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Nonlocal third-order shear deformation plate theory with application to bending and vibration of plates

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ABSTRACT

The third-order shear deformation plate theory of Reddy [A simple higher-order theory for laminated composite plates, *J. Appl. Mech*. 51 (1984) 745–752] is reformulated using the nonlocal linear elasticity theory of Eringen. This theory has ability to capture the both small scale effects and quadratic variation of shear strain and consequently shear stress through the plate thickness. Analytical solutions of bending and free vibration of a simply supported rectangular plate are presented using this theory to illustrate the effect of nonlocal theory on deflection and natural frequency of the plates. Finally, the relations between nonlocal third-order, first-order and classical theories are discussed by numerical results.

1. Introduction

Due to the rapid development of technology, especially in micro- and nano-scale fields, one must consider small scale effects and atomic forces to obtain solutions with acceptable accuracy. Neglecting these effects in some cases may result in completely incorrect solutions and hence wrong designs. Some methods (e.g., molecular dynamics [1–3]) are presented in recent years which consider size effects and atomic lengths. Nonetheless, all of these method involve solving a large number of equations, hence, they have difficulties in handling systems with large length and time scales. Therefore, modeling of the large systems is left to continuum mechanics approach. One of the well known continuum mechanics theory that includes small scale effects with good accuracy is the nonlocal theory of Eringen [4,5].

Compared to classical continuum mechanics theories, nonlocal theory of Eringen has capability to predict behavior of the large nano-sized structures, while it avoids solving the large number of equations. Nonlocal theory of Eringen is based on this assumption that the stress at a point is considered as a function of the strain field at all neighbor points in the continuum body. The inter-atomic forces and atomic length scales directly come to the constitutive equations as material parameters [6]. Thus, it appears that nonlocal continuum mechanics could potentially play a useful role in near future. Hence, many papers have been published on this topic, especially for analyzing of nano-structures (see, for example, the nonlocal theory of longitudinal waves in an elastic circular bar [7], nonlocal theory solution of two collinear cracks in the functionally graded materials [8], buckling analysis of CNT based on nonlocal theory [9], nonlocal theories of beams [10,11]). Contrary to one-dimensional nonlocal theories, there are only a few studies on two-dimensional ones [12].

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Recently, some researchers have studied micro and nano plates for different applications [13,14]. These studies were based on classical and first-order theory of plates. To analyse two-dimensional nano plate with accurate stress fields, further studies are needed. In the classical plate theory, the effect of the transverse shear strain and shear stress are neglected. In the first-order shear deformation theory, the transverse shear strain and consequently transverse shear stress are represented as constant through the plate thickness, which is a gross approximation of the true variation that vanishes on the top and bottom plane of the plate. To validate this discrepancy between the true variation and the constant state of stress, shear correction factor is introduced such that the energy due to the transverse shear stress in both cases is the same.

The third-order shear deformation theory of Reddy [15] is based on a displacement field that includes the cubic term in the thickness coordinate, hence the transverse shear strain and hence stress are represented as quadratic through the plate thickness and vanish on the bounding planes of the plate. Consequently, the shear correction factor is avoided in this theory. In spite of relatively more complex algebraic equations and computational effort compared to the classical and first-order theories, the third-order shear deformation theory yields results that are close to 3-D elasticity solutions [16,17]. There are some articles that incorporate the third-order plate theory to obtain more accurate results [15–18]. Therefore, it is useful to study the extension of the third-order plate theory to include the size effects. The present study deals with the use of the nonlocal third-order plate theory in bending and vibration response of plates.

2. Review of nonlocal elasticity

In nonlocal elasticity theory, it is assumed that the stress at a point in a continuum body is function of the strain at all neighbor points of the continuum, hence the effects of small scale and atomic forces are considered as material parameters in the constitutive equation. Following experimental observations, Eringen proposed a constitutive model that expresses the nonlocal stress tensor at point **x** as

$$t_{ij} = \int_v \alpha(|x' - x|)\sigma_{ij}(x')\,\mathrm{d}v(x'), \tag{1}$$

where the volume integral in Eq. (1) is over the region v occupied by the body and σ_{ij} is the Hookean stress tensor defined as

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl} \tag{2}$$

and $\alpha(|x' - x|)$ is the kernel function which is normalized over the volume of the body, i.e., $\int_v \alpha(|x'|)\,\mathrm{d}v = 1$. This function can be obtained by matching the lattice dynamics with nonlocal results [5]. For example, the kernel function for 2-D problems has the form

$$\alpha(|x|) = (2\pi l^2\tau^2)^{-1}K_0(|x|/l\tau), \quad \tau = e_0 a/l, \tag{3}$$

where K_0 is the modified Bessel function, a and l are internal and external characteristic lengths, and e_0 is material constant which is defined by the experiment. On the other hand, nonlocal elasticity involves spatial integrals that represent weighted averages of the contributions of the strain of all points in the continuum body to the stress tensor at a point.

In the nonlocal linear elasticity, equations of motion can be obtained from nonlocal balance law

$$t_{ij,j} + f_i = \rho\ddot{u}_i, \tag{4}$$

where i, j take the symbols x, y, z and f_i, ρ and u_i are the components of the body force, mass density and displacement vector [5]. By substituting Eq. (1) into Eq. (4), the integral form of nonlocal constitutive equation is obtained. Because solving an integral equation is more difficult than a differential equation, Eringen [5,6] proposed a differential form of the nonlocal constitutive equation as

$$\sigma_{ij,j} + \mathscr{L}(f_i - \rho\ddot{u}_i) = 0 \tag{5}$$

in which the linear differential operator $\mathscr{L}$ defined by

$$\mathscr{L} = 1 - \mu\nabla^2, \quad \mu = (e_0 a)^2. \tag{6}$$

By applying this operator on Eq. (1), the constitutive equation can be simplified to

$$[1 - \mu\nabla^2]t_{ij} = \sigma_{ij} \tag{7}$$

Eq. (7) is simpler and more convenient than the integral relation (1) to apply to various linear elasticity problems.

3. Plate equations of nonlocal elasticity

Using Eqs. (2) and (7), stress resultants introduced in plate and shell theories can be reformulated in term of strain for the nonlocal theory. In plate theories based on plane-stress assumption, we take $\sigma_{zz} = 0$ and the resulting theory becomes two-dimensional.

Consider a (x, y, z) coordinate system with the xy-plane coinciding with the mid-plane of the plate. So the stress–strain relations of plane-stress can be expressed as

$$\sigma_{\alpha\beta} = \hat{c}_{\alpha\beta\omega\rho}\varepsilon_{\omega\rho}, \tag{8}$$

where $\hat{c}_{\alpha\beta\omega\rho} = c_{\alpha\beta\omega\rho} - c_{\varepsilon\beta zz}c_{zz\omega\rho}/c_{zzzz}$.

and transverse shear stress–strain relation is expressed as

$$\sigma_{\alpha z} = 2\hat{c}_{\alpha z\omega z}\varepsilon_{\omega z}, \tag{9}$$

where α, β, ω and ρ take the symbols x, y.

The relations between stress resultants in local theory and nonlocal theory are defined by integrating Eq. (7) through the plate thickness:

$$\begin{aligned} [1-\mu\nabla^2]N_{ij} &= N^L_{ij}, \\ [1-\mu\nabla^2]M_{ij} &= M^L_{ij}, \\ [1-\mu\nabla^2]P_{ij} &= P^L_{ij}, \end{aligned} \tag{10}$$

where

$$\begin{Bmatrix} N_{\alpha\beta} \\ M_{\alpha\beta} \\ P_{\alpha\beta} \end{Bmatrix} = \int_{-h/2}^{h/2} t_{\alpha\beta} \begin{Bmatrix} 1 \\ z \\ z^3 \end{Bmatrix} \mathrm{d}z, \quad \begin{Bmatrix} N_{\alpha z} \\ R_{\alpha z} \end{Bmatrix} = \int_{-h/2}^{h/2} t_{\alpha z} \begin{Bmatrix} 1 \\ z^2 \end{Bmatrix} \mathrm{d}z., \tag{11}$$

$$\begin{Bmatrix} N^L_{\alpha\beta} \\ M^L_{\alpha\beta} \\ P^L_{\alpha\beta} \end{Bmatrix} = \int_{-h/2}^{h/2} \sigma_{\alpha\beta} \begin{Bmatrix} 1 \\ z \\ z^3 \end{Bmatrix} \mathrm{d}z, \quad \begin{Bmatrix} N^L_{\alpha z} \\ R^L_{\alpha z} \end{Bmatrix} = \int_{-h/2}^{h/2} \sigma_{\alpha z} \begin{Bmatrix} 1 \\ z^2 \end{Bmatrix} \mathrm{d}z. \tag{12}$$

The superscript L denoted the quantities in local third-order shear deformation theory and h is the thickness of the plate.

The governing equation of the plate in nonlocal theory can be determined by integrating Eq. (4) through the plate thickness and noting Eq. (11)

$$N_{i\alpha,\alpha} + F_i = \int_{-h/2}^{h/2} \rho \ddot{u}_i \,\mathrm{d}z, \tag{13}$$

where $F_i = \int_{-h/2}^{h/2} f_i dz$. By multiplying Eq. (4) by z and then integrating from it through plate thickness and using integration-by-parts, we obtain

$$M_{\alpha\beta,\beta} - N_{\alpha z} = \int_{-h/2}^{h/2} \rho \ddot{u}_\alpha \cdot z \,\mathrm{d}z. \tag{14}$$

Similarly, for higher-order stress resultants, Eq. (4) is multiplied by higher powers of the thickness coordinate z and integrating through the plate thickness. We obtain

$$P_{\alpha\beta,\beta} - 3R_{\alpha z} = \int_{-h/2}^{h/2} \rho \ddot{u}_\alpha \cdot z^3 \,\mathrm{d}z. \tag{15}$$

In general, differential operator ∇ in Eq. (7) is the 3-D Laplace operator. For 2-D problems, the operator ∇ may be reduced to 2-D one. Thus, the linear differential operator $\mathscr{L}$ becomes

$$\mathscr{L}' = 1 - \mu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right). \tag{16}$$

It is clear that the operator $\mathscr{L}'$ is independent of the z direction.

To express the governing equations of motion in terms of local stress resultants, the reduced linear differential operator $\mathscr{L}'$ is used in Eqs. (13)–(15)

$$N^L_{i\alpha,\alpha} = \left[1 - \mu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right]\left(\int_{-h/2}^{h/2} \rho \ddot{u}_i \,\mathrm{d}z - F_i\right), \tag{17}$$

$$M^L_{\alpha\beta,\beta} - N^L_{\alpha z} = \left[1 - \mu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right]\left(\int_{-h/2}^{h/2} \rho \ddot{u}_\alpha \cdot z \,\mathrm{d}z\right), \tag{18}$$

$$P^L_{\alpha\beta,\beta} - 3R^L_{\alpha z} = \left[1 - \mu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right]\left(\int_{-h/2}^{h/2} \rho \ddot{u}_\alpha \cdot z^3\,\mathrm{d}z\right). \tag{19}$$

Later, the equations of motion for nonlocal third-order shear deformation plate theory will be presented based on Eqs. (17)–(19).

4. Nonlocal third-order shear deformation theory

The third-order shear deformation theory (TSDT) extends the first-order theory by assuming that shear strain and consequently shear stress are not constant through the plate thickness (see Reddy [15,17]). The displacement field of the third-order theory of plates is given by

$$u_\alpha = u^0_\alpha + z\phi_\alpha - \frac{4z^3}{3h^2}(\phi_\alpha + w^0_{,\alpha}), \quad u_z = w^0, \tag{20}$$

where u_α are the inplane displacements of point on the mid-plane (i.e., $z = 0$) at $t = 0$, u_z is the transverse displacement of the mid-plane of the plate, and ϕ_α denotes the slope of the transverse normal on mid-plane.

The nonzero components of the strain can be defined by substituting Eq. (20) into the linear strain–displacement relations of the TSDT [17].

$$\varepsilon_{\alpha\beta} = \varepsilon^0_{\alpha\beta} + z\varepsilon^1_{\alpha\beta} + z^3\varepsilon^3_{\alpha\beta}, \tag{21}$$

with

$$\varepsilon^0_{\alpha\beta} = \tfrac{1}{2}(u^0_{\alpha,\beta} + u^0_{\beta,\alpha}),$$

$$\varepsilon^1_{\alpha\beta} = \tfrac{1}{2}(\phi_{\alpha,\beta} + \phi_{\beta,\alpha}),$$

$$\varepsilon^3_{\alpha\beta} = \frac{c_1}{2}(\phi_{\alpha,\beta} + \phi_{\beta,\alpha} + 2w^0_{,\alpha\beta}).$$

The transverse shear strain components are of the form

$$\gamma_{\alpha z} = \gamma^0_{\alpha z} + z^2\gamma^2_{\alpha z}, \tag{22}$$

with

$$\gamma^0_{\alpha z} = \tfrac{1}{2}(\phi_\alpha + w^0_{,\alpha}), \quad \gamma^2_{\alpha z} = -\frac{c_2}{2}(\phi_\alpha + w^0_{,\alpha}).$$

The parameters c_1, c_2 are defined as

$$c_1 = \frac{4}{3h^2}, \quad c_2 = 3c_1 = \frac{4}{h^2}.$$

By substituting the displacement field into Eqs. (13)–(15), we obtain

$$N_{i,\alpha,\alpha} + F_i = I_0\ddot{u}^0_i, \tag{23}$$

$$M_{\alpha\beta,\beta} - N_{\alpha z} = I_2\ddot{\phi}_\alpha - c_1 I_4(\ddot{\phi}_\alpha + \ddot{w}^0_{,\alpha}), \tag{24}$$

$$P_{\alpha\beta,\beta} - 3R_{\alpha z} = I_4\ddot{\phi}_\alpha - c_1 I_6(\ddot{\phi}_\alpha + \ddot{w}^0_{,\alpha}). \tag{25}$$

Then Eq. (23) for $i = 3$ and Eq. (24) can be combined with Eq. (25) to drive the following governing equations for flexural response of the nonlocal third-order plate theory,

$$N_{\alpha z,\alpha} - c_2 R_{\alpha z,\alpha} + q_z + c_1 P_{\alpha\beta,\alpha\beta} = I_0\ddot{w}^0 + c_1[I_4\ddot{\phi}_{\alpha,\alpha} - c_1 I_6(\ddot{\phi}_{\alpha\alpha} + \ddot{w}^0_{,\alpha\alpha}), \tag{26}$$

$$M_{\alpha\beta,\beta} - c_1 P_{\alpha\beta,\beta} - N_{\alpha z} + c_2 R_{\alpha z} = I_2\ddot{\phi}_\alpha - c_1 I_4(\ddot{\phi}_\alpha + \ddot{w}^0_{,\alpha}) - c_1[I_4\ddot{\phi}_\alpha + \ddot{w}^0_{,\alpha}). \tag{27}$$

The local resultant forces and moments for the third-order plate theory can be obtained by substituting Eqs. (8), (9), (21) and (22) into Eq. (12)

$$N^L_{\alpha\beta} = A_{\alpha\beta\omega\rho}\varepsilon^0_{\omega\rho} + B_{\alpha\beta\omega\rho}\varepsilon^1_{\omega\rho} + D_{\alpha\beta\omega\rho}\varepsilon^3_{\omega\rho}, \tag{28}$$

$$N^L_{z\beta} = 2A_{z\beta z\rho}\gamma^0_{z\rho} + 2C_{z\beta z\rho}\gamma^2_{z\rho}, \tag{29}$$

$$M^L_{\alpha\beta} = B_{\alpha\beta\omega\rho}\varepsilon^0_{\omega\rho} + C_{\alpha\beta\omega\rho}\varepsilon^1_{\omega\rho} + E_{\alpha\beta\omega\rho}\varepsilon^3_{\omega\rho}, \tag{30}$$

$$R^{L}_{z\beta} = 2C_{z\beta z\rho}\gamma^{0}_{z\rho} + 2E_{z\beta z\rho}\gamma^{2}_{z\rho}, \tag{31}$$

$$P^{L}_{\alpha\beta} = D_{\alpha\beta\omega\rho}\varepsilon^{0}_{\alpha\rho} + E_{\alpha\beta\omega\rho}\varepsilon^{1}_{\omega\rho} + F_{\alpha\beta\omega\rho}\varepsilon^{3}_{\omega\rho}, \tag{32}$$

where

$$(A_{\alpha\beta\omega\rho}, B_{\alpha\beta\omega\rho}, C_{\alpha\beta\omega\rho}, D_{\alpha\beta\omega\rho}, E_{\alpha\beta\omega\rho}, F_{\alpha\beta\omega\rho}) = \int_{h/2}^{h/2} \hat{c}_{\alpha\beta\omega\rho}(1, z, z^2, z^3, z^4, z^6)\,\mathrm{d}z.$$

In Eqs. (23), (26) and (27), we derived the governing equations for nonlocal third-order plate theory. By applying linear differential operator $\mathscr{L}$ on Eqs. (23), (26) and (27) and noting Eqs. (28)–(32), the equations of motion for nonlocal third-order shear deformation theory can be obtained in terms of displacements as

$$A_{\alpha\beta\omega\rho}u^{0}_{\omega,\rho B} + B_{\alpha\beta\omega\rho}\phi_{\omega,\rho B} - c_1 D_{\alpha\beta\omega\rho}(\phi_{\omega,\rho B} + w^{0}_{,\beta\omega\rho}) + [1-\mu\nabla^2](F_\alpha - I_0\ddot{u}^{0}_{\alpha}) = 0, \tag{33}$$

$$\begin{aligned}
&A_{z\alpha z\rho}(\phi_{\rho,\alpha} + w^{0}_{,\alpha\rho}) - c_2 C_{z\alpha z\rho}(\phi_{\rho,\alpha} + w^{0}_{,\alpha\rho})\\
&\quad - c_2[c_{z\alpha z\rho}(\phi_{\rho,\alpha} + w^{0}_{,\alpha\rho}) - c_2 E_{z\alpha z\rho}(\phi_{\rho,\alpha} + w^{0}_{,\alpha\rho})]\\
&\quad + c_2[D_{\alpha\beta\omega\rho}u^{0}_{\omega,\rho\alpha\beta} + E_{\alpha\beta\omega\rho}\phi_{\omega,\rho\alpha\beta} - c_1 F_{\alpha\beta\omega\rho}(\phi_{\omega,\rho\alpha\beta} + w^{0}_{,\alpha\beta\omega\rho})]\\
&\quad + [1-\mu\nabla^2](q_z - I_0\ddot{w}^0 + c_1]I_4\ddot{\phi}_{\alpha,\alpha} - c_1 I_6(\ddot{\phi}_{\alpha,\alpha} + \ddot{w}^{0}_{,\alpha,\alpha})]) = 0,
\end{aligned} \tag{34}$$

$$\begin{aligned}
&B_{\alpha\beta\omega\rho}u^{0}_{\omega,\rho B} + c_{\alpha\beta\omega\rho}\phi_{\omega,\rho B} - c_1 E_{\alpha\beta\omega\rho}(\phi_{\omega,\rho B} + w^{0}_{,\beta\omega\rho})\\
&\quad - c_1[D_{\alpha\beta\omega\rho}u^{0}_{\omega,\rho B} + E_{\alpha\beta\omega\rho}\phi_{\omega,\rho B} - c_1 F_{\alpha\beta\omega\rho}(\phi_{\omega,\rho B} + w^{0}_{,\beta\omega\rho})]\\
&\quad - A_{z\alpha z\rho}(\phi_\rho + w^{0}_{,\rho}) + c_2 c_{z\alpha z\rho}(\phi_\rho + w^{0}_{,\rho}) + c_2[c_{z\alpha z\rho}(\phi_\rho + w^{0}_{,\rho})\\
&\quad - c_2 E_{z\alpha z\rho}(\phi_\rho + w^{0}_{,\rho}) - [1-\mu\nabla^2][I_2\ddot{\phi}_\alpha - c_1 I_4(\ddot{\phi}_\alpha + \ddot{w}^{0}_{,\alpha})\\
&\quad - c_2(I_4\ddot{\phi}_\alpha - c_1 I_6(\ddot{\phi}_\alpha + \ddot{w}^{0}_{,\alpha}))] = 0,
\end{aligned} \tag{35}$$

where $I_k = \int_{-h/2}^{h/2}\rho(z)^k\,\mathrm{d}z \quad (k = 0, 2, 4, 6)$.

5. Variational statements

The variational statements facilitate the direct derivation of the equations of motion in term of the displacements. Hence, we also present the variational form of governing equations which is useful in integral formulations and displacement finite element formulations.

The governing equations of the third-order nonlocal plate theory can be derived using dynamic version of the principle of virtual displacement (Hamilton's principle)

$$0 = \int_0^T (\delta U + \delta V - \delta K)\,\mathrm{d}t.$$

The Hamilton principle in case of nonlocal third-order plate theory takes the form

$$\begin{aligned}
0 = \int_0^t \int_\Omega & N_{xx}\delta\varepsilon^{0}_{xx} + M_{xx}\delta\varepsilon^{1}_{xx} - P_{xx}\delta\varepsilon^{3}_{xx} + N_{yy}\delta\varepsilon^{0}_{yy}\\
& + M_{yy}\delta\varepsilon^{1}_{yy} - P_{xx}\delta\varepsilon^{3}_{yy} + N_{xy}\delta\gamma^{0}_{yy} + M_{xy}\delta\gamma^{1}_{xy} - P_{xx}\delta\gamma^{3}_{xy}\\
& + N_{xz}\delta\gamma^{0}_{xz} - R_{xz}\delta\gamma^{1}_{xz} + N_{yz}\delta\gamma^{0}_{yz} - R_{yz}\delta\gamma^{1}_{yz} - q_z\delta w^0\\
& + \mu(q_{z,x}\delta w^{0}_{,x} + q_{z,y}\delta w^{0}_{,y}) - I_0\dot{u}^{0}_{x}\delta\dot{u}^{0}_{x} + I_0\mu(\dot{u}^{0}_{x,x}\delta\dot{u}^{0}_{x,x} + \dot{u}^{0}_{x,y}\delta\dot{u}^{0}_{x,y})\\
& - I_0\dot{u}^{0}_{y}\delta\dot{u}^{0}_{y} + I_0\mu(\dot{u}^{0}_{y,x}\delta\dot{u}^{0}_{y,x} + \dot{u}^{0}_{y,y}\delta\dot{u}^{0}_{y,y}) - I_0\dot{w}^0\delta\dot{w}^0\\
& + I_0\mu(\dot{w}^{0}_{,x}\delta\dot{w}^{0}_{,x} + \dot{w}^{0}_{,y}\delta\dot{w}^{0}_{,y}) - (I_2\dot{\phi}_x + \dot{w}^{0}_{,x})\delta\dot{\phi}_x\\
& + \mu[(I_2\dot{\phi}_{x,x} - c_1 I_4(\dot{\phi}_{x,x} + \dot{w}^{0}_{,xx}))\delta\dot{\phi}_{x,x}\\
& + (I_2\dot{\phi}_{x,y} - c_1 I_4(\dot{\phi}_{x,y} + \dot{w}^{0}_{,xy}))\delta\dot{\phi}x,y - (I_2\dot{\phi}_y - c_1 I_4(\dot{\phi}_y + \dot{w}^{0}_{,y}))\delta\dot{\phi}_y))\delta\dot{\phi}_y
\end{aligned}$$

$$
\begin{aligned}
&+\mu[(I_2\dot{\phi}_{y,x}-c_1I_4(\dot{\phi}_{y,x}+\dot{w}^0_{,yx}))\delta\dot{\phi}_{y,x}\\
&+(I_2\dot{\phi}_{y,y}-c_1I_4(\dot{\phi}_{y,y}+\dot{w}^0_{,yy}))\delta\dot{\phi}_{y,y}\\
&+c_1(I_4\dot{\phi}_{x,x}-c_1I_6(\dot{\phi}_{x,x}+\dot{w}^0_{,xx}))(\delta\dot{\phi}_{x,x}+\dot{w}^0_{,xx}\\
&-\mu[c_1(I_4\dot{\phi}_{x,x}-c_1I_6(\dot{\phi}_{x,y}+\dot{w}^0_{,xy}))(\delta\dot{\phi}_{x,y}+\delta\dot{w}^0_{,xy})\\
&+c_1(I_4\dot{\phi}_{x,y}-c_1I_6(\dot{\phi}_{x,y}+\dot{w}^0_{,xy}))(\delta\dot{\phi}_{x,y}+\delta\dot{w}^0_{,xy})]\\
&+c_1(I_4\dot{\phi}_{y}-c_1I_6(\dot{\phi}_{y}+\dot{w}^0_{,y}))(\delta\dot{\phi}_{y}+\delta\dot{w}^0_{y})\\
&-\mu[c_1(I_4\dot{\phi}_{y,x}-c_1I_6(\dot{\phi}_{y,x}+\dot{w}^0_{,yx}))(\delta\dot{\phi}_{y,x}+\delta\dot{w}^0_{,yx})\\
&+c_1(I_4\dot{\phi}_{y,y}-c_1I_6(\dot{\phi}_{y,y}+\dot{w}^0_{,yy}))(\delta\dot{\phi}_{y,y}+\delta\dot{w}^0_{,yy})]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.
\end{aligned}
\tag{36}
$$

By substituting nonlocal stress resultants in term of the displacements into the principle of virtual displacements and integrate by part, the equations of motion can be obtain

$$\delta u_x:\quad N_{xx,x}+N_{xy,y}-(1-\mu\nabla^2)I_0\ddot{u}^0_x=0, \tag{37}$$

$$\delta u_y:\quad N_{xy,x}+N_{yy,y}-(1-\mu\nabla^2)I_0\ddot{u}^0_y=0, \tag{38}$$

$$
\begin{aligned}
\delta w^0:\ &N_{zx,x}+N_{zy,y}-c_2(R_{zx,x}+R_{zy,y})+c_1(P_{xx,xx}+P_{yy,yy}+2P_{xy,xy})\\
&-(1-\mu\nabla^2[q_z-I_0\ddot{w}^0-c_1I_4(\ddot{\phi}_{x,x}+\ddot{\phi}_{y,y}-c_1^2I_6(\ddot{\phi}_{x,x}+\ddot{\phi}_{y,y}+\ddot{w}^0_{,xx}+\ddot{w}^0_{,yy}=0,
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
\delta\phi_x:\ &M_{xx,y}+M_{xy,y}-c_1(P_{xx,x}+P_{xy,y})N_{zx}+c_2R_{zx}\\
&-(1-\mu\nabla^2)[I_2\ddot{\phi}_x-c_1I_4(\ddot{\phi}_x+\ddot{w}^o_{,x})-c_2(I_4\ddot{\phi}_x-c_1I_6(\ddot{\phi}_x+\ddot{w}^o_{,x}))]=0,
\end{aligned}
\tag{40}
$$

$$
\begin{aligned}
\delta\phi_y:\ &M_{xy,y}+M_{yy,y}-c_1(P_{xy,x}+P_{yy,y})N_{zy}+c_2R_{zy}\\
&-(1-\mu\nabla^2)[I_2\ddot{\phi}_y-c_1I_4(\ddot{\phi}_y+\ddot{w}^o_{,y})-c_2(I_4\ddot{\phi}_y-c_1I_6(\ddot{\phi}_y+\ddot{w}^o_{,y}))]=0.
\end{aligned}
\tag{41}
$$

It can be verified that equations of motion associated with the variational statements are the same as Eqs. (33)–(35). In addition, the equations of motion of the conventional third-order plate theory are obtained from above equations by setting $\mu=0$ (see Reddy [17]).

6. The Navier solutions of nonlocal third-order shear deformation theory

Here, analytical solutions for bending and vibration of simply supported plates are presented using the nonlocal third-order plate theory to illustrate the small scale effects on deflections and natural frequencies of the plates. In order to simplify the governing equations, we consider the case of orthotropic rectangular plate with simply supported boundary conditions. For this case, In-plane displacements are uncoupled from the bending deflections (i.e., the coupling stiffness $B_{\alpha\beta\omega\rho}$ and $D_{\alpha\beta\omega\rho}$ are zero in Eqs. (34) and (35) for an orthotropic plate). For the static case, all time derivative terms are set to zero.

For an orthotropic plate, the local bending moments and higher-order stress resultants are related to the flexural (i.e., bending) deflections by

$$
\begin{aligned}
\begin{bmatrix} M^L_{xx}\\ M^L_{yy}\\ M^L_{xy}\\ N^L_{yz}\\ N^L_{xz}\end{bmatrix}
&=\begin{bmatrix} C_{11} & C_{12} & 0 & 0 & 0\\ C_{12} & C_{22} & 0 & 0 & 0\\ 0 & 0 & C_{66} & 0 & 0\\ 0 & 0 & 0 & A_{44} & 0\\ 0 & 0 & 0 & 0 & A_{55}\end{bmatrix}
\begin{bmatrix} \phi_{x,x}\\ \phi_{y,y}\\ \phi_{x,y}+\phi_{y,x}\\ w^0_{,y}+\phi_y\\ w^0_{,x}+\phi_x\end{bmatrix}\\
&-\begin{bmatrix} c_1E_{11} & c_1E_{12} & 0 & 0 & 0\\ c_1E_{12} & c_1E_{22} & 0 & 0 & 0\\ 0 & 0 & c_1C_{66} & 0 & 0\\ 0 & 0 & 0 & c_2C_{44} & 0\\ 0 & 0 & 0 & 0 & c_2C_{55}\end{bmatrix}
\begin{bmatrix} \phi_{x,x}+w^0_{,xx}\\ \phi_{y,y}+w^0_{,yy}\\ \phi_{x,y}+\phi_{y,x}+2w^0_{,xy}\\ w^0_{,y}+\phi_y\\ w^0_{,x}+\phi_x\end{bmatrix},
\end{aligned}
\tag{42a}
$$

$$\begin{bmatrix} P^L_{xx} \\ P^L_{yy} \\ P^L_{xy} \\ R^L_{yz} \\ R^L_{xz} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & 0 & 0 & 0 \\ E_{12} & E_{22} & 0 & 0 & 0 \\ 0 & 0 & E_{66} & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & C_{55} \end{bmatrix} \begin{bmatrix} \phi_{x,x} \\ \phi_{y,y} \\ \phi_{x,y} + \phi_{y,x} \\ w^0_{,y} + \phi_y \\ w^0_{,x} + \phi_x \end{bmatrix}$$

$$- \begin{bmatrix} c_1F_{11} & c_1F_{12} & 0 & 0 & 0 \\ c_1F_{12} & c_1F_{22} & 0 & 0 & 0 \\ 0 & 0 & c_1F_{66} & 0 & 0 \\ 0 & 0 & 0 & c_2E_{44} & 0 \\ 0 & 0 & 0 & 0 & c_2E_{55} \end{bmatrix} \begin{bmatrix} \phi_{x,x} + w^0_{,xx} \\ \phi_{y,y} + w^0_{,yy} \\ \phi_{x,y} + \phi_{y,x} + 2w^0_{,xy} \\ w^0_{,y} + \phi_y \\ w^0_{,x} + \phi_x \end{bmatrix}. \tag{42b}$$

Then the decoupled bending equations of motion can be obtained from Eqs. (34) and (35) as

$$\begin{aligned} & A_{55}(\phi_{x,x} + w^0_{,xx}) + A_{44}(\phi_{y,y} + w^0_{,yy}) - 2c_2C_{55}(\phi_{x,x} + w^0_{,xx}) \\ & - 2c_2C_{44}(\phi_{y,y} + w^0_{,yy}) + c_2^2E_{55}(\phi_{x,x} + w^0_{,xx}) + c_2^2E_{44}(\phi_{y,y} + w^0_{,yy}) \\ & + c_1E_{11}\phi_{x,xxx} + c_1E_{66}\phi_{x,xyy} + c_1(E_{12} + E_{66})\phi_{y,xxy} - c_1^2F_{11}(\phi_{x,xxx} + \omega^0_{,xxxx}) \\ & - c_1^2F_{12}(\phi_{y,xxy} + w^0_{,xxyy}) - c_1^2F_{66}(\phi_{x,xyy} + \phi_{y,xxy} + 2w^0_{,xxyy}) \\ & + c_1E_{22}\phi_{y,yyy} + c_1E_{66}\phi_{y,xxy} + c_1(E_{12} + E_{66})\phi_{x,xyy} - c_1^2F_{22}(\phi_{y,yyy} + \omega^0_{,yyyy}) \\ & - c_1^2F_{12}(\phi_{x,xyy} + w^0_{,yyxx}) - c_1^2F_{66}(\phi_{y,xxy} + \phi_{x,xyy} + 2w^0_{,yyxx}) \\ & + [1 - \mu\nabla^2](q_z - I_0\ddot{w}^0 - c_1[I_4(\ddot{\phi}_{y,y} + \ddot{\phi}_{x,x}) \\ & - c_1I_6(\ddot{\phi}_{x,x} + \ddot{w}^0_{,xx} + \ddot{\phi}_{y,y} + \ddot{w}^0_{,yy})]) = 0, \end{aligned} \tag{43}$$

$$\begin{aligned} & c_{11}\phi_{x,xx} + C_{66}\phi_{x,yy} + (C_{12} + C_{66})\phi_{y,xy} - c_1E_{11}(\phi_{x,xx} + w^0_{,xxx}) \\ & - c_1E_{12}(\phi_{y,xy} + w^0_{,xyy}) + c_1E_{66}(\phi_{x,yy} + \phi_{y,xy} + 2w^0_{,xyy}) \\ & - c_1E_{11}\phi_{x,xx} - c_1E_{66}\phi_{x,yy} - c_1(E_{12} + E_{66})\phi_{y,xy} + c_1^2F_{11}(\phi_{x,xx} + w^0_{,xxx}) \\ & + c_1^2F_{12}(\phi_{y,xy} + w^0_{,xyy}) + c_1^2F_{66}(\phi_{x,yy} + \phi_{y,xy} + 2w^0_{,xyy}) \\ & - A_{55}(\phi_x + w^0_{,x}) + 2c_2C_{55}(\phi_x + w^0_{,x}) - c_2^2E_{55}(\phi_x + w^0_{,x}) \\ & - [1 - \mu\nabla^2](I_2\ddot{\phi}_x - c_1I_4(\ddot{\phi}_x + \ddot{w}^0_{,x}) - c_1(I_4\ddot{\phi}_x - c_1I_6(\ddot{\phi}_x + \ddot{w}^0_{,x}))) = 0, \end{aligned} \tag{44}$$

$$\begin{aligned} & C_{22}\phi_{y,yy} + C_{66}\phi_{y,xx} + (C_{12} + C_{66})\phi_{x,xy} - c_1E_{22}(\phi_{y,yy} + w^0_{,yyy}) \\ & - c_1E_{12}(\phi_{x,xy} + w^0_{,yxx}) - c_1E_{66}(\phi_{y,xx} + \phi_{x,xy} + 2w^0_{,yxx}) \\ & - c_1E_{22}\phi_{y,yy} - c_1E_{66}\phi_{y,xx} - c_1(E_{12} + E_{66})\phi_{x,xy} + 2w^0_{,yxx} \\ & + c_1^2F_{12}(\phi_{x,xy} + w^0_{,yxx}) + c_1^2F_{66}(\phi_{y,xx} + \phi_{x,xy} + 2w^0_{,yxx}) \\ & - A_{44}(\phi_y + w^0_{,y}) + 2c_2C_{44}(\phi_y + w^0_{,y}) - c_2^2E_{44}(\phi_y + w^0_{,y}) \\ & - [1 - \mu\nabla^2](I_2\ddot{\phi}_y - c_1I_4(\ddot{\phi}_y + \ddot{w}^0_{,y}) - c_1(I_4\ddot{\phi}_y - c_1I_6(\ddot{\phi}_y + \ddot{w}^0_{,y}))) = 0. \end{aligned} \tag{45}$$

These equations may be reduced to those of a nonlocal third-order beam theory when the case of cylindrical bending is considered (see Reddy [10]).

For a simply supported plate, the boundary conditions are of the form

$$\begin{aligned} & w^0(x,0,t) = w^0(x,b,t) = w^0(0,y,t) = w^0(a,y,t) = 0, \\ & \phi_x(x,0,t) = \phi_x(x,b,t) = \phi_y(0,y,t) = \phi_x(a,y,t) = 0, \\ & M_{xx}(0,y,t) = M_{xx}(a,y,t) = M_{yy}(x,0,t) = M_{yy}(x,b,t) = 0, \\ & P_{xx}(0,y,t) = P_{xx}(a,y,t) = P_{yy}(x,0,t) = P_{yy}(x,b,t) = 0. \end{aligned} \tag{46}$$

For this set of boundary conditions, the Navier solution can be obtained [17]. According to the Navier solution theory, the generalized displacements at middle of the plane ($z=0$) are expanded in double Fourier series as

$$w^0(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}W_{mm}\cdot\sin(\zeta_n x)\cdot\sin(\eta_m y)\cdot e^{i\omega_{nm}t},$$

$$\phi_x(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}W_{nm}\cdot\cos(\zeta_n x)\cdot\sin(\eta_m y)\cdot e^{i\omega_{nm}t},$$

$$\phi_y(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}Y_{nm}\cdot\sin(\zeta_n x)\cdot\cos(\eta_m y)\cdot e^{i\omega_{nm}t}. \tag{47}$$

So that the boundary conditions in Eq. (46) are identically satisfied. In addition, the distributed transverse load is also expanded in double Fourier series as

$$q_z(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}Q_{nm}\cdot\sin(\zeta_n x)\cdot\sin(\eta_m y)\cdot e^{i\omega_{nm}t}, \tag{48}$$

where Q_{nm} has different values for different kind of loading [17]. For example, for uniform load with magnitude of q_0,

$$Q_{nm}=\frac{16q_0}{\pi^2 mn}.$$

By substituting Eqs. (47) and (48) into Eqs. (34) and (35), matrix form is as follow

$$\begin{bmatrix} K_{11}-m_{11}(\omega_{nm}G_{nm})^2 & K_{12}-m_{12}(\omega_{nm}G_{nm})^2 & K_{13}-m_{13}(\omega_{nm}G_{nm})^2 \\ K_{21}-m_{12}(\omega_{nm}G_{nm})^2 & K_{22}-m_{22}(\omega_{nm}G_{nm})^2 & K_{23} \\ K_{31}-m_{13}(\omega_{nm}G_{nm})^2 & K_{32} & K_{33}-m_{33}(\omega_{nm}G_{nm})^2 \end{bmatrix}\begin{Bmatrix} W_{nm} \\ X_{nm} \\ Y_{nm} \end{Bmatrix}=\begin{bmatrix} (G_{nm})^2 \\ 0 \\ 0 \end{bmatrix}Q_{nm}, \tag{49}$$

$$K_{11}=(A_{55}-2c_2C_{55}+c_2^2E_{55})\zeta_n^2+(A_{44}-2c_2C_{44}+c_2^2E_{55})\eta_m^2 \\ +c_1^2[F_{11}\zeta_n^4+2F_{12}\zeta_n^2\eta_m^2+4F_{66}\zeta_n^2\eta_m^2+F_{22}\eta_m^4],$$

$$K_{12}=(A_{55}-2c_2C_{55}+c_2^2E_{55})\zeta_n-c_1[E_{11}-c_1F_{11})\zeta_n^3+(E_{12}-c_1F_{12}+2E_{66}-2c_1F_{66})\zeta_n\eta_m^2],$$

$$K_{13}=(A_{44}-2c_2C_{44}+c_2^2E_{44})\eta_m-c_1[E_{22}-c_1F_{22})\zeta_n^3+(E_{12}-c_1F_{12}+2E_{66}-2c_1F_{66})\zeta_n\eta_m^2],$$

$$K_{22}=(C_{11}-2c_1E_{11}+c_1^2F_{11})\zeta_n^2+(C_{66}-2c_1E_{66}+c_1^2F_{66})\eta_m^2+(A_{55}-2c_2C_{55}+c_2^2E_{55}),$$

$$K_{23}=(C_{12}-2c_1E_{12}+c_1^2F_{12}+c_{66}-2c_1E_{66}+c_1^2F_{66})\zeta_n\eta_m,$$

$$K_{33}=(C_{22}-2c_1E_{22}+c_1^2F_{22})\eta_m^2+(C_{66}-2c_1E_{66}+c_1^2F_{66})\zeta_n^2+(A_{44}-2c_2C_{44}+c_2^2E_{44}), \tag{50a}$$

$$m_{11}=I_0-c_1^2I_6(\zeta_n^2+\eta_m^2),\quad m_{12}=(c_1I_4-c_1^2I_6)\zeta_n,\quad m_{13}=(c_1I_4-c_2^2I_6)\eta_m,$$

$$m_{22}=(I_2-2c_1I_4+c_1^2I_6),\quad m_{33}=(I_2-2c_1I_4+c_1^2I_6), \tag{50b}$$

$$G_{nm}=\sqrt{1+\mu(\zeta_n^2+\eta_m^2)}. \tag{50c}$$

Note that G_{nm} is greater than unity.

6.1. Bending solution

For static bending analysis, we consider W_{nm}, X_{nm}, Y_{nm} and Q_{nm} to be time-independent; consequently, the time derivative terms in Eq. (45) and therefore, terms containing ω_{mn} are omitted. Eq. (49) has been solved for each pair of integers (m,n) to determine the magnitude of W_{nm}, X_{nm}, Y_{nm} and the total solution is obtained from Eq. (47) (without the time terms) as

$$W_{nm}=(G_{nm})^2\cdot(W_{nm}^L)_{\max},$$

$$X_{nm}=(G_{nm})^2\cdot(X_{nm}^L)_{\max},$$

$$Y_{nm} = (G_{nm})^2 \cdot (Y^L_{nm})_{\max}, \tag{51}$$

where

$$(W^L_{nm})_{\max} = \frac{K_{22}K_{33}K_{23}^2}{\lambda} Q_{nm},$$

$$(X^L_{nm})_{\max} = \frac{K_{13}K_{23} - K_{12}K_{33}}{\lambda} Q_{nm},$$

$$(Y^L_{nm})_{\max} = \frac{K_{12}K_{23} - K_{22}K_{13}}{\lambda} Q_{nm},$$

$$\lambda = K_{11}K_{22}K_{33} - K_{23}^2K_{11} - K_{12}^2K_{33} - K_{13}^2K_{22} + 2K_{12}K_{13}K_{23}. \tag{52}$$

Since $G_{nm} > 1$, it can be seen that the displacement predicted by nonlocal theory is larger than local (or conventional) theory.

6.2. Free vibration solution

To obtain natural frequencies of a simply supported rectangular plate using the nonlocal third-order theory, we assume that $Q_{nm} = 0$. Following the same procedure as in the case of bending (i.e., expanding the displacement field in double Fourier series and substituting into the equations of motion)

We obtain the following eigenvalue problem for the determination of the natural frequencies:

$$\left| \begin{bmatrix} K_{11} - m_{11}(\omega_{nm}G_{nm})^2 & K_{12} - m_{12}(\omega_{nm}G_{nm})^2 & K_{13} - m_{13}(\omega_{nm}G_{nm})^2 \\ K_{21} - m_{12}(\omega_{nm}G_{nm})^2 & K_{22} - m_{22}(\omega_{nm}G_{nm})^2 & K_{23} \\ K_{31} - m_{11}(\omega_{nm}G_{nm})^2 & K_{32} & K_{33} - m_{33}(\omega_{nm}G_{nm})^2 \end{bmatrix} \right| = 0. \tag{53}$$

The natural frequencies of simply supported plate in nonlocal third-order plate theory are obtained as

$$(\omega_{nm})_1 = \frac{(\omega^L_{nm})_1}{G_{nm}}, \quad (\omega_{nm})_2 = \frac{(\omega^L_{nm})_2}{G_{nm}}, \quad (\omega_{nm})_3 = \frac{(\omega^L_{nm})_3}{G_{nm}}, \tag{54}$$

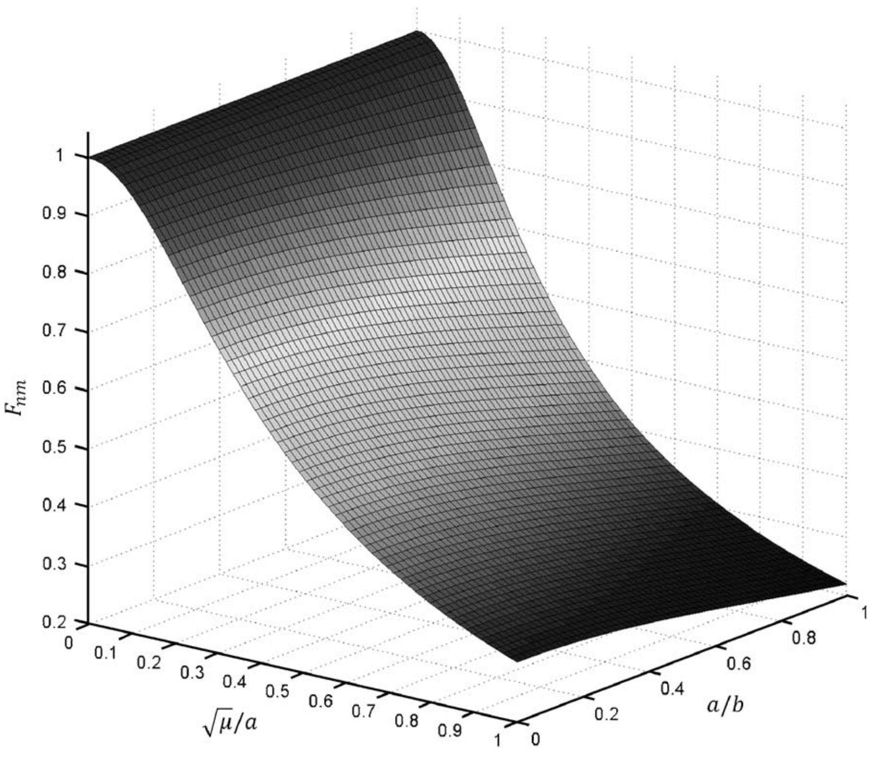

Fig. 1. Variation of nonlocal coefficient F_{nm} with respect to non-dimensional nonlocal parameter $\sqrt{\mu}/a$ and aspect ratio.

Table 1
Nonlocal coefficient F_{nm} in terms of nonlocal parameter $\sqrt{\mu}/a$ and aspect ratio a/b.

$\sqrt{\mu}/a$	0	0.2	0.4	0.6
(n,m)	$a/b=1$			
(1,1)	1.0000	0.7477	0.4904	0.3512
(1,2)	1.0000	0.5801	0.3353	0.2309
(1,3)	1.0000	0.4497	0.2440	0.1655
(2,1)	1.0000	0.5801	0.3353	0.2309
(2,2)	1.0000	0.4906	0.2708	0.1844
(2,3)	1.0000	0.4040	0.2155	0.1456
(3,1)	1.0000	0.4497	0.2440	0.1655
(3,2)	1.0000	0.4040	0.2155	0.1456
(3,3)	1.0000	0.3514	0.1844	0.1241
	$a/b=0.5$			
(1,1)	1.0000	0.8183	0.5799	0.4287
(1,2)	1.0000	0.7475	0.4904	0.3512
(1,3)	1.0000	0.6618	0.4038	0.2823
(2,1)	1.0000	0.6111	0.3601	0.2492
(2,2)	1.0000	0.5799	0.3353	0.2309
(2,3)	1.0000	0.5370	0.3033	0.2076
(3,1)	1.0000	0.4637	0.2531	0.1718
(3,2)	1.0000	0.4496	0.2440	0.1655
(3,3)	1.0000	0.4287	0.2309	0.1562

Table 2
Non-dimensional maximum center deflection ($\bar{w}=-w\times(Eh^2/q_0a^4)\times10^2$) in simply supported plate subjected to uniform load q_0 ($q_0=1$, $a=10$, $E=30\times10^6$, $v=0.3$, 100 term series).

a/b	a/h	μ	Third-order	First-order	Classical
1	10	0	4.1853	4.1853	4.0083
		0.5	4.5607	4.5608	4.3702
		1	4.9362	4.9363	4.7322
		1.5	5.3116	5.3118	5.0942
		2	5.6871	5.6873	5.4561
		2.5	6.0625	6.0628	5.8181
		3	6.4380	6.4383	6.1800
	50	0	4.0154	4.0154	4.0083
		0.5	4.3779	4.3779	4.3702
		1	4.7404	4.7404	4.7322
		1.5	5.1029	5.1029	5.0942
		2	5.4654	5.4654	5.4561
		2.5	5.8279	5.8279	5.8181
		3	6.1904	6.1904	6.1800
	100	0	4.0100	4.0100	4.0083
		0.5	4.3721	4.3721	4.3702
		1	4.7342	4.7342	4.7322
		1.5	5.0963	5.0963	5.0942
		2	5.4584	5.4584	5.4561
		2.5	5.8205	5.8205	5.8181
		3	6.1826	6.1826	6.1800
2	10	0	0.7169	0.7170	0.6483
		0.5	0.8767	0.8768	0.7946
		1	1.0364	1.0366	0.9408
		1.5	1.1961	1.1965	1.0870
		2	1.3558	1.3563	1.2332
		2.5	1.5155	1.5161	1.3794
		3	1.6752	1.6759	1.5256
	50	0	0.6511	0.6511	0.6483
		0.5	0.7978	0.7978	0.7946
		1	0.9446	0.9446	0.9408
		1.5	1.0914	1.0914	1.0870
		2	1.2381	1.2381	1.2332
		2.5	1.3849	1.3849	1.3794
		3	1.5316	1.5316	1.5256
	100	0	0.6490	0.6490	0.6483
		0.5	0.7954	0.7954	0.7946
		1	0.9417	0.9417	0.9408
		1.5	1.0881	1.0881	1.0870
		2	1.2344	1.2344	1.2332
		2.5	1.3808	1.3808	1.3794
		3	1.5271	1.5271	1.5256

where the superscript L denoted the quantities in local third-order shear deformation theory. Between these three frequencies for each combination of n and m, the lowest one is related to transverse deflection and two remained frequencies related to shear modes [19,20].

7. Numerical results and discussion

The analytical bending and free vibration solutions presented in Eqs. (51) and (54) are numerically evaluated here for an isotropic plate to discuss the effects of nonlocal parameter μ on the plate bending and vibration response. We consider the following parameter that defines the relation between nonlocal and local theory in both bending and free vibration cases

$$F_{nm} = \frac{1}{G_{nm}} = \frac{1}{\sqrt{1+\mu(\zeta_n^2+\eta_m^2)}} - \frac{1}{\sqrt{1+\pi^2(\sqrt{\mu}/a)^2(n^2+m^2(a/b)^2)}}, \tag{55}$$

where $\sqrt{\mu}/a$ is a non-dimensional nonlocal parameter and a/b is aspect ratio of the plate. Note that, because $G_{nm} \geq 1$, F_{nm} is less than or equal to unity. Therefore, it can be seen that natural frequencies predicted based on the nonlocal third-order plate theory are smaller than those based on local theory, while The maximum deflections predicted by nonlocal theory are larger than one predicted by local theory.

Table 3
Non-dimensional maximum center deflection ($\bar{w} = -w \times (Eh^2/Q_0a^4) \times 10^2$) of simply supported plate subjected to point load Q_0 at center ($Q_0 = 1$, $a = 10$, $E = 30 \times 10^6$, $\nu = 0.3$, 100 term series).

a/b	a/h	μ	Third-order	First-order	Classical
1	10	0	0.5137	0.5147	0.4609
		0.5	0.8072	0.821	0.5752
		1	1.1008	1.1274	0.6894
		1.5	1.3944	1.4337	0.8037
		2	1.688	1.7401	0.918
		2.5	1.9816	2.0465	1.0322
		3	2.2751	2.3528	1.1465
	50	0	0.463	0.463	0.4609
		0.5	0.585	0.585	0.5752
		1	0.7069	0.707	0.6894
		1.5	0.8288	0.8289	0.8037
		2	0.9508	0.9509	0.918
		2.5	1.0727	1.0728	1.0322
		3	1.1947	1.1948	1.1465
	100	0	0.4614	0.4614	0.4609
		0.5	0.5776	0.5776	0.5752
		1	0.6938	0.6938	0.6894
		1.5	0.81	0.81	0.8037
		2	0.9262	0.9262	0.918
		2.5	1.0424	1.0424	1.0322
		3	1.1586	1.1586	1.1465
2	10	0	0.2165	0.2183	0.1685
		0.5	0.6528	0.7092	0.2753
		1	1.089	1.2002	0.3821
		1.5	1.5253	1.6911	0.4889
		2	1.9616	2.182	0.5957
		2.5	2.3979	2.6729	0.7025
		3	2.8341	3.1638	0.8093
	50	0	0.1705	0.1705	0.1685
		0.5	0.2926	0.2927	0.2753
		1	0.4146	0.4148	0.3821
		1.5	0.5367	0.537	0.4889
		2	0.6587	0.6592	0.5957
		2.5	0.7808	0.7813	0.7025
		3	0.9029	0.9035	0.8093
	100	0	0.169	0.169	0.1685
		0.5	0.2796	0.2796	0.2753
		1	0.3903	0.3903	0.3821
		1.5	0.5009	0.5009	0.4889
		2	0.6115	0.6116	0.5957
		2.5	0.7222	0.7222	0.7025
		3	0.8328	0.8328	0.8093

In Fig. 1, the variation of F_{11} respect to $\sqrt{\mu}/a$ and a/b is presented. It can be seen that where $\sqrt{\mu}/a \ll 1$, the effect of nonlocal parameter F_{nm} can be neglected, while it has significant effect on the results when $\sqrt{\mu}/a \gg 1$. Also it has been shown that F_{nm} will be increases with increasing aspect ratio a/b.

In comparison with the first-order plate theory, difference between the maximum deflection in nonlocal first-order and nonlocal third-order theories is larger than one in the corresponding local theories whereas the difference between natural frequencies in nonlocal first-order and nonlocal third-order theories is smaller than one in the corresponding local theories. Furthermore nonlocal theory can predict the higher-order frequencies more accurately than local theory. As a result, the nonlocal theory softens the plate and makes it more flexible.

Table 1 shows that the coefficient F_{nm} decreases with decreasing nonlocal parameter $\sqrt{\mu}/a$ or increasing aspect ratio a/b. Therefore, the natural frequencies decrease when nonlocal parameter decreases. Table 2–4 contain the numerical results of

Table 4
Non-dimensional first mode frequency ($\overline{\omega_{11}} = \omega_{11} h\sqrt{\rho/G}$) of simply supported plate. ($a = 10$, $E = 30\times10^6$, $\nu = 0.3$).

a/b	a/h	μ	Third-order	First-order	Classical
1	10	0	0.0935	0.0930	0.0963
		1	0.0854	0.0850	0.0880
		2	0.0791	0.0788	0.0816
		3	0.0741	0.0737	0.0763
		4	0.0699	0.0696	0.0720
		5	0.0663	0.0660	0.0683
	20	0	0.0239	0.0239	0.0241
		1	0.0218	0.0218	0.0220
		2	0.0202	0.0202	0.0204
		3	0.0189	0.0189	0.0191
		4	0.0179	0.0178	0.0180
		5	0.017	0.0169	0.0171
2	10	0	0.0591	0.0589	0.0602
		1	0.0557	0.0556	0.0568
		2	0.0529	0.0527	0.0539
		3	0.0505	0.0503	0.0514
		4	0.0483	0.0482	0.0493
		5	0.0464	0.0463	0.0473
	20	0	0.0150	0.0150	0.0150
		1	0.0141	0.0141	0.0142
		2	0.0134	0.0134	0.0135
		3	0.0128	0.0128	0.0129
		4	0.0123	0.0123	0.0123
		5	0.0118	0.0118	0.0118

Table 5
Non-dimensional higher order frequencies ($\bar{\omega} = \omega h\sqrt{\rho/G}$) of simply supported plate. ($E = 30\times10^6$, $\nu = 0.3$, $a = 10$, $a/b = 1$, $a/h = 10$).

Frequencies	μ	Third-order	First-order	Classical
ω_{11}	0	0.0935	0.0930	0.0963
	1	0.0854	0.0850	0.0880
	2	0.0791	0.0788	0.0816
	3	0.0741	0.0737	0.0763
	4	0.0699	0.0696	0.0720
	5	0.0663	0.0660	0.0683
ω_{22}	0	0.3458	0.3414	0.3853
	1	0.2585	0.2552	0.288
	2	0.2153	0.2126	0.2399
	3	0.1884	0.186	0.2099
	4	0.1696	0.1674	0.1889
	5	0.1555	0.1535	0.1732
ω_{33}	0	0.702	0.6889	0.8669
	1	0.4213	0.4134	0.5202
	2	0.329	0.3228	0.4063
	3	0.279	0.2738	0.3446
	4	0.2466	0.242	0.3045
	5	0.2233	0.2191	0.2757

nonlocal classical, first-order and third-order theories. The following parameters are used to obtain the numerical values:

$$a = 10, \quad E = 30 \times 10^6, \quad \nu = 0.3, \quad \rho = 1, \tag{56}$$

where a, E, ν, ρ are plate length, Young modules, Poisson's ratio and plate density.

The non-dimensional quantities for maximum deflection and natural frequencies are:

$$\bar{w} = -w \times \left(\frac{Eh^3}{q_0 a^4}\right) \times 10^2, \quad \bar{w} = -w \times \left(\frac{Eh^3}{Q_0 a^4}\right) \times 10^2, \quad \bar{\omega} = \omega h \sqrt{\frac{\rho}{G}}, \tag{57}$$

where q_0 and Q_0 are the magnitude of the uniform and point load. (Reddy, [10])

Tables 2 and 3 compare the results of non-dimensional maximum deflection in three nonlocal plate theories for different values of nonlocal parameter and thickness for square and rectangular plates subjected to uniform and point loads. According to these results, it can be seen that the nonlocal theory predicts larger deflections. The results of first-order and third-order theories are almost the same for all values of μ.

Tables 4 and 5 contain natural frequencies of third-order nonlocal plate theories for different values of nonlocal parameter and aspect ratio. Again, it can be seen that nonlocal theories predict smaller values of natural frequencies than local theories especially for higher order frequencies (Table 5). Thus the local theories overestimate the frequencies. From the results presented in Tables 2–4, it follows that the difference between the results of the third-order theory and other theories increase with the thickness, nonlocal parameter, and mode number. Furthermore, it is observed that the difference between nonlocal third-order and first-order is slightly increased with increasing the nonlocal parameter and thickness of the plate but it is still negligible.

8. Conclusions

In this paper, equations of motion for nonlocal third-order shear deformation plate theory have been derived, and analytical solutions for bending and free vibration are also presented to bring out the effect of nonlocal parameters. Numerical results are presented for simply supported rectangular plates to illustrate the effects of nonlocal theories on plate response compared to the local theories. Nonlocal theory can be applied for the analysis of nano plates where the small size effects are significant. The effect of nonlocal constitutive relations is to increase the magnitude of deflections and decrease the amplitude of frequencies. In addition, the difference between nonlocal theories and local theories is significant for high value of the nonlocal parameter.

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