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Vibration of plates in contact with viscous fluid: Extension of Lamb's model

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ABSTRACT

The natural frequency of a plate decreases when in contact with a fluid. An analytical calculation using the Navier–Stokes equations for Newtonian fluids is performed to find the dependency of the frequency decrease on the fluid viscosity. The analysis follows the rationale of Lamb's model, and gives the viscosity contribution to the added virtual mass. The magnitude of the viscous effect is found to be inversely related to the thickness of the plate. Therefore, while the viscous effect is negligible for macroscopic plates, it is significant for thin plates incorporated in micro-sensors fabricated by silicon technology. The analytical results are shown to be well correlated with measured natural frequencies taken from published data of a micro-resonator in contact with viscous water–glycol mixtures.

1. Introduction

The natural frequency of a vibrating plate decreases when in contact with a fluid [1], a phenomenon which arises from the induced vibration of the fluid. While the vibration extends spatially throughout the fluid, it can be integrated and expressed as a layer of fluid adjacent to the plate, which vibrates with the plate. Intuitively, the plate vibrates as if its mass is increased by the mass of the virtual layer of vibrating fluid, and consequently its natural frequency decreases. This phenomenon is termed the added virtual mass effect.

The original classic analysis of the problem by Lamb [1] is elegant and simple. It proceeds along the following outline. The subject of analysis is the vibration of a thin circular elastic plate that is clamped to a rigid wall. Its natural mode in vacuum is well known from the elasticity theory [2]. When in contact with a fluid, it is approximated that the plate vibration mode remains unchanged. In order to find the fluid vibration, the fluid is assumed incompressible and inviscid, and therefore its velocity can be derived from a velocity potential. At the boundary with the plate, the fluid is required to match the normal velocity of the plate. Finally, the kinetic energies of the plate and of the fluid are determined, and the ratio of the latter to the former gives the added virtual mass. The classic Lamb's model has been extended and improved. For example, researchers have considered more accurate plate vibration modes [3], other geometries, such as annular [4] and rectangular [5] plates, and other boundary conditions for the plate [6]. Experiments have shown adequate agreement between theory and experiments [7]. However, the effect of viscosity has not been theoretically analyzed.

This work extends Lamb's model, and considers a Newtonian viscous fluid in contact with a vibrating plate in order to elucidate the effect of viscosity on the added virtual mass. It is natural to associate viscosity with dissipation, and indeed the energy dissipated by viscosity will be calculated, but its effect on the added virtual mass has another origin. Through

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the "no slip" boundary condition, viscosity couples the plate vibration to the tangential velocity of the fluid, thus increasing the fluid movement and its kinetic energy. Theoretical analysis of such motion might seems difficult at first, as it requires the nonlinear Navier–Stokes equations. However, it will be shown that a linear form of the equations, without the nonlinear convective terms, is adequate for this analysis.

We first repeat the analysis by Lamb [1] of a vibrating plate in contact with an inviscid fluid, but along a different mathematical technique in the manner of [3], then we analyze the same plate in contact with a viscous Newtonian fluid to find the fluid velocity, its kinetic energy, the added virtual mass and the energy dissipated by viscosity. Finally, the results are compared with the natural frequencies measured in an experiment of a micro-resonator in contact with a very viscous fluid [8].

2. Plate vibration

The analysis starts with a plate vibrating in vacuum. The plate is circular, thin, and elastic and is clamped to a rigid wall. The plate has radius a, density ρ_1, and thickness h. The normal velocity profile [1] of the first mode of vibration of a clamped elastic thin plate is approximately given by an assumed mode shape

$$u(r) = u_0 \Theta(a-r)(1-(r/a)^2)^2 \mathrm{e}^{\mathrm{i}\omega t}, \tag{1}$$

where $\Theta(a-r)$ is the unit step function, and only the real part of the expression has physical significance. It will be assumed that the plate will retain this mode shape even when in contact with a fluid. It is noted that the exact mode shape in vacuum is a combination of Bessel functions [3], and moreover, the mode shape is influenced by the fluid in contact with the vibrating plate [3]. However, as Section 4.6 argues, the assumed mode shape given here is an adequate approximation.

In this work distances will be nondimensional in units of a while velocities will be nondimensional in units of u_0, the amplitude of the velocity profile. The nondimensional velocity profile is

$$f(r) = \Theta(1-r)(1-r^2)^2. \tag{2}$$

The maximal kinetic energy of the plate is

$$T_1^d = \frac{1}{2}\rho_1 h \int |u(r)|^2 \,\mathrm{d}S = \pi\rho_1 h u_0^2 a^2 \int_0^1 f(r)^2 r\,\mathrm{d}r = \pi\rho_1 h a^2 u_0^2/10. \tag{3}$$

The superscript d in T_1^d denotes that this energy quantity is dimensional, in contrast to the nondimensional quantities defined below. The Hankel transform of the velocity profile is introduced and will be used later

$$F(k) = \int_0^\infty f(r) J_0(kr) r\,\mathrm{d}r = 8\frac{J_3(k)}{k^3}, \tag{4}$$

where $J_n(x)$ is the Bessel function of order n, and the integral was evaluated by repeatedly integrating by parts. The inverse Hankel transform is

$$f(r) = \int_0^\infty F(k) J_0(kr) k\,\mathrm{d}k. \tag{5}$$

3. Plate in contact with an inviscid fluid

The analysis proceeds to consider a plate clamped to an infinite rigid wall in contact with a fluid on one side, with the fluid taken to be inviscid and incompressible. Cylindrical coordinates (r,θ,z) are used with the origin set at the centre of the plate, and the wall and the plate are in the $z=0$ plane. As stated before, the plate radius a sets the length scale of the system, in particular that of the cylindrical coordinates r and z. An incompressible, inviscid fluid is irrotational, and the velocity of the fluid can be given in terms of a velocity potential, $\mathbf{v} = \nabla\phi$. The potential satisfies the Laplace equation $\nabla^2\phi = 0$, with the boundary conditions $\mathbf{v}(r\to\infty) = \mathbf{v}(z\to\infty) = 0$ and

$$v_z|_{z=0} = \left.\frac{\partial\phi}{\partial z}\right|_{z=0} = f(r)\mathrm{e}^{\mathrm{i}\omega t}. \tag{6}$$

Note that the treatment of the fluid as incompressible amounts to the condition that the wavelength of the acoustic waves in the fluid is much larger than the plate's dimension, $c/\omega \gg a$, where c is the fluid sound velocity [1,9]. The velocity potential will be found in a manner similar to [3]. The general solution for the Laplace equation having azimuthal symmetry for which the velocity vanishes far from the origin is

$$\phi = \mathrm{e}^{\mathrm{i}\omega t}\int_0^\infty A(k)\mathrm{e}^{-kz} J_0(kr)\,\mathrm{d}k. \tag{7}$$

Having $A(k) = -F(k)$ ensures that the boundary condition Eq. (6) is satisfied. The nondimensional kinetic energy of the fluid, in units of $\rho a^3 u_0^2$, where ρ is the fluid density, is

$$T_2 = \frac{1}{2}\int |\mathbf{v}|^2\,\mathrm{d}V = \frac{1}{2}\int |\nabla\phi|^2\,\mathrm{d}V = -\frac{1}{2}2\pi\int_0^\infty \left(\phi^* \frac{\partial\phi}{\partial z}\right)\Big|_{z=0} r\,\mathrm{d}r. \tag{8}$$

The above transformation of the volume integral into an integral over the surface of the plate is standard. Inserting ϕ, the integral is evaluated to be

$$T_2 = \pi\int_0^\infty F(k)^2\,\mathrm{d}k = \pi 0.06689. \tag{9}$$

The numeric result was obtained by the formula below taken from Section 13.41 of [13]

$$\int_0^\infty \frac{J_m(ar)J_n(ar)}{r^p}\,\mathrm{d}r = \frac{\left(\frac{a}{2}\right)^{p-1}\Gamma(p)\Gamma\left(\frac{m}{2}+\frac{n}{2}-\frac{p}{2}+\frac{1}{2}\right)}{2\Gamma\left(\frac{p}{2}+\frac{m}{2}-\frac{n}{2}+\frac{1}{2}\right)\Gamma\left(\frac{p}{2}+\frac{n}{2}-\frac{m}{2}+\frac{1}{2}\right)\Gamma\left(\frac{p}{2}+\frac{m}{2}+\frac{n}{2}+\frac{1}{2}\right)}, \tag{10}$$

where $\Gamma(x)$ is the gamma function. The added virtual mass factor β gives the ratio between the total kinetic energy to the kinetic energy of the plate [1]

$$1+\beta = \frac{\text{Total kinetic energy}}{\text{Plate kinetic energy}} = 1 + \frac{T_2\rho a^3 u_0^2}{T_1^d}. \tag{11}$$

Its value is

$$\beta = 0.6689\frac{\rho a}{\rho_1 h}. \tag{12}$$

This is the same result as obtained by Lamb [1]. The increase in the kinetic energy of the plate and fluid system is equivalent to a virtual mass increase of the plate. Analogous to a harmonic oscillator, as the mass increases, the natural frequency decreases, and the relation between the natural angular frequency of the plate when in contact with a fluid ω, and when in vacuum ω_{vacuum}, is

$$\omega = \frac{\omega_{\mathrm{vacuum}}}{\sqrt{1+\beta}}. \tag{13}$$

The analysis presented thus far is well known. The following sections present our original analysis.

4. Plate in contact with a Newtonian fluid

4.1. Stream function method

Finally, the analysis considers an incompressible Newtonian fluid characterised by a kinematical viscosity ν. Because the problem has azimuthal symmetry, the stream function method [10] can be used. The method defines a scalar stream function $\psi(r,z)$, from which the velocity is derived:

$$v_r = -\frac{1}{r}\frac{\partial\psi}{\partial z}, \quad v_z = \frac{1}{r}\frac{\partial\psi}{\partial r}. \tag{14}$$

The advantage of this approach is that the continuity equation of an incompressible fluid is automatically satisfied:

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{\partial}{\partial z}v_z = 0. \tag{15}$$

The vorticity is $\boldsymbol{\omega} = \nabla\times\mathbf{v}$. The azimuthal symmetry means that the vorticity has only one component, which is denoted here as Ω.

$$\boldsymbol{\omega} = \omega_\theta\hat{\boldsymbol{\theta}} \equiv \Omega\hat{\boldsymbol{\theta}}. \tag{16}$$

The relation between the vorticity and the stream function is

$$-\Omega = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) + \frac{1}{r}\frac{\partial^2\psi}{\partial z^2}. \tag{17}$$

The vorticity satisfies a nonlinear equation derived from the Navier–Stokes equations

$$\frac{\partial}{\partial t}\boldsymbol{\omega} = -(\mathbf{v}\cdot\nabla)\boldsymbol{\omega} + (\boldsymbol{\omega}\cdot\nabla)\mathbf{v} + \nu\nabla^2\boldsymbol{\omega}. \tag{18}$$

Showing the dimensions explicitly and rearranging so that the last term, the diffusion term, will be of unit order, gives

$$\mathrm{i}\frac{\omega a^2}{\nu}\boldsymbol{\omega}=\frac{au_0}{\nu}[-(\mathbf{v}\cdot\nabla)\boldsymbol{\omega}+(\boldsymbol{\omega}\cdot\nabla)\mathbf{v}]+\nabla^2\boldsymbol{\omega}. \tag{19}$$

All terms contain the vorticity, whose dimension is therefore excluded. In the limit of small plate velocity,

$$u_0 \ll \nu/a, \tag{20}$$

the nonlinear terms in square brackets (Eq. (19)) are negligible compared to the diffusion term. They are also negligible compared to the time-derivative term, if $u_0 \ll \omega a$. Denoting the plate velocity amplitude as $u_0=\omega A$, where A is the amplitude of the vertical displacement of the plate, and recalling that plate elastic theory [2] requires that $A \ll a$, the condition is verified. The linearised vorticity equation in terms of the component Ω is

$$\mathrm{i}\frac{1}{\xi^2}\Omega=\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}r\Omega\right)+\frac{\partial^2}{\partial z^2}\Omega, \tag{21}$$

where ξ is a nondimensional parameter defined as

$$\xi=\sqrt{\frac{\nu}{\omega a^2}}. \tag{22}$$

Note that ω is the actual angular frequency when the plate is in contact with a fluid, and not the angular frequency in vacuum. The parameter ξ represents a nondimensional length, which alters the dimensional analysis that demonstrated that the nonlinear terms are negligible. The magnitude of the nonlinear terms is rescaled to $u_0 a/\nu\xi$, while both linear terms have magnitudes of $1/\xi^2$. However, the velocity requirement Eq. (20) ensures that the nonlinear term is still negligible, as long as ξ is not large. In fact, the derivations presented in the following sections require that ξ is small. Therefore, it will be assumed that

$$\xi \ll 1. \tag{23}$$

4.2. General solution of the vorticity and stream function

The equation for Ω is linear and can be solved by separation of variables. Substitution will prove that the solution is

$$\Omega=\mathrm{e}^{\mathrm{i}\omega t}J_1(kr)\mathrm{e}^{-\kappa z},\quad \kappa^2=\mathrm{i}\xi^{-2}+k^2. \tag{24}$$

Another possible independent solution contains the Bessel function of the second kind, which diverges at the origin, and therefore is not admissible. In order to determine whether κ and k are real or complex, the asymptotic expansion of a Bessel function with a complex argument $z=x+\mathrm{i}y$ is shown

$$J_n(z)\underset{|z|\to\infty}{\longrightarrow}\sqrt{\frac{2}{\pi z}}\cos\left(x+\mathrm{i}y-\frac{n\pi}{2}-\frac{\pi}{4}\right)=\sqrt{\frac{2}{\pi z}}\left(\cos\left(x-\frac{n\pi}{2}-\frac{\pi}{4}\right)\cosh y-\mathrm{i}\sin\left(x-\frac{n\pi}{2}-\frac{\pi}{4}\right)\sinh y\right). \tag{25}$$

The expansion diverges for large y unless the argument is real, that is $y=0$. Therefore, k must be real and κ must be complex. The stream function is

$$\psi=A\,\mathrm{e}^{\mathrm{i}\omega t}rJ_1(kr)\mathrm{e}^{-\kappa z}. \tag{26}$$

When substituted in Eq. (17), it gives

$$(\kappa^2-k^2)A\,\mathrm{e}^{\mathrm{i}\omega t}J_1(kr)\mathrm{e}^{-\kappa z}=-\Omega=-\mathrm{e}^{\mathrm{i}\omega t}J_1(kr)\mathrm{e}^{-\kappa z}. \tag{27}$$

This equation is inhomogeneous. The homogeneous solution is obtain by having $\kappa=k$, while the inhomogeneous solution is obtained by having $\kappa^2=\mathrm{i}\xi^{-2}+k^2$ and $A=-(\kappa^2-k^2)^{-1}=\mathrm{i}\xi^2$. The general solution is

$$\psi=\mathrm{e}^{\mathrm{i}\omega t}\int_0^\infty[A(k)\mathrm{e}^{-\kappa z}+B(k)\mathrm{e}^{-kz}]rJ_1(kr)\,\mathrm{d}k, \tag{28}$$

where κ, the complex wavenumber of the first term, is given by

$$\kappa^2=\mathrm{i}\xi^{-2}+k^2. \tag{29}$$

The corresponding vorticity component is

$$\Omega=\mathrm{e}^{\mathrm{i}\omega t}\frac{-\mathrm{i}}{\xi^2}\int_0^\infty A(k)J_1(kr)\mathrm{e}^{-\kappa z}\,\mathrm{d}k. \tag{30}$$

By Eqs. (14) and (28), the radial velocity is

$$v_r = \mathrm{e}^{\mathrm{i}\omega t}\int_0^\infty [\kappa A(k)\mathrm{e}^{-\kappa z} + kB(k)\mathrm{e}^{-kz}]J_1(kr)\,\mathrm{d}k, \tag{31}$$

and the z component of the velocity is

$$v_z = \mathrm{e}^{\mathrm{i}\omega t}\int_0^\infty [kA(k)\mathrm{e}^{-\kappa z} + kB(k)\mathrm{e}^{-kz}]J_0(kr)\,\mathrm{d}k. \tag{32}$$

4.3. Boundary conditions

There are two conditions at the boundary $z = 0$. The first condition is that the normal velocity equals the velocity of the plate and wall, as with the inviscid problem. The second is that the tangential velocity vanishes: the no-slip condition.

$$\begin{aligned} v_z|_{z=0} &= \frac{1}{r}\frac{\partial\psi}{\partial r}\bigg|_{z=0} = f(r)\mathrm{e}^{\mathrm{i}\omega t} \\ v_r|_{z=0} &= -\frac{1}{r}\frac{\partial\psi}{\partial z}\bigg|_{z=0} = 0. \end{aligned} \tag{33}$$

Applying the boundary conditions with Eqs. (5), (31) and (32) gives

$$\int_0^\infty [A(k)\mathrm{e}^{-\kappa z} + B(k)\mathrm{e}^{-kz}]kJ_0(kr)\,\mathrm{d}k = \int_0^\infty F(k)kJ_0(kr)\,\mathrm{d}k$$

$$\int_0^\infty [\kappa A(k)\mathrm{e}^{-\kappa z} + kB(k)\mathrm{e}^{-kz}]J_1(kr)\,\mathrm{d}k = 0, \tag{34}$$

whose solution is $A(k) + B(k) = F(k)$ and $\kappa A(k) + kB(k) = 0$. Rearranging gives

$$B = \frac{F}{1 - k/\kappa}, \quad A = \frac{F}{1 - \kappa/k}. \tag{35}$$

In order to check these results, the coefficients are expanded to leading order in small ξ

$$B \approx F(k)\left(1 + k\xi\frac{1-\mathrm{i}}{\sqrt{2}}\right), \quad A \approx -F(k)k\xi\frac{1-\mathrm{i}}{\sqrt{2}}. \tag{36}$$

As ξ tends to zero, the inviscid solution is obtained.

4.4. Fluid kinetic energy and added virtual mass

The calculations of the kinetic energy of the fluid are somewhat involved. For brevity, only a few intermediary results are shown. The nondimensional kinetic energy of the fluid, in units of $\rho a^3 u_0^2$, is

$$T_2 = \frac{1}{2}\int |\mathbf{v}|^2\,\mathrm{d}V = \frac{1}{2}\int \frac{1}{r^2}\psi\frac{\partial\psi^*}{\partial n}\,\mathrm{d}S + \frac{1}{2}\int\frac{1}{r}\psi\Omega^*\,\mathrm{d}V. \tag{37}$$

The surface integral vanishes, while the volume integral equals

$$T_2 = \pi\int_0^\infty F^2(k)\frac{|\kappa|^2 - k(\kappa+\kappa^*)/2}{|\kappa|^2 - k(\kappa+\kappa^*) + k^2}\,\mathrm{d}k. \tag{38}$$

The fraction in the integrand can be expanded as a Taylor series in the variable $k\xi$.

$$T_2 = \pi\int_0^\infty F^2(k)\left(1 + \frac{k\xi}{\sqrt{2}} - \frac{k^3\xi^3}{2\sqrt{2}} + \frac{k^5\xi^5}{8\sqrt{2}} + \dots\right)\mathrm{d}k. \tag{39}$$

The expansion is legitimate for a small $k\xi$, while for large k the factor multiplying the series $F^2(k)$ decays as k^{-7}, as Eq. (4) shows. Note, this expansion necessitates that ξ is small. Substituting $F(k)$ and using Eq. (10) gives

$$T_2 = \pi 0.06689 + \frac{\pi}{10}\frac{1}{\sqrt{2}}\xi - \frac{\pi}{3\sqrt{2}}\xi^3 + O(\xi^5). \tag{40}$$

The added virtual mass, as defined by Eq. (11), is viscosity dependent

$$\beta = 0.6689\frac{\rho a}{\rho_1 h}(1 + 1.057\xi + O(\xi^3)) \tag{41}$$

This is the main result of this work. It shows that viscosity increases the added virtual mass and decreases the natural frequency.

4.5. Energy dissipated by viscosity

Viscosity dissipates mechanical energy into heat. The power dissipated is $\int \tau_{ij}\partial_i v_j \, \mathrm{d}V$, where τ_{ij} is the viscous stress tensor [10]. The nondimensional energy dissipated per cycle in units of $\rho a^3 u_0^2$ is

$$U_v = 2\pi\xi^2 \int \mathrm{d}V \left[\left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{v_r}{r}\right)^2 + \left(\frac{\partial v_z}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right)^2 \right], \tag{42}$$

which can be rearranged to give

$$U_v = \pi\xi^2 \int |\Omega|^2 \, \mathrm{d}V = 2\pi^2 \int_0^\infty F^2(k) \frac{1}{\xi^2} \frac{k^2}{|\kappa|^2 - k(\kappa + \kappa^*) + k^2} \frac{1}{k(\kappa + \kappa^*)} \, \mathrm{d}k. \tag{43}$$

Expanding the integrand as a Taylor series gives

$$U_v = 2\pi^2 \int_0^\infty F^2(k) \left(\frac{k\xi}{\sqrt{2}} + k^2\xi^2 + \frac{k^3\xi^3}{2\sqrt{2}} + \dots \right) \mathrm{d}k. \tag{44}$$

The leading order term is

$$U_v = 2\pi^2 \frac{\xi}{10\sqrt{2}} = 0.44\pi\xi. \tag{45}$$

The influence of dissipation on the vibrations is characterised by the Q factor, which is defined as [9]

$$Q = 2\pi \frac{\text{energy stored}}{\text{energy dissipated per cycle}}. \tag{46}$$

For systems with large β, most of the energy is stored in the fluid, so the system energy equals the maximal kinetic energy of the fluid, T_2. The Q factor becomes

$$Q \approx 2\pi \frac{T_2}{U_v} = \frac{0.95}{\xi}. \tag{47}$$

The vibrations necessitate that the Q factor is larger than 1, which is satisfied by the assumption that ξ is small. Dissipation lowers the natural frequency, but for a large Q factor this effect is proportional to $1/Q^2 \approx \xi^2$, and does not modify the leading order viscosity contribution to the added virtual mass (Eq. (41)).

A plate vibrating in contact with a fluid dissipates energy also through acoustic radiation. Lamb [1] analyzed the vibrating plate as a simple acoustic source, and derived the power radiated. The nondimensional energy dissipated per cycle by acoustic radiation in units of $\rho a^3 u_0^2$ is

$$U_r = \frac{\omega a}{c} \frac{\pi^2}{18}. \tag{48}$$

The ratio between the energy dissipated by acoustic radiation to the energy dissipated by viscosity is

$$\frac{U_r}{U_v} = 0.4 \frac{\omega a}{c\xi} = 0.4 \frac{\omega a}{c} \sqrt{\frac{\omega a^2}{v}}. \tag{49}$$

The treatment of the fluid as incompressible requires the condition that the acoustic wavelength in the fluid is larger than the plate dimension, which can be written as $\omega a \ll c$, while the analysis here assumed that ξ is small, that is $\omega a \gg v/a$. Therefore, this energy ratio can be either small or large.

4.6. Analysis of approximations

The assumed mode shape of the velocity profile, given by Eq. (2), involves two approximations. First, the polynomial mode shape is an approximation to the exact mode shape in vacuum, and, second, the mode shape is modified when the plate is in contact with a fluid. Both approximations were corrected by Amabili and Kwak [3] for a circular plate vibrating in contact with an inviscid fluid. By adopting their results, the first approximation can be corrected quite easily for the viscous fluids considered here. The exact mode shape of a plate vibrating in vacuum is [3]

$$f(r) = \Theta(1 - r)(AJ_0(\lambda r) + CI_0(\lambda r)), \tag{50}$$

where I_0 is the modified Bessel function of the first kind. The parameters λ, A, and C are set by the clamped boundary conditions at $\lambda = 3.196$ and $C = 0.05571A$. Noting that the equivalent of $F(k)$ in [3] is $H_{00}(\eta)$, Eq. (39) can be re-calculated to give the added virtual mass factor

$$\beta = 0.6538\frac{\rho a}{\rho_1 h}(1 + 1.082\xi). \tag{51}$$

Re-calculating Eq. (43) gives the energy dissipated per cycle

$$U_\nu = 0.45\pi\xi. \tag{52}$$

Comparison with Eqs. (41) and (45) shows that the simple mode shape assumed by Lamb gives excellent results.

Correcting the second approximation, which consists of calculating the influence of the fluid on the mode shape, is much more difficult, and is not performed here. However, the results for the inviscid case [3] show that for the first natural mode, which is the only mode considered in this work, the mode shape is almost unaffected by the fluid, and the added virtual mass factor is modified by less than 1 percent. A simple analysis suggests that the viscous forces acting on the plate do not alter these results. Viscosity produces reactive and dissipative forces. The reactive forces were the main focus of this work as they are responsible for the added virtual mass. Eq. (41) shows that viscosity reactive forces are smaller than the reactive forces of an inviscid fluid if ξ is small. The dissipative forces are proportional to the viscous stresses, and therefore their magnitude is proportional to the amplitude of the plate velocity u_0. The dissipative forces are tangential to the plate surface, and only their component that is normal to the flat plate at the equilibrium state affects the mode shape. The normal component is proportional to the plate slope, which is proportional to u_0. Therefore, the influence of the dissipative forces on the mode shape is second order in u_0 and can be ignored.

Another approximation employed in this work is the assumption that the fluid is incompressible. Compressibility is excepted to decrease the natural frequency by a term proportional to $(\omega a/c)^2$, which is assumed to be small [1,9].

The errors introduced by the approximations discussed above are probably less important than the errors due to discrepancies between actual plates and the idealised theoretical model. Actual plates are only approximately subject to the ideal boundary conditions of a plate clamped to a rigid infinite plane in contact with a semi-infinite volume of fluid, and actual plates are only approximately cylinders that have an exact radius a and thickness h.

5. Discussion and conclusions

This work investigated the small amplitude vibrations of an elastic plate in contact with a viscous fluid. The key results obtained are the determinations of the viscosity contribution to the added virtual mass, given by Eq. (41), and of the energy dissipation by viscosity, given by Eq. (45). Both results are proportional to the nondimensional parameter $\xi = \sqrt{\nu/(\omega a^2)}$. The question of which circumstances these results are significant is an important one. More specifically we want to address how the viscosity effects depend on the dimensions of the plate. Consider different plates with varying thickness h (each plate has a constant thickness), but all have the same radius to thickness ratio a/h, are all composed of the same material, and are all in contact with the same fluid. The natural frequency of a clamped plate in a vacuum is [2]

$$f_{\text{vacuum}} = A\frac{h}{a^2}. \tag{53}$$

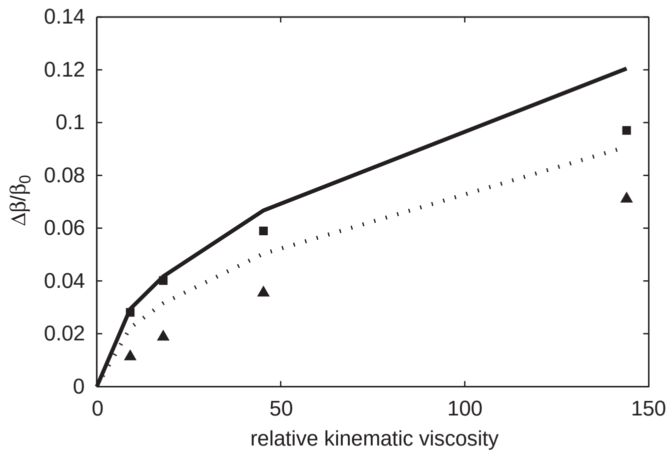

Fig. 1. Relative contribution of viscosity to added virtual mass, $(\beta - \beta_0)/\beta_0$, as a function of kinematic viscosity in the experiment of [8]. β_0 is the added virtual mass factor of an inviscid fluid $\nu = 0$. The theoretical relative contribution is found from Eq. (41) to be $1.057\sqrt{\nu/(\omega a^2)}$. The kinematic viscosity is given relative to water, $\nu = 1.14 \times 10^{-6}\,\text{m}^2\,\text{s}^{-1}$. The graphs are for two plate radii: $a = 100\,\mu\text{m}$: ▲—experiment, dotted line—theory; $a = 150\,\mu\text{m}$: ■—experiment, solid line—theory.

where A is a constant that depends on the material elastic properties. As an example, $A = 4000\,\mathrm{Hz\,m}$ for silicon plates. The natural frequency of the plate in contact with a fluid at one side is given by Eq. (13), which depends on the added virtual mass factor β. Because β depends only on the ratio a/h, it will be the same for all plates. Therefore, the dependence of both the viscosity contribution to the added virtual mass and the energy dissipation on the plate dimensions is proportional to

$$\xi = \sqrt{\nu/(\omega a^2)} \propto (fa^2)^{-1/2} \propto h^{-1/2}. \tag{54}$$

The viscosity effects are proportional to $h^{-1/2}$: for thinner plates, the viscosity effects are larger. A silicon plate with a dimensions ratio $a/h = 100$ in contact with water, whose plate thickness is $h = 1\,\mu\mathrm{m}$, has $\xi = 0.025$, which is probably on the verge of being significant. State of the art micro-electro-mechanical systems (MEMS) sensors reach this thickness [11,12], and future sensors will probably be even thinner.

Increasing viscosity will also enhance the effect, as Ayela and Nicu [8] found experimentally. They investigated the natural frequency of a plate incorporated in a micro-machined piezoelectric sensor, which was in contact with viscous water–glycerol mixtures. Their goal was to validate Lamb's model. While they obtained excellent agreement between experiments and Lamb's model for small viscosity, they found a discernable influence on the natural frequency at large viscosities. That influence was observed for water–glycerol mixtures at viscosities 10–200 times the that of water. Their results were used to find the deviation of the measured natural frequencies from the values predicted by the viscosity-free Lamb's model, and were compared with the deviation predicted by the analysis presented here, Eq. (41). This comparison is shown in Fig. 1. The analytical results predict slightly larger deviations, but are still close enough to the experimental results. Further investigation is needed in order to analyze this discrepancy.

In conclusion, the viscosity contribution to the added virtual mass, while insignificant for macroscopic plates, will have increasing importance for thin plates incorporated in micro-sensors; this will be the trend as technological progress continues to miniaturise them.

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