]\leqslant\mathbb{E}[V_g(x_0,y_0)]\exp\left(\frac{K_\beta}{2\alpha}t\right) \tag{38}$$

for $t\geqslant 0$. This estimate also confirms exponential mean square stability of y with upper (moment Lyapunov) exponent not greater than $K_\beta/2$ whenever $K_\beta<0$ and $\alpha>0$. It remains to apply Theorem 6.1 from Schurz [26] (p. 513) to conclude almost sure stability of y-component. (Similarly, one could argue with stochastic Lasalle Theorem 2.1 from Mao [35] when f uniformly bounded and g of linear growth.) Thus, the assertion of Theorem 8 is proven. □

Example 1 (*Randomly forced damped pendulum*). Consider the damped pendulum with locally Gaussian randomly forced linear resistance

$$\ddot{x} + \left(\frac{1}{2} + c\right)\dot{x} + \sin(x) = \sigma\dot{x}\xi_t, \tag{39}$$

where $c > 0, \sigma \in \mathbb{R}^1$ and ξ Gaussian white noise. Hence, we have

$$g(x) = \sin(x), \quad f(x) = \frac{1}{2} + c, \sigma_0 = \sigma_1 = \sigma_3 = 0, \quad \sigma_2 = \sigma. \tag{40}$$

Take $a \in \mathbb{R}^1$, $K_V = a^2$ and consider

$$V_g(x,y) = \frac{y^2}{2} + ay\sin(x) + a^2 + 1 - \cos(x)\left(= \frac{(y + a\sin(x))^2}{2} - \frac{a^2}{2}\sin^2(x) + a^2 + 1 - \cos(x) \right.$$
$$\left. = \frac{y^2}{4} + \left(\frac{y}{2} + a\sin(x)\right)^2 + a^2(1 - \sin^2(x)) + 1 - \cos(x) \right). \tag{41}$$

Obviously $V_g(x,y) \geqslant y^2/4 \geqslant 0$ for all $a, x, y \in \mathbb{R}^1$. Thus, we can apply Theorem 8 with

$$\alpha = 0, \quad K_\alpha = -2|a| - 1 - \sigma^2 - 2c, \quad K_\beta = 2|a| - 1 + \sigma^2 - 2c, \tag{42}$$

and its condition for uniform boundedness of moments along V_g reads as

$$\forall x \in \mathbb{R}^1 : K_\alpha \leqslant 2a\cos(x) + \sigma^2 - 1 - 2c \leqslant K_\beta \leqslant 0 \tag{43}$$

which is trivially fulfilled whenever $\sigma^2 \leqslant 2c$ and $-0.5 \leqslant a \leqslant 0.5$. Moreover, Theorem 8 says that the limit

$$\frac{a^2}{2} \leqslant \lim_{t\to+\infty} \mathbb{E}[V_g(x(t), y(t))] \leqslant \mathbb{E}[V_g(x_0, y_0)] < +\infty \tag{44}$$

exists, the limits $\lim_{t\to+\infty}\mathbb{E}[y^2(t)] = 0$ and (with probability one) $\lim_{t\to+\infty} y^2(t) = 0$ if $\sigma^2 < 2c$ and $-0.5 \leqslant a \leqslant 0.5$ (since this guarantees that $K_\beta < 0$ can be chosen). Actually, it suffices to require $K_\beta < 0$ for the validity of this statement on the limits. Furthermore, let us apply also Theorem 7. For this purpose, consider

$$V_p(x,y) = (y^2 + 2(1 - \cos(x)))^{p/2} \tag{45}$$

as defined by Eq. (25). Let $p = 2(1+\varepsilon)$ with real constant $\varepsilon > 0$. Suppose that $\mathbb{E}[(y_0^2 + 2(1 - \cos(x_0)))^{p/2}] < +\infty$. Then one easily can also conclude the uniform boundedness of

$$\mathbb{E}[([y(t)]^2 + 2(1 - \cos(x(t))))^{1+\varepsilon}] \leqslant \mathbb{E}[(y_0^2 + 2(1 - \cos(x_0)))^{1+\varepsilon}]. \tag{46}$$

Moreover, the assumptions of stochastic LaSalle-type Theorem 2.1 from Mao [35] (p. 177) are fulfilled. Therefore, we may conclude that the finite limit

$$\lim_{t\to+\infty}([y(t)]^2 + 2(1 - \cos(x(t))))^{1+\varepsilon} \tag{47}$$

exists (a.s.) and

$$\lim_{t\to+\infty} y^2(t) = 0 \ (a.s.) \tag{48}$$

whenever $-(1+2c) + (1+2\varepsilon)\sigma^2 < 0$. This guarantees almost sure stability of its y-component whenever

$$-(1+2c) + \sigma^2 < 0 \tag{49}$$

by choosing $\varepsilon > 0$ to be sufficiently small in above computations and replacing $V_p(x,y)$ by $V(x,y) = (1 + y^2 + 2(1 - \cos(x)))^{p/2}$. In this case we can also conclude uniform boundedness of 1st moments of x-component by integrating y with respect to t.

4. A modified stochastic Liénard oscillator

The comments of Burton [23, p. 228] give rise to a modification of our previously studied Liénard oscillator and related Lyapunov functional. We shall see that this modification is done such that the Lyapunov functional and its Liénard system are nearly "optimally tuned" to reach stability on infinite time-intervals. Consider

$$\ddot{x} + f(x)\dot{x} + g(x) = \sigma_0\xi_0 + \sigma_1 x\xi_1 + \sigma_2\dot{x}\xi_2 + \sigma_3\sqrt{|f(x)|}\dot{x}\xi_3 + \sigma_4 h(x)\xi_4 \tag{50}$$

perturbed by multiplicative independent white noise $\xi_j(t)$ $(j = 0, \ldots, 4)$ and local Lipschitz-continuous noise intensity function h. Define $V_{p,c} : \mathbb{R}^2 \to \mathbb{R}^1$ by

$$V_{p,c}(x,y) := \left(K_V + cy^2 + \left(y + \int_0^x f(z)\,dz \right)^2 + 2(c+1) \int_0^x g(z)\,dz \right)^{p/2} \tag{51}$$

for $x, y \in \mathbb{R}^1$, where $K_V, c \geqslant 0, p \geqslant 2$ are real constants. Set

$$m(p,c) := 2(p-2)\max\{c, 1\} + c + 1. \tag{52}$$

Theorem 9 (*Uniform boundedness of pth moments*). *Assume that*

(i) x_0, y_0 *are independent of naturally generated* σ*-algebra* $\mathscr{F}$,
(ii) $f, g, h \in C^0_{\text{locLip}}(\mathbb{R}^1)$,
(iii) $\exists$ *constants* $K_V, c \geqslant 0$ *and* $\forall x \in \mathbb{R}^1 : K_V + 2(c+1)\int_0^x g(z)\,dz \geqslant 0$,
(iv) $\mathbb{E}[V_{p,c}(x_0, y_0)] < +\infty$ *where* $V_{p,c}$ *is defined by Eq.* (51),
(v) $\exists K^1_{p,c}, K^2_{p,c} \in \mathbb{R}^1$ (*constants*) $\forall x \in \mathbb{R}^1$:

$$\begin{aligned} & -2cf(x) + m(p,c)(\sigma_2^2 + \sigma_3^2 |f(x)|) \leqslant 2cK^1_{p,c} \quad \text{and} \\ & m(p,c)(\sigma_0^2 + \sigma_1^2 x^2 + \sigma_4^2 (h(x))^2) - 2g(x)\int_0^x f(z)\,dz \leqslant 2K^2_{p,c}\left(K_V + 2(c+1)\int_0^x g(z)\,dz \right). \end{aligned} \tag{53}$$

Then, for Eq. (50), *the moments of* $V_{p,c}$ *are bounded on* $[0, T]$, *i.e.*

$$\sup_{0 \leqslant t \leqslant T} \mathbb{E}[V_{p,c}(x(t), y(t))] \leqslant \mathbb{E}[V_{p,c}(x_0, y_0)] \exp(p[\max\{K^1_{p,c}, K^2_{p,c}\}]_+ T) < +\infty, \tag{54}$$

where $[\cdot]_+$ *represents the positive part of the inscribed expression.*

Proof. Recall the form of infinitesimal generator $\mathscr{L}$ given by Eq. (10). For abbreviation, define

$$d(x,y) := \sigma_0^2 + \sigma_1^2 x^2 + \sigma_2^2 y^2 + \sigma_3^2 |f(x)| y^2 + \sigma_4^2 (h(x))^2 \tag{55}$$

as diffusion part of $\mathscr{L}$. Use condition (v) to calculate

$$\begin{aligned} \mathscr{L}V_{p,c}(x,y) &= y\frac{p}{2}(V_{p,c}(x,y))^{(p-2)/p}\left[2\left(y + \int_0^x f(z)\,dz \right) f(x) + 2(c+1)g(x) \right] \\ &\quad - (f(x)y + g(x))\frac{p}{2}(V_{p,c}(x,y))^{(p-2)/p}\left[2cy + 2\left(y + \int_0^x f(z)\,dz \right) \right] \\ &\quad + \frac{1}{2}d(x,y)\left[\frac{p}{2}\left(\frac{p}{2} - 1 \right)(V_{p,c}(x,y))^{(p-4)/p} 4\left((c+1)y + \int_0^x f(z)\,dz \right)^2 \right] \\ &\quad + \frac{1}{2}d(x,y)\left[\frac{p}{2}(V_{p,c}(x,y))^{(p-2)/p} 2(c+1) \right] \\ &= -p(V_{p,c}(x,y))^{(p-2)/p}\left[g(x)\int_0^x f(z)\,dz + cf(x)y^2 \right] \\ &\quad + p\frac{d(x,y)}{2}\left[(p-2)(V_{p,c}(x,y))^{(p-4)/p}\left((c+1)y + \int_0^x f(z)\,dz \right)^2 \right] \\ &\quad + p\frac{d(x,y)}{2}[(c+1)(V_{p,c}(x,y))^{(p-2)/p}] \\ &\leqslant p(V_{p,c}(x,y))^{(p-2)/p}[-2cf(x) + m(p,c)(\sigma_2^2 + \sigma_3^2|f(x)|)]\frac{y^2}{2} \\ &\quad + p(V_{p,c}(x,y))^{(p-2)/p}\left[m(p,c)(\sigma_0^2 + \sigma_1^2 x^2 + \sigma_4^2(h(x))^2) - 2g(x)\int_0^x f(z)\,dz \right] \\ &\leqslant p(V_{p,c}(x,y))^{(p-2)/p}\left[K^1_{p,c}cy^2 + K^2_{p,c}\left(K_V + 2(c+1)\int_0^x g(z)\,dz \right) \right] \\ &\leqslant p\max\{K^1_{p,c}, K^2_{p,c}, 0\}V_{p,c}(x,y) \end{aligned} \tag{56}$$

for all $x, y \in \mathbb{R}^1$. Therefore, Dynkin's formula Eq. (11) with $\tau_r(t) = t$ and Gronwall–Bellman inequality provide the conclusion of Theorem 9. □

Remark 10. Theorem 9 is particularly efficient for stability analysis when the resistance function $f(x)$ is nonnegative. If $f(x)$ possesses negative values then the only efficient application can be seen with the choice of constants $c = 0$, $\sigma_2 = 0$ and

$\sigma_3 = 0$ due to the form of conditions (v). While looking at the conditions (v), we also observe that the assumption

$$\forall x \in \mathbb{R}^1 : g(x) \int_0^x f(z)\,dz \geqslant 0 \tag{57}$$

is essentially needed to have stable moment dynamics along the functions $V_{p,c}$ (to compensate the destabilizing part of terms connected to σ_0, σ_1 and σ_4). This exhibits a kind of "*matching condition*" of interaction between resistance f and restoring parts g to achieve boundedness and stability. If all $K^i_{p,c} \leqslant 0$ we have uniform boundedness of first moments along $V_{p,c}$ on infinite time-intervals $[0, +\infty)$ thanks to Theorem 9. More precisely, we have the following conclusions.

Theorem 11 (*On moment limit behavior*). *Assume that*

(i) x_0, y_0 *are independent of naturally generated* σ-*algebra* $\mathscr{F}$,
(ii) $f, g, h \in C^0_{\text{locLip}}(\mathbb{R}^1)$,
(iii) $\exists$ *constants* $K_V, c \geqslant 0$ *and* $\forall x \in \mathbb{R}^1 : K_V + 2(c+1)\int_0^x g(z)\,dz \geqslant 0$,
(iv) $\mathbb{E}[V_{p,c}(x_0, y_0)] < +\infty$ *where* $V_{p,c}$ *is defined by Eq.* (51),
(v) $\forall x \in \mathbb{R}^1$:

$$u(x) := -2cf(x) + m(p,c)(\sigma_2^2 + \sigma_3^2|f(x)|) \leqslant 0 \tag{58}$$

$$\textit{and} \tag{59}$$

$$v(x) := m(p,c)(\sigma_0^2 + \sigma_1^2 x^2 + \sigma_4^2 (h(x))^2) - 2g(x)\int_0^x f(z)\,dz \leqslant 0. \tag{60}$$

Then, for Eq. (50), *the moments of* $V_{p,c}$ *are uniformly bounded in time* t, *i.e.*

$$\sup_{0 \leqslant t \leqslant +\infty} \mathbb{E}[V_{p,c}(x(t), y(t))] = \mathbb{E}[V_{p,c}(x_0, y_0)] < +\infty, \tag{61}$$

and $t \in \mathbb{R}^1_+ \mapsto z(t) := \mathbb{E}[V_{p,c}(x(t), y(t))]$ *is a decreasing function in t with existing finite limits* $\lim_{t\to+\infty} z(t)$. *In addition we have the convergence of improper integrals satisfying*

$$-\frac{2}{p}\mathbb{E}[V_{p,c}(x_0, y_0)] \leqslant \int_0^{+\infty} \mathbb{E}\left[(V_{p,c}(x(t), y(t)))^{(p-2)/p}(u(x(t))(y(t))^2 + 2v(x(t)))\right] dt \leqslant 0$$

with vanishing integrands as $t \to +\infty$. *Moreover, in probability, we find the limit*

$$\lim_{t\to+\infty} [V_{p,c}(x(t), y(t))]^{(p-2)/p}(u(x(t))(y(t))^2 + 2v(x(t))) = 0. \tag{62}$$

Proof. Recall $m(p,c)$ defined by Eq. (52) and from previous proof of Theorem 9 that

$$\begin{aligned} \mathscr{L}V_{p,c}(x,y) &\leqslant p(V_{p,c}(x,y))^{(p-2)/p}[-2cf(x) + m(p,c)(\sigma_2^2 + \sigma_3^2|f(x)|)]\frac{y^2}{2} \\ &\quad + \frac{p}{2}(V_{p,c}(x,y))^{(p-2)/p}\left[m(p,c)(\sigma_0^2 + \sigma_1^2 x^2 + \sigma_4^2(h(x))^2) - 2g(x)\int_0^x f(z)\,dz\right] \\ &= \frac{p}{2}(V_{p,c}(x,y))^{(p-2)/p} u(x) y^2 + p(V_{p,c}(x,y))^{(p-2)/p} v(x) \end{aligned} \tag{63}$$

for all $x, y \in \mathbb{R}^1$, where $u(x)$ and $v(x)$ are defined as in Eqs. (58) and (60), respectively. Note that

$$\forall x, y \in \mathbb{R}^1 : \quad \mathscr{L}V_{p,c}(x,y) \leqslant 0 \tag{64}$$

under the condition of Eqs. (58) and (59). Apply Dynkin's formula Eq. (11) with $\tau_r(t) = t$ in order to arrive at

$$\mathbb{E}[V_{p,c}(x(t), y(t))] = \mathbb{E}[V_{p,c}(x(s), y(s))] + \mathbb{E}\left[\int_s^t \mathscr{L}V_{p,c}(x(r), y(r))\,dr\right] \leqslant \mathbb{E}[V_{p,c}(x(s), y(s))] \tag{65}$$

for all $t \geqslant s \geqslant 0$. Hence the function $t \in \mathbb{R}^1_+ \mapsto z(t) := \mathbb{E}[V_{p,c}(x(t), y(t))]$ is decreasing and bounded by $\mathbb{E}[V_{p,c}(x_0, y_0)]$. The monotone convergence theorem from calculus implies that the finite limit $\lim_{t\to+\infty} z(t) \geqslant 0$ must exist. Thus, Eq. (61) is confirmed under Eqs. (58) and (59). Now, put the negative parts of the right-hand side of Eq. (65) to the left-hand side of obtained moment inequality Eq. (65) and replace the positive term of $\mathbb{E}[V_{p,c}(x(t), y(t))]$ by the trivial estimate 0 from below. This yields that

$$0 \leqslant -\frac{p}{2}\int_0^t \mathbb{E}[[V_{p,c}(x(s), y(s))]^{(p-2)/p}[u(x(s))(y(s))^2 + 2v(x(s))]\,ds \leqslant \mathbb{E}[V_{p,c}(x(0), y(0))] = \mathbb{E}[V_{p,c}(x_0, y_0)] < +\infty. \tag{66}$$

Notice that this chain of inequalities is uniformly bounded in t by the finite constant $\mathbb{E}[V_{p,c}(x_0, y_0)]$. Moreover, these integrals are nondecreasing in t thanks to the set of conditions (v). So, by bounded convergence theorem from calculus, we know about the existence and finiteness of improper integrals

$$\int_0^{+\infty} \mathbb{E}[-[V_{p,c}(x(s),y(s))]^{(p-2)/p}(u(x(s))(y(s))^2 + 2v(x(s)))]\,ds \geqslant 0. \tag{67}$$

Note that the integrand

$$f_p(s) = \mathbb{E}[-[V_{p,c}(x(s),y(s))]^{(p-2)/p}(u(x(s))(y(s))^2 + 2v(x(s)))] \tag{68}$$

is nonnegative and Lipschitz-continuous (hence uniformly continuous). Therefore, an application of standard convergence theory of improper integrals over nonnegative, Lipschitz-continuous functions f_p leads to the fact that those integrands f_p (moment expressions) have to converge to 0 as their argument s tends to $+\infty$ (with respect to the Lebesgue measure on the real line). Furthermore, from probability theory (e.g. see [36]), we know that this property implies convergence in probability of involved random variables as claimed in the conclusion of Theorem 11. □

Theorem 12 (*On pathwise limit behavior*). *Let $V_{p,c}$ be as in* (51). *Assume that conditions* (i)–(v) *from Theorem* 11 *are satisfied for a constant $p>2$, and*

$$\sup_{x\in\mathbb{R}^1} |f(x)| + \sup_{x\in\mathbb{R}^1} \frac{g(x)}{1+|x|} < +\infty. \tag{69}$$

Then, the limit $\lim_{t\to+\infty} V_{p,c}(x(t),y(t))$ exists almost surely, and the limits

$$\lim_{t\to+\infty} [V_{p,c}(x(t),y(t))]^{(p-2)/p} u(x(t))(y(t))^2 = 0, \tag{70}$$

$$\lim_{t\to+\infty} [V_{p,c}(x(t),y(t))]^{(p-2)/p} v(x(t)) = 0 \tag{71}$$

hold almost surely, where u and v are defined as in Theorem 11.

Proof. We only need to apply the version of stochastic LaSalle-type Theorem 2.1 from Mao [35] since all its assumptions are satisfied, in particular, the uniform boundedness of p-th moments of solutions $(x(t),y(t))$. One may use the nonnegative wedge function

$$w(x,y) = -\frac{p}{2}[V_{p,c}(x,y)]^{(p-2)/p} u(x)y^2 - p[V_{p,c}(x,y)]^{(p-2)/p} v(x) \geqslant 0, \tag{72}$$

where $u(x)$ and $v(x)$ are defined as in Eqs. (58) and (59), respectively. Then, the conclusion on almost sure limits along both $V_{p,c}$ and w follows immediately. □

Example 2 (*Random pendulum with oscillating resistance*). Consider the pendulum with locally Gaussian randomly perturbed nonlinear restoring force

$$\ddot{x} + a\cos(x)\dot{x} + b\sin(x) = \sigma\sin(x)\xi_t, \tag{73}$$

where $a,b,\sigma \in \mathbb{R}^1$ and ξ Gaussian white noise. Hence, we have

$$g(x) = b\sin(x), \quad f(x) = a\cos(x), \quad h(x) = \sin(x) \tag{74}$$

and parameters

$$\sigma_0 = \sigma_1 = \sigma_2 = \sigma_3 = 0, \quad \sigma_4 = \sigma. \tag{75}$$

Suppose that the "matching condition of sign-stability"

$$ab > 0 \tag{76}$$

is met. Take $p>2$, $c=0$, $m(p,c) = 2p-3$. Consider

$$V_{p,0}(x,y) = (K_V + (y + a\sin(x))^2 + 2b(1-\cos(x)))^{p/2}, \tag{77}$$

where

$$K_V = \begin{cases} 0 & \text{if } b \geqslant 0, \\ -4b & \text{if } b < 0. \end{cases} \tag{78}$$

Then, applying Theorem 11, the condition for stability of moments along $V_{p,0}$ essentially reduces to

$$v(x) = [(2p-3)\sigma^2 - 2ab](\sin(x))^2 \leqslant 0, \tag{79}$$

which is trivially fulfilled whenever $\sigma^2 \leqslant 2ab/(2p-3)$, i.e. when the product of resistance parameter a and restoring parameter b is large enough or, on the other hand, the noise intensity σ is sufficiently small. Note that $u(x)=0$ in Eq. (58) vanishes for all x here (since $c=0$ and $\sigma_2=\sigma_3=0$). Suppose that

$$\mathbb{E}[(K_V + (y_0 + a\sin(x_0))^2 + 2b(1-\cos(x_0)))^{p/2}] < +\infty. \tag{80}$$

Then, by Theorem 11, one can easily conclude the uniform boundedness of all moments

$$\mathbb{E}[(K_V + [y(t) + a\sin(x(t))]^2 + 2b(1-\cos(x(t))))^{p/2}] \leqslant \mathbb{E}[(K_V + (y_0 + a\sin(x_0))^2 + 2b(1-\cos(x_0)))^{p/2}]. \quad (81)$$

Furthermore, if additionally $\sigma^2 < 2ab/(2p-3)$ holds then we may apply Theorem 12 too. In this case, we may conclude that the finite limit

$$\lim_{t\to+\infty}([y(t) + a\sin(x(t))]^2 + b(1-\cos(x(t)))) \quad (82)$$

exists (a.s.) and

$$\lim_{t\to+\infty} V^{(p-2)/p}(x(t), y(t))\sin(x(t)) = 0 \quad \text{(a.s.)}. \quad (83)$$

The latter limit can be only established if

$$\lim_{t\to+\infty}\sin(x(t)) = 0 \quad \text{(a.s.)} \quad (84)$$

or

$$\lim_{t\to+\infty}(K_V + [y(t) + a\sin(x(t))]^2 + b(1-\cos(x(t)))) = 0 \quad (85)$$

hold almost surely. Consequently, the x-limit $\lim_{t\to+\infty}x(t)$ must be contained in the limit set

$$\{k\pi : k \in Z\} \quad (86)$$

(Z is the set of all integers). Thus, we may conclude that

$$\lim_{t\to+\infty}[y(t)]^{(p-2)/p}\sin(x(t)) = 0 \quad (a.s.) \quad (87)$$

by evaluating the assertions of Theorem 12. Moreover, we know that the limit $\lim_{t\to+\infty}\mathbb{E}[y(t)]^2 = 0$ from uniform boundedness of moments along $V_{p,0}$. That is this random oscillator with sufficiently small noise intensity σ of unbounded noise terms will eventually rest in its rest point $(0,0)$ with probability one as time advances to + infinity, even though we may observe the periodic possibility of negative resistance coefficients $f(x) = a\cos(x)$ (recall that solutions of such oscillators are Hölder-continuous almost surely, see Mao [11]).

Example 3 (*A modified Van der Pol oscillator*). Consider the modified Van der Pol oscillator with locally Gaussian randomly perturbed nonlinear restoring force

$$\ddot{x} + \varepsilon(x^2-1)\dot{x} + \omega^2 x^{2n}(x^3-3x) = \sigma h(x)\xi_t, \quad (88)$$

where $\varepsilon > 0, \omega, \sigma \in \mathbb{R}^1$, $n \in \mathbb{N}$ and ξ Gaussian white noise. Let us briefly discuss the role of noise intensity $h(x)$ to guarantee moment-stable dynamics of these oscillators. We may apply Theorem 11. For this purpose, set

$$f(x) = \varepsilon(x^2-1), \quad g(x) = \omega^2 x^{2n}(x^3-3x). \quad (89)$$

Note that

$$\int_0^x f(z)\,\mathrm{d}z = \frac{\varepsilon}{3}(x^3-3x), \quad (90)$$

hence, for all $x \in \mathbb{R}^1$, we have

$$g(x)\int_0^x f(z)\,\mathrm{d}z = \frac{\omega^2\varepsilon}{3}x^{2n}(x^3-3x)^2 \geqslant 0. \quad (91)$$

This observation of nonnegativity allows us to formulate a sufficient criterion for the uniform boundedness of moments along

$$V_{p,0}(x,y) = \left(K_V + \left(y + \frac{\varepsilon}{3}(x^3-3x)\right)^2 + 2\omega^2\left(\frac{x^{2n+4}}{2n+4} - \frac{3x^{2n+2}}{2n+2}\right)\right)^{p/2} \quad (92)$$

and the existence of certain limits in probability by the help of Theorem 11, whereas Theorem 12 is not applicable here due to the obvious violation of condition (69). We have to choose $c = 0$ and $\sigma_2 = \sigma_3 = 0$ since f may take on negative values. Therefore, the auxiliary function $u(x) = 0$ defined by (58) vanishes for all $x \in \mathbb{R}^1$. However, we may gain the conclusion from Theorem 11 that either $\lim_{t\to+\infty}V_{p,0}(x(t), y(t)) = 0$ or $\lim_{t\to+\infty}v(x(t)) = 0$, provided that $\mathbb{E}[V_{p,0}(x(0), y(0))] < +\infty$ and

$$\forall x \in \mathbb{R}^1: \quad v(x) = (2p-3)\sigma^2(h(x))^2 - \frac{2\varepsilon\omega^2}{3}x^{2n}(x^3-3x)^2 \leqslant 0. \quad (93)$$

Thus, for nonrandom initial values (x_0, y_0), $v(x) \leqslant 0$ for all $x \in \mathbb{R}^1$ clearly represents a sufficient condition on the noise intensity $h(x)$ and all involved parameters σ, $p \geqslant 2$, $\varepsilon > 0$ and ω in order to guarantee moment stability and convergence of solutions $(x(t), y(t))$ of modified Van der Pol system Eq. (88) in probability on infinite time-intervals $[0, +\infty)$. For example, if

$p=2$ then we need to require that

$$|\sigma h(x)| \leqslant |\omega| \frac{\sqrt{6\varepsilon}}{3} |x|^n |x^3 - 3x|. \tag{94}$$

Moreover, we may choose the constant $K_V = K_V(n,\omega) > 0$ in the Lyapunov-type functional $V_{p,0}$ sufficiently large such that $V_{p,0}$ has no real root and is strictly positive. Therefore, in this case, we even conclude that

$$\lim_{t\to+\infty} v(x(t)) = 0 \tag{95}$$

in probability and $\lim_{t\to+\infty}\mathbb{E}[v(x(t))]=0$, where $v(x)$ is as defined by Eq. (60). In particular, the case $h(x)=x^{n+1}(x^2-3)$ with $3\sigma^2 < 2\varepsilon\omega^2$ leads to the conclusion that the x-limit must be contained in the set $\{0,+\sqrt{3},-\sqrt{3}\}$ with probability one and y-limit is identical to 0 as time t tends to $+\infty$ (recall that $x(t)=x(0)+\int_0^t y(s)\,ds$ and use theory of improper integration).

5. Conclusion

The idea of using Lyapunov functions for qualitative analysis of dynamical systems is known a long time. However, the explicit construction remains to be a challenging task in stochastic settings. In this respect, we overcome this dilemma by constructing and verifying the use of certain Lyapunov functions for the quite general class of stochastic Liénard oscillators in this paper. Indeed, we have found a new type of matching condition (57) between damping terms and restoring force to ensure meaningful models with stable scenarios under stochastic perturbations. This condition guarantees us boundedness and stability of its solutions.

Further research could treat systems of coupled stochastic Liénard oscillators by appropriate Lyapunov functions or functionals as they occur in the space-discretization of stochastic partial differential equations, in stochastic systems with memory or other types of noises such as locally non-Gaussian ones.

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