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Free vibrations of elastically connected stretched beams

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A R T I C L E I N F O

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A B S T R A C T

A general theory for the determination of natural frequencies and mode shapes for a set of elastically connected axially loaded Euler–Bernoulli beams is developed. A normal-mode solution is applied to a set of non-dimensional coupled partial differential equations. The natural frequencies are the eigenvalues of a matrix of differential operators. The matrix operator is shown to be self-adjoint leading to an orthogonality condition for the mode shapes.

In the special case of identical beams, it is shown that the natural frequencies are organized into sets of intramodal frequencies in which each mode shape is a product of a spatial mode and a discrete mode. An exact solution is available for the general case. However the natural frequencies and mode shapes are then determined using a complicated numerical method. A Rayleigh–Ritz method using mode shapes of the corresponding unstretched beams is developed as an alternative.

1. Introduction

The problem considered is the free vibrations of a set of n axially loaded Euler–Bernoulli beams connected by elastic layers and connected to a Winkler type foundation as illustrated in Fig. 1.The linear free vibrations of a single axially loaded Euler–Bernoulli beam is well documented [1,2].

Vibrations of composite material have been modeled by elastically connected beams [3]. Researchers have studied the free and forced vibrations of elastically connected double Euler–Bernoulli beams [4–8], and double and triple Timoshenko beams [9,10]. Chen and Sheu [3] studied an axially loaded double Timoshenko beam. Most researchers used a normal-mode assumption leading to a set of ordinary differential equations from which the natural frequencies and mode shapes are attained. Some, [3,10], employed a dynamic stiffness matrix approach.

Researchers have used elastically connected concentric beams as continuous system models for carbon nanotubes [11–13]. The elastic layers provide a linear model for inter-atomic Van der Waals forces. The tensile strength and elastic modulus of multiwall carbon nanotubes have been shown to increase when the tubes are stretched [14,15]. Manufacturing of such tubes often occurs in a polymer gel. A model of elastically connected stretched beams may be used as a first attempt at continuous system modeling of vibrations of stretched nanotubes. An elastic layer surrounding the tube is used to model the polymer gel.

This paper presents a general theory for the free response of elastically connected axially loaded beams. A set of coupled partial differential equations governing the free response of a set of n elastically connected axially loaded beams is presented and non-dimensionalized. For generality, the beams may also be attached to a Winkler foundation. The general mathematical theory for a normal-mode solution is developed in Section 2 and discussed in Section 3. The resulting

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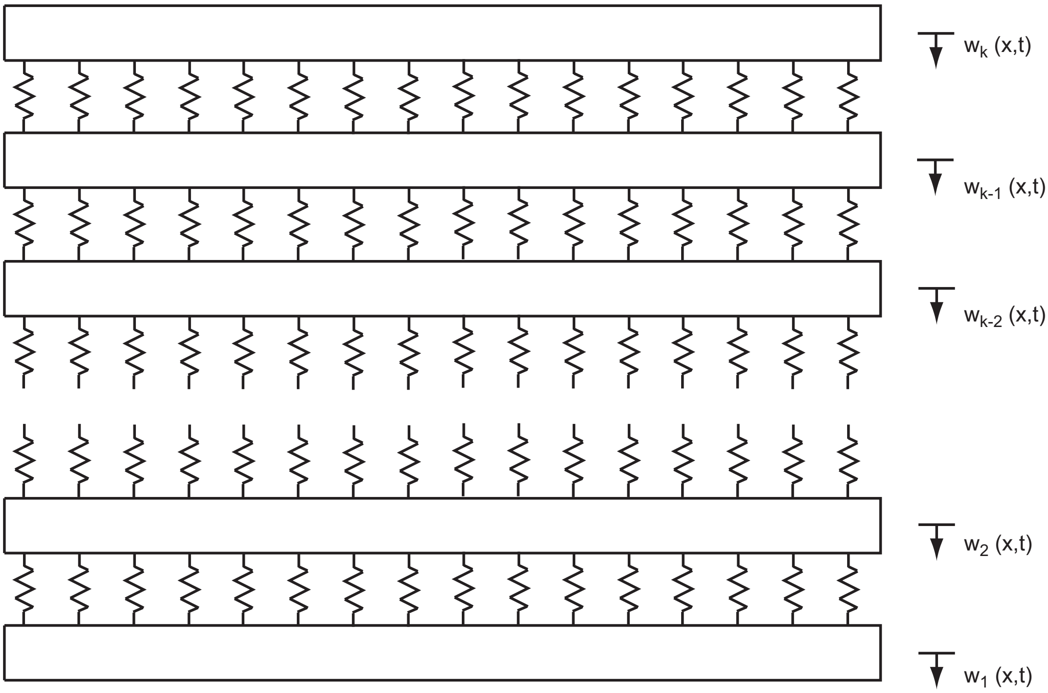

Fig. 1. A set of n elastically connected free–free beams.

ordinary differential equations are summarized in a matrix form leading to an eigenvalue–eigenvector problem to determine the natural frequencies and mode shapes. The general theory of the eigenvalue problems is discussed and an exact solution obtained. However, as noted by Williams [16], when studying the equation for a single axially loaded beam, numerical difficulties arise in the determination of natural frequencies due to the presence of exponentially large terms. The special case of a set of identical beams is considered in Section 4, while a Rayleigh–Ritz method which may be used for a general set of beams is developed in Section 5.

2. General theory

The problem considered is that of a set of n elastically connected axially loaded Euler–Bernoulli beams. The ith beam is made of a material of elastic modulus E_i and mass density ρ_i and has a cross-section with a uniform cross-section of area A_i and moment of inertia I_i. Each beam is subject to the same tensile axial load P. The ith beam and the i plus first beam are connected by a continuous linear elastic layer of the Winkler type of stiffness per length k_i. The first and nth beams are connected to Winkler foundations of stiffness per length of k_0 and k_n, respectively. Each beam is of length L and all beams have identical supports. The transverse displacement of the ith beam is $w_i(x,t)$.

Application of extended Hamilton's principle to a differential element of each beam leads to the following set of coupled differential equations:

$$E_1 I_1 \frac{\partial^4 w_1}{\partial x^4} - P\frac{\partial^2 w_1}{\partial x^2} + k_0 w_1 + k_1(w_1 - w_2) + \rho_1 A_1 \frac{\partial^2 w_1}{\partial t^2} = 0, \tag{1a}$$

$$E_i I_i \frac{\partial^4 w_i}{\partial x^4} - P\frac{\partial^2 w_i}{\partial x^2} + k_{i-1}(w_i - w_{i-1}) + k_i(w_i - w_{i+1}) + \rho_i A_i \frac{\partial^2 w_i}{\partial t^2} = 0 \tag{1b}$$

for $i = 2, 3, \ldots, n-1$, and

$$E_n I_n \frac{\partial^4 w_n}{\partial x^4} - P\frac{\partial^2 w_n}{\partial x^2} + k_n w_n + k_{n-1}(w_n - w_{n-1}) + \rho_n A_n \frac{\partial^2 w_n}{\partial t^2} = 0. \tag{1c}$$

Non-dimensional variables are defined according to

$$x^* = \frac{x}{L}, \tag{2a}$$

$$t^* = t\sqrt{\frac{E_1 I_1}{\rho_1 A_1 L^4}}, \tag{2b}$$

and

$$w_i^* = \frac{w_i}{L}. \tag{2c}$$

Introduction of Eq. (2) in Eq. (1) leads to

$$\mu_1 \frac{\partial^4 w_1}{\partial x^4} - \varepsilon \frac{\partial^2 w_1}{\partial x^2} + \lambda_0 w_1 + \lambda_1 (w_1 - w_2) + \beta_1 \frac{\partial^2 w_1}{\partial t^2} = 0, \tag{3a}$$

$$\mu_j \frac{\partial^4 w_j}{\partial x^4} - \varepsilon \frac{\partial^2 w_j}{\partial x^2} + \lambda_{j-1}(w_j - w_{j-1}) + \lambda_j (w_j - w_{j+1}) + \beta_j \frac{\partial^2 w_j}{\partial t^2} = 0 \tag{3b}$$

for $i = 2, 3, \ldots, n-1$, and

$$\mu_n \frac{\partial^4 w_n}{\partial x^4} - \varepsilon \frac{\partial^2 w_n}{\partial x^2} + \lambda_{n-1}(w_n - w_{n-1}) + \lambda_n w_n + \beta_n \frac{\partial^2 w_n}{\partial t^2} = 0. \tag{3c}$$

The * have been dropped from all variables in Eq. (3); all dependent and independent variables are non-dimensional. Non-dimensional parameters are defined as

$$\mu_j = \frac{E_j I_j}{E_1 I_1}, \tag{4a}$$

$$\beta_j = \frac{\rho_j A_j}{\rho_1 A_1}, \tag{4b}$$

$$\lambda_j = \frac{k_j L^4}{E_1 I_1}, \tag{4c}$$

and

$$\varepsilon = \frac{PL^2}{E_1 I_1}. \tag{4d}$$

A normal-mode solution of Eq. (3) is assumed as

$$w_i(x,t) = u_i(x) \mathrm{e}^{\mathrm{i}\omega t}, \tag{5}$$

where $u_i(x)$ is the spatially dependent mode shape of the ith beam corresponding to the natural frequency ω. The result of substitution of Eq. (5) in Eq. (3) is summarized in a system of simultaneous ordinary differential equations summarized in matrix form as

$$\mathbf{K} + \mathbf{K_c u} - \omega^2 \mathbf{M u} = \mathbf{0}, \tag{6}$$

where $\mathbf{u}$ is a vector whose components are the spatial mode shapes and $\mathbf{K}$ the structural stiffness operator matrix which can be written as

$$\mathbf{K} = \mathbf{K_b} + \mathbf{K_a} \tag{7}$$

in which $\mathbf{K_b}$ is a diagonal operator matrix representing the bending stiffness with

$$(K_b)_{i,i} = \mu_i \frac{\partial^4}{\partial x^4}, \tag{8}$$

$\mathbf{K_a}$ is a diagonal operator matrix representing the axial stiffness with

$$(K_a)_{i,i} = -\varepsilon \frac{\partial^2}{\partial x^2}, \tag{9}$$

$\mathbf{K_c}$ is a tri-diagonal matrix of coupling stiffnesses due to the elastic layers with

$$(K_c)_{i,i-1} = -\lambda_{i-1}, \quad i = 2, 3, \ldots, n,$$

$$(K_c)_{i,i} = \lambda_{i-1} + \lambda_i, \quad i = 1, 2, \ldots, n,$$

$$(K_c)_{i,i+1} = -\lambda_i, \quad i = 1, 2, \ldots, n-1, \tag{10}$$

and $\mathbf{M}$ is a diagonal mass matrix with

$$m_{i,i} = \beta_i. \tag{11}$$

Eq. (6) may be rewritten as

$$\mathbf{M}^{-1}(\mathbf{K}+\mathbf{K_c})\mathbf{u} = \omega^2\mathbf{u}. \tag{12}$$

It is clear from Eq. (12) that the natural frequencies are the square roots of the eigenvalues of $\mathbf{M}^{-1}(\mathbf{K}+\mathbf{K_c})$ and the mode shapes are the corresponding eigenvectors.

Consider the operator

$$\mathbf{L_i} = \mu_i \frac{\partial^4}{\partial x^4} - \varepsilon \frac{\partial^2}{\partial x^2}. \tag{13}$$

It is easy to show [2] that this operator is self-adjoint with respect to the standard inner product on $C^4[0,1]$ when the ends are fixed, pinned, free or connected to an elastic element. This implies that for any $f(x)$ and $g(x)$ that satisfy boundary conditions for such supports

$$\int_0^1 [\mathbf{L_i} f(x)]g(x)\,\mathrm{d}x = \int_0^1 f(x)[\mathbf{L_i} f(x)]\,\mathrm{d}x. \tag{14}$$

Furthermore, unless the beams have free–free or pinned–free end conditions the operator is also positive definite with respect to this inner product. In these exceptional cases the operator is non-negative definite.

The solution vector $\mathbf{u} = [u_1(x)\ \ u_2(x)\ \ \ldots\ \ u_n(x)]^{\mathrm{T}}$ is an element of the vector space $W = C^4[0,1] \times R^n$ whose elements are n-dimensional vectors of functions in $C^4[0,1]$. It can be shown [2] that if the operators for the individual structural elements ($\mathbf{L_i}$) are self-adjoint then the matrix operator $\mathbf{K}+\mathbf{K_c}$ is self-adjoint with respect to the inner product defined for all $\mathbf{u}$ and $\mathbf{v}$ in W defined by

$$(\mathbf{u},\mathbf{v})_W = \int_0^1 \mathbf{v}^{\mathrm{T}}\mathbf{u}\,\mathbf{dx}. \tag{15}$$

Since $\mathbf{M}$ is a diagonal matrix this leads to the conclusion that $\mathbf{M}^{-1}(\mathbf{K}+\mathbf{K_c})$ is self-adjoint with respect to a kinetic energy inner product [2] defined as

$$(\mathbf{u},\mathbf{v})_M = \int_0^1 \mathbf{v}^{\mathbf{T}}\mathbf{M}\mathbf{u}\,\mathbf{dx}. \tag{16}$$

The eigenvalues of a self-adjoint operator are real and the eigenvectors satisfy an orthogonality condition. If $\mathbf{u_i}$ and $\mathbf{u_j}$ are mode shapes corresponding to distinct frequencies then

$$(\mathbf{u_i},\mathbf{u_j})_M = 0. \tag{17}$$

The matrix $\mathbf{K_c}$ is positive definite unless $\lambda_0 = 0$ and $\lambda_n = 0$ in which case it is positive semi-definite. If either $\mathbf{K}$ or $\mathbf{K_c}$ is positive definite then $\mathbf{K}+\mathbf{K_c}$ is positive definite. Then since $\mathbf{M}$ is positive definite, all eigenvalues of $\mathbf{M}^{-1}(\mathbf{K}+\mathbf{K_c})$ are positive. If $\mathbf{K}$ and $\mathbf{K_c}$ are positive semi-definite then $\mathbf{K}+\mathbf{K_c}$ is positive semi-definite and the lowest natural frequency is zero which has a corresponding rigid-body mode.

3. General case

In the most general case the coupled differential equations of Eq. (12) are linear ordinary differential equations with constant coefficients. A solution is assumed of the form

$$\mathbf{u}(x) = \mathbf{a}\,\mathrm{e}^{\alpha x}, \tag{18}$$

where $\mathbf{a}$ is a $n \times 1$ vector of constants and α the constant to be determined. Substitution of Eq. (18) into Eq. (12) leads to a set of algebraic equations of the form

$$(\mathbf{K}_\alpha + \mathbf{K_c})\mathbf{a} = \omega^2\mathbf{M}\mathbf{a}, \tag{19}$$

where $\mathbf{K}_\alpha$ is a diagonal matrix with $K_{\alpha_{i,i}} = \mathbf{L_i}(\alpha)$, where $\mathbf{L_i}(\alpha)$ is a polynomial obtained by replacing the differential operator $\mathrm{d}/\mathrm{d}x$ by α in the operator $\mathbf{L_i}$.

Given a natural frequency, Eq. (19) may be used to determine the appropriate values of α. However the natural frequencies are not known a priori and must be determined from Eq. (12), dependent on the values of α. Thus an iterative procedure is necessary. A suggested procedure is

- Guess a value of ω.
- For this value of ω determine all values of α by setting $|(\mathbf{K}_\alpha + \mathbf{K_c}) - \omega^2 \mathbf{M}| = 0$. This leads to a polynomial equation order $4n$ in α where n is the number of elastically connected components. Thus there are $4n$ values of α. Since all coefficients are real, when complex roots occur, they occur in complex conjugate pairs. Furthermore, since the differential operators include only even order derivatives then the polynomial only includes even powers of α, in which case the roots occur in opposite signed pairs; if α is a root then so is $-\alpha$.
- A general solution is built using the calculated values of α. Purely imaginary values of α lead to trigonometric solutions. Real values of α lead to hyperbolic trigonometric functions and complex conjugate roots lead to products of trigonometric and hyperbolic trigonometric solutions. The most general solution is a linear combination of all solutions. The coefficients in the linear combination are the constants of integration.
- Boundary conditions are applied to establish a set of homogeneous algebraic equations which must be satisfied by the constants of integration.
- If the correct solution has been obtained then the determinant of the set of homogeneous equations must be zero. If not the assumed value of ω is not a natural frequency.
- The determinant of this homogeneous set of equations is a function of the parameter ω. If ω is a natural frequency then the determinant must be zero, in order for a non-trivial solution for the constants of integration to exist.
- An iterative procedure is best employed to determine the natural frequencies, perhaps using a bracketing method. The system has an infinite number of natural frequencies and it is generally desired to determine the frequencies sequentially. Thus employing an open method such as the Newton–Raphson method will likely miss some frequencies.
- Computational problems will occur in determining larger frequencies due to evaluation of hyperbolic trigonometric functions of large arguments.
- Once a natural frequency is obtained the corresponding mode shape is obtained by solving for the constants of integration realizing that a unique solution does not exist.

A numerical algorithm may be developed to implement the above scheme. However computations are very difficult due to very large exponential terms. A small change in a guess for the natural frequency can lead to a large change in the determinant.

4. Identical beams

When all beams are identical ($\mu_i = 1,\ \beta_i = 1,\ i = 1, 2, \ldots, n$),

$$\mathbf{K_b} + \mathbf{K_a} = \mathbf{I}\left(\frac{\mathrm{d}^4}{\mathrm{d}x^4} - \varepsilon \frac{\mathrm{d}^2}{\mathrm{d}x^2}\right), \tag{20}$$

and $\mathbf{M} = \mathbf{I}$, where $\mathbf{I}$ is the $n \times n$ identity matrix.

Consider the problem for the natural frequencies, δ and corresponding mode shapes $\phi(x)$ of a single stretched beam

$$\phi^{iv} - \varepsilon\phi'' = \delta^2\phi, \tag{21}$$

where appropriate boundary conditions are applied. When solved the problem yields an infinite, but countable, number of natural frequencies, $\delta_1 < \delta_2 < \cdots < \delta_{k-1} < \delta_k < \delta_{k+1} < \cdots$. Each natural frequency has a corresponding mode shape $\phi_k(x)$. Mode shape orthogonality implies that

$$\int_0^1 \phi_j(x)\phi_k(x)\,\mathrm{d}x = 0 \tag{22}$$

for $j \neq k$. Mode shapes are normalized by requiring

$$\int_0^1 [\phi_k(x)]^2\,\mathrm{d}x = 1. \tag{23}$$

A solution of Eq. (12) when the operator stiffness matrix is of the form of Eq. (20) is assumed as

$$\mathbf{u} = \mathbf{a}\phi(\mathbf{x}) \tag{24}$$

which leads to

$$\mathbf{I}(\phi^{\mathbf{iv}} - \varepsilon\phi'')\mathbf{a} + \mathbf{K_c a}\phi = \omega^2 \mathbf{I}\phi. \tag{25}$$

Using Eq. (21) in Eq. (25) leads to

$$\mathbf{K_c a} = [\omega^2 - \delta_k^2]\mathbf{a}. \tag{26}$$

Eq. (26) is that of a matrix eigenvalue problem. The natural frequencies are of the form

$$\omega_{k,j} = (\delta_k^2 + v_j)^{1/2}, \tag{27}$$

where $k = 1, 2, \ldots, j = 1, 2, \ldots, n$, and $v_j j = 1, 2, \ldots, n$ are the eigenvalues of the coupling matrix $\mathbf{K_c}$.

For each value of δ there are n natural frequencies and mode shapes. Thus the natural frequencies can be indexed by $\omega_{k,j}$ where $k = 1, 2, \ldots,$ and $j = 1, 2, \ldots, n$ and the corresponding mode shapes are

$$\mathbf{u}_{k,j} = \mathbf{a}_j \phi_k(x), \tag{28}$$

where $\mathbf{a}_j$ is the eigenvector of $\mathbf{K_c}$ corresponding to the eigenvalue v_j. The eigenvectors are normalized by requiring $\mathbf{a}_j^{\mathrm{T}} \mathbf{a}_j = 1$.

Eqs. (27) and (28) show that the modes for a set of identical beams can be organized into an infinite number of sets, each with n modes. Modes with the same first index have the same spanwise displacement, but with a different amplitude. These are called intramodal modes. Modes corresponding to different values of the first index have different spanwise variations and are called intermodal modes.

5. Rayleigh–Ritz approximations

The natural frequencies and mode shapes are difficult to obtain for the general case. Application of the Rayleigh–Ritz method provides an alternative to obtain good approximations for the natural frequencies and mode shapes. Let $\mathbf{q_i}$ $i = 1, 2, \ldots, p$ be k n-dimensional vectors of functions of x, each of which satisfy all geometric boundary conditions. For simplicity assume

$$\mathbf{q_i} = \mathbf{z_i} \psi_i(x) \tag{29}$$

for $i = 1, 2, \ldots, p$ and where $\mathbf{z_i}$ is a n-dimensional vector of constants and $\psi_i(x)$ satisfies all geometric boundary conditions. The Rayleigh–Ritz approximation for a mode shape is

$$\mathbf{w} = \sum_{i=1}^{k} c_i \mathbf{z_i} \psi_i(x). \tag{30}$$

Minimization of Rayleigh's quotient leads to the following matrix eigenvalue problem for the natural frequency and mode shape approximations as

$$\mathbf{K}_r \mathbf{c} = \omega^2 \mathbf{M}_r \mathbf{c}, \tag{31}$$

where $\mathbf{c} = [c_1 \;\; c_2 \;\; \ldots \;\; c_k]^{\mathrm{T}}$, the elements of the $p \times p$ Rayleigh stiffness matrix are

$$\mathbf{K}_{r,\mathbf{i},\mathbf{j}} = \int_0^1 \mathbf{z_j^T} \psi_j (\mathbf{K_c} + \mathbf{K_a}) \mathbf{z_i} \psi_i \, \mathrm{d}x, \tag{32}$$

and the elements of the $k \times k$ Rayleigh mass matrix are

$$\mathbf{M_{r,i,j}} = \int_0^1 \mathbf{z_j^T} \psi_j \mathbf{M} \mathbf{z_i} \psi_i \, \mathrm{d}x. \tag{33}$$

Eq. (32) is used when each $\psi_i(x)$ satisfies all geometric and natural boundary conditions. Integration by parts leads to an alternate form of Eq. (32) as

$$\begin{aligned}\mathbf{K}_{r,i,j} = {} & \left(\int_0^1 \frac{\mathrm{d}^2\psi_i}{\mathrm{d}x^2} \frac{\mathrm{d}^2\psi_j}{\mathrm{d}x^2} \mathrm{d}x + \psi_j(1)\psi_i'''(1) - \psi_j(0)\psi_i'''(0) - \psi_j'(1)\psi_i''(1) + \psi_j'(0)\psi_i''(0) \right) \mathbf{z_j^T} \Delta \mathbf{z_i} \\ & + \varepsilon \left(\int_0^1 \frac{\mathrm{d}\psi_i}{\mathrm{d}x} \frac{\mathrm{d}\psi_j}{\mathrm{d}x} \mathrm{d}x - \psi_j(1)\psi_i'(1) + \psi_j(0)\psi_i'(0) \right) \mathbf{z_j^T z_i} + \left(\int_0^1 \psi_i \psi_j \, \mathrm{d}x \right) \mathbf{z_j^T K_c z_i},\end{aligned} \tag{34}$$

where $\Delta = \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_n)$.

If the beam is fixed–fixed all boundary conditions are geometric and Eqs. (32) and (34) are identical. If the beam is pinned–pinned or fixed–pinned it is likely that the basis functions are chosen to satisfy the natural boundary condition at a pinned support and the both equations are satisfied. If the beam has a free end, say at $x = 1$, then the appropriate non-dimensional boundary condition is $\mu_i w_i'''(1) = \varepsilon w_i'(1)$.

An obvious choice of basis vectors is the partial set of mode shape vectors for elastically connected Euler–Bernoulli beams without stretching. The problem for the natural frequencies and mode shapes for a set of elastically coupled

Euler–Bernoulli beams is

$$(\mathbf{K_b} + \mathbf{K_c})\mathbf{w} = \omega^2 \mathbf{M}\mathbf{w}. \tag{35}$$

A procedure similar to that used for identical elastically connected stretched beams is employed leading to solutions of Eq. (35) of the form

$$\mathbf{w_{k,j}} = \mathbf{a_{k,j}}\phi_k(x) \tag{36}$$

for $k = 1, 2, \ldots,$and $j = 1, 2, \ldots, n$ and where $\phi_k(x)$ $k = 1, 2, \ldots$ are the mode shapes for a single beam and $\mathbf{a_{k,j}}$ $j = 1, 2, \ldots, n$ are the eigenvectors of the matrix eigenvalue problem

$$(\delta_k^4 \Delta + \mathbf{K_c})\mathbf{a} = \omega^2 \mathbf{M}\mathbf{a}, \tag{37}$$

where $\phi_k^{iv} = \delta_k^4 \phi_k$ and all boundary conditions are satisfied. Note that the beam functions satisfy the same boundary conditions as the stretched beam for fixed and pinned ends, but not for free ends.

6. Examples

6.1. Identical pinned–pinned beams

Consider a set of n identical elastically connected pinned–pinned axially loaded beams. The exact natural frequencies for a single axially loaded pinned–pinned beam are $\omega_k = \sqrt{k^4\pi^4 + \varepsilon k^2\pi^2}$ which have corresponding normalized mode shape vectors of $u_k(x) = \sqrt{2}\,\sin(k\pi x)$, $k = 1, 2, \ldots$. Then from Eq. (27) the exact natural frequencies of the elastically connected beams are

$$\omega_{k,j} = \sqrt{k^4\pi^4 + \varepsilon k^2\pi^2 + v_j}, \tag{38}$$

where $k = 1, 2, \ldots,$ and for each k, $j = 1, 2, \ldots, n$. The normalized mode shape corresponding to the natural frequency $\omega_{k,j}$ is

$$\mathbf{u_{k,j}} = \sqrt{2}\mathbf{a_j}\,\sin(k\pi x), \tag{39}$$

where v_j $j = 1, 2, \ldots, n$ are the eigenvalues of $\mathbf{K_c}$ and $\mathbf{a_j}$ are their corresponding normalized eigenvectors.

The set of natural frequencies $\omega_{k,j}$ for each k with $j = 1,2,\ldots,n$ constitute a set of n intramodal frequencies for the kth mode. For large k, all intramodal frequencies approach $k^2\pi^2$, the natural frequencies of an Euler–Bernoulli beam without axial loading. Each mode shape is a product of a spatial mode shape $\phi_k(x) = \sqrt{2}\,\sin(k\pi x)$ and a discrete mode shape $\mathbf{a_j}$. The coupling only has a small effect on the numerical values of large frequencies but provides the discrete component of the mode shapes.

Consider a coupling matrix of the form

$$\mathbf{K_c} = \begin{bmatrix} 1000 & -1000 & 0 & 0 & 0 \\ -1000 & 2000 & -1000 & 0 & 0 \\ 0 & -1000 & 2000 & -1000 & 0 \\ 0 & 0 & -1000 & 2000 & 1000 \\ 0 & 0 & 0 & -1000 & 2000 \end{bmatrix}. \tag{40}$$

The eigenvalues and eigenvectors of the coupling matrix are determined such that the kth set of intramodal natural frequencies are

$$\omega_{k,1} = \sqrt{k^4\pi^4 + \varepsilon k^2\pi^2}, \tag{41a}$$

$$\omega_{k,2} = \sqrt{k^4\pi^4 + \varepsilon k^2\pi^2 + 382.0}, \tag{41b}$$

$$\omega_{k,3} = \sqrt{k^4\pi^4 + \varepsilon k^2\pi^2 + 1.382 \times 10^3}, \tag{41c}$$

$$\omega_{k,4} = \sqrt{k^4\pi^4 + \varepsilon k^2\pi^2 + 2.618 \times 10^3}, \tag{41d}$$

and

$$\omega_{k,5} = \sqrt{k^4\pi^4 + \varepsilon k^2\pi^2 + 3.618 \times 10^3}. \tag{41e}$$

The normalized mode shape vectors for the kth mode are

$$\mathbf{u}_{\mathbf{k},1}=\sqrt{2}\begin{bmatrix}0.4472\\0.4472\\0.4472\\0.4472\\0.4472\end{bmatrix}\sin(k\pi x),\quad \mathbf{u}_{\mathbf{k},2}=\sqrt{2}\begin{bmatrix}0.6015\\0.3717\\0\\-0.3717\\-0.6015\end{bmatrix}\sin(k\pi x),\mathbf{u}_{\mathbf{k},3}=\sqrt{2}\begin{bmatrix}0.5117\\-0.1954\\-0.6325\\-0.1954\\0.5117\end{bmatrix}\sin(k\pi x),$$

$$\mathbf{u}_{\mathbf{k},4}=\sqrt{2}\begin{bmatrix}0.3717\\-0.6015\\0\\0.6015\\-0.3717\end{bmatrix}\sin(k\pi x),\quad \text{and}\quad \mathbf{u}_{\mathbf{k},5}=\sqrt{2}\begin{bmatrix}0.1954\\-0.5717\\0.6325\\-0.5717\\0.1954\end{bmatrix}\sin(k\pi x). \tag{42}$$

The natural frequencies of a set of identical pinned–pinned Euler–Bernoulli beams without stretching are $\omega_{k,j}=\sqrt{k^4\pi^4+v_j}$ and their corresponding mode shapes are the same as those for the stretched beams. When these mode shapes are used as basis functions for a Rayleigh–Ritz approximation to approximate the natural frequencies and mode shapes for the corresponding stretched beams the exact solution results.

6.2. Identical fixed–free beams

Now consider a set of identical fixed–free beams. The natural frequencies for a single stretched fixed–free beam are the solutions of

$$\begin{aligned}&\eta^5+\eta v^4-\eta\varepsilon v^2-\frac{\eta^4\varepsilon}{v}\cosh^2(\eta)-\varepsilon\eta^3\sinh^2(\eta)+(2\eta^3v^2-\eta^3\varepsilon-\eta^2v\varepsilon)\cosh(\eta)\cos(v)\\&\quad+(\eta^2v^3-\eta^4v-2\varepsilon\eta^2v)\sinh(\eta)\sin(v)=0,\end{aligned} \tag{43}$$

where

$$\eta=\left[\frac{\sqrt{\varepsilon+4\omega^2}+\varepsilon}{2}\right]^{1/2} \tag{44a}$$

and

$$v=\left[\frac{\sqrt{\varepsilon+4\omega^2}-\varepsilon}{2}\right]^{1/2}. \tag{44b}$$

For $\varepsilon=1$ the first three sets of intramodal frequencies for a set of five elastically connected axially loaded beams are determined from Eq. (27) as

$$\omega_{1,j}=\sqrt{21.169+v_j}, \tag{45a}$$

$$\omega_{2,\mathrm{j}}=\sqrt{515.29+v_j}, \tag{45b}$$

and

$$\omega_{3,j}=\sqrt{1.764x10^3+v_j} \tag{45c}$$

for $j=1,2,\ldots,5$.

The characteristic equation for the natural frequencies of a set of unstretched Euler–Bernoulli fixed–free beams is $\cos\delta\cosh\delta=-1$. The corresponding normalized mode shapes are of the form

$$\psi_k(x)=\cosh(\delta_kx)-\cos(\delta_kx)-\alpha_k[\sinh(\delta_kx)-\sin(\delta_kx)], \tag{46}$$

where

$$\alpha_k=\frac{\cos\delta_k+\cosh\delta_k}{\sin\delta_k+\sinh\delta_k}. \tag{47}$$

A set of 15 basis functions for a Rayleigh–Ritz approximation for the natural frequencies and mode shapes of the stretched beam are of the form of Eq. (36) for $k = 1, 2, 3$ and $j = 1, 2, \ldots, 5$, where $\phi_k(x)$ are given by Eq. (46) and $\mathbf{a_j}$'s are the same as the discrete mode shape vectors of Eq. (42).

The elements of the Rayleigh stiffness matrix are of the form

$$\mathbf{K}_{r,i,j} = \left(\int_0^1 \frac{\mathrm{d}^2\psi_i}{\mathrm{d}x^2}\frac{\mathrm{d}^2\psi_j}{\mathrm{d}x^2}\mathrm{d}x + \psi_j(1)\psi_i'''(1) - \psi_j'(1)\psi_i''(1)\right)\mathbf{z_j^T}\Delta\mathbf{z_i} + \varepsilon\left(\int_0^1 \frac{\mathrm{d}\psi_i}{\mathrm{d}x}\frac{\mathrm{d}\psi_j}{\mathrm{d}x}\mathrm{d}x - \psi_j(1)\psi_i'(1)\right)\mathbf{z_j^T z_i} + \left(\int_0^1 \psi_i\psi_j\,\mathrm{d}x\right)\mathbf{z_j^T K_c z_i}. \tag{48}$$

6.3. Non-identical fixed–fixed beams

Consider a set of five fixed–fixed beams with $\mu_1 = 1,\ \mu_2 = 4,\ \mu_3 = 9,\ \mu_4 = 16,\ \mu_5 = 25, \beta_1 = 1,\ \beta_2 = 2,\ \beta_3 = 3,\ \beta_4 = 4$, and $\beta_5 = 5$. The beams are elastically connected with a coupling matrix of

$$\mathbf{K_c} = \begin{bmatrix} 800 & -800 & 0 & 0 & 0 \\ -800 & 1600 & -800 & 0 & 0 \\ 0 & -800 & 1600 & -800 & 0 \\ 0 & 0 & -800 & 1600 & -800 \\ 0 & 0 & 0 & -800 & 2400 \end{bmatrix}. \tag{49}$$

Eq. (49) is the coupling matrix for a system of five elastically connected beams with $\lambda_0 = 0$, $\lambda_1 = 800$, $\lambda_2 = 800$, $\lambda_3 = 800$, $\lambda_4 = 800$, and $\lambda_5 = 800$. The fifth beam is connected to a Winkler foundation.

Fifteen basis functions are used in the Rayleigh–Ritz approximation to approximate the natural frequencies and mode shapes for the set of five elastically connected beams. They are chosen as the normalized mode shapes for the first three sets of intramodal frequencies for a set of beams with $\varepsilon = 0$. The mode shapes for an unstretched fixed–fixed Euler–Bernoulli beam are

$$\phi_k(x) = \cosh(\delta_k x) - \cos(\delta_k x) - \frac{\cosh(\delta_k) - \cos(\delta_k)}{\sinh(\delta_k) - \sin(\delta_k)}[\sinh(\delta_k x) - \sin(\delta_k x)]. \tag{50}$$

The three lowest values of δ_k are $\delta_1 = 4.730$, $\delta_2 = 7.852$, and $\delta_3 = 10.996$. The discrete part of the basis functions are the eigenvectors of $\mathbf{K_c} + \delta_k^4 \Delta$ normalized using the inner product of Eq. (16). The 15 basis functions are

$$\mathbf{z}_1 = \begin{bmatrix} -0.189 \\ -0.252 \\ -0.326 \\ -0.321 \\ -0.146 \end{bmatrix}\phi_1(x), \quad \mathbf{z}_2 = \begin{bmatrix} 0.354 \\ 0.371 \\ 0.193 \\ -0.197 \\ -0.258 \end{bmatrix}\phi_1(x), \quad \mathbf{z}_3 = \begin{bmatrix} 0.321 \\ 0.239 \\ -0.112 \\ -0.239 \\ 0.321 \end{bmatrix}\phi_1(x), \quad \mathbf{z}_4 = \begin{bmatrix} 0.442 \\ 0.182 \\ -0.405 \\ 0.225 \\ -0.083 \end{bmatrix}\phi_1(x),$$

$$\mathbf{z}_5 = \begin{bmatrix} -0.736 \\ 0.457 \\ -0.116 \\ 0.019 \\ -0.003 \end{bmatrix}\phi_1(x), \quad \mathbf{z}_6 = \begin{bmatrix} 0.008 \\ 0.035 \\ 0.134 \\ 0.330 \\ 0.319 \end{bmatrix}\phi_2(x), \quad \mathbf{z}_7 = \begin{bmatrix} -0.020 \\ -0.081 \\ -0.244 \\ -0.291 \\ 0.306 \end{bmatrix}\phi_2(x), \quad \mathbf{z}_8 = \begin{bmatrix} 0.067 \\ 0.235 \\ 0.462 \\ -0.236 \\ 0.068 \end{bmatrix}\phi_2(x),$$

$$\mathbf{z}_9 = \begin{bmatrix} -0.262 \\ -0.633 \\ 0.205 \\ -0.032 \\ 0.004 \end{bmatrix}\phi_2(x), \quad \mathbf{z_{10}} = \begin{bmatrix} -0.963 \\ 0.190 \\ -0.017 \\ 0.001 \\ -0.00004 \end{bmatrix}\phi_2(x), \quad \mathbf{z_{11}} = \begin{bmatrix} 0.0001 \\ 0.001 \\ 0.017 \\ 0.129 \\ 0.432 \end{bmatrix}\phi_3(x), \quad \mathbf{z_{12}} = \begin{bmatrix} 0.0007 \\ 0.010 \\ 0.096 \\ 0.470 \\ -0.116 \end{bmatrix}\phi_3(x),$$

$$\mathbf{z_{13}} = \begin{bmatrix} -.006 \\ -0.085 \\ -0.565 \\ 0.085 \\ -0.006 \end{bmatrix}\phi_3(x), \quad \mathbf{z_{14}} = \begin{bmatrix} 0.077 \\ 0.700 \\ -0.070 \\ 0.085 \\ -0.006 \end{bmatrix}\phi_3(x), \quad \text{and} \quad \mathbf{z_{15}} = \begin{bmatrix} 0.997 \\ -0.054 \\ 0.001 \\ -0.00002 \\ 0.0000003 \end{bmatrix}\phi_3(x). \tag{51}$$

Table 1
Rayleigh–Ritz coefficients corresponding to $\omega_{3,3}$ for $\varepsilon = 250$.

j	α_j
1	0.012
2	0.036
3	0.048
4	−0.026
5	9.65×10^{-3}
6	8.51×10^{-6}
7	-1.49×10^{-4}
8	-3.98×10^{-4}
9	1.82×10^{-4}
10	-9.02×10^{-5}
11	-1.56×10^{-4}
12	−0.073
13	−0.99
14	0.102
15	0.014

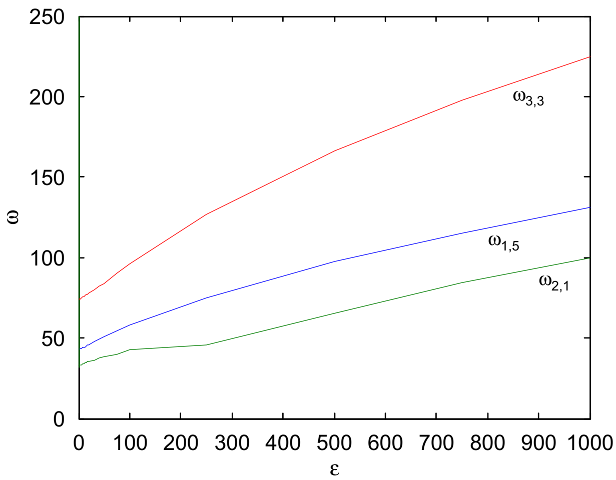

Fig. 2. Variation of natural frequencies of five elastically connected axially loaded beams with the non-dimensional stretching parameter ε.

Since the basis functions satisfy the boundary conditions for a fixed–fixed beam Eq. (34) simplifies to

$$\mathbf{K}_{r,i,j} = \left(\int_0^1 \frac{\mathrm{d}^2\psi_i}{\mathrm{d}x^2}\frac{\mathrm{d}^2\psi_j}{\mathrm{d}x^2}\,\mathrm{d}x\right)\mathbf{z_j^T}\Delta\mathbf{z_i}\left(\int_0^1 \psi_i\psi_j\,\mathrm{d}x\right)\mathbf{z_j^T}\mathbf{K_c}\mathbf{z} + \varepsilon\left(\int_0^1 \frac{\mathrm{d}\psi_i}{\mathrm{d}x}\frac{\mathrm{d}\psi_j}{\mathrm{d}x}\,\mathrm{d}x\right)\mathbf{z_j^T}\mathbf{z_i}. \tag{52}$$

Furthermore the basis functions satisfy the orthogonality conditions such that

$$\int_0^1 \phi_i(x)\phi_j(x)\,\mathrm{d}x = \delta_{i,j}$$

and for each set of intramodal mode shape vectors $\mathbf{u}_{k,j}$, $k = 1,2,3$ and $j = 1,2,\ldots,5$, $\mathbf{u}_{k,j}^T\mathbf{M}\mathbf{u}_{\mathbf{k},\ell} = \delta_{j,\ell}$ and $\mathbf{u}_{k,j}^T(\mathbf{K_b} + \Delta)\mathbf{u}_{\mathbf{k},\ell} = \omega_{k,j}^2\delta_{j,\ell}$. Application of the orthogonality conditions reduces Eq. (51) to

$$\mathbf{K}_{r,i,j} = \Omega + \varepsilon\left(\int_0^1 \frac{\mathrm{d}\psi_i}{\mathrm{d}x}\frac{\mathrm{d}\psi_j}{\mathrm{d}x}\,\mathrm{d}x\right)\mathbf{z_j^T}\mathbf{z_i}, \tag{53}$$

where Ω is a diagonal matrix with the squares of the natural frequencies for the unstretched beams along the diagonal. The Rayleigh–Ritz mass matrix of Eq. (33) is the identity matrix.

Use of the basis functions of Eq. (51) in the Rayleigh–Ritz method leads to approximations for 15 natural frequencies and mode shapes. Three of the approximations are illustrated in Fig. 1 as functions of the parameter ε. The natural frequencies are labeled according to their number corresponding to $\varepsilon = 0$ for which there are a set of intramodal modes for each spatially varying mode However for the non-identical stretched beams the modes cannot truly be classified using the terms intramodal and intermodal. The Rayleigh–Ritz approximation for each mode is a linear combination of all mode shapes of

Eq. (51). The coefficients multiplying mode shapes not corresponding to the spatial mode would all be zero for all modes in an intramodal set. This is not the case as evidenced by Table 1 which contains the components of the mode shape vector corresponding to $\omega_{3,3}$ for $\varepsilon = 250$. This mode shape has significant components from the first set of intramodal modes as well as the third set. The components for the second set are much smaller as the first and third modes are even functions when reflected about $x = 0.5$ while the second mode is an odd function when reflected about $x = 0.5$ (Fig. 2).

7. Conclusions

The problem of stretched beams connected by elastic layers is considered. A normal-mode solution is applied to the governing partial differential equations to derive a set of coupled ordinary differential equations which are used to determine the natural frequencies and mode shapes. It is shown that the set of differential equations can be written in s self-adjoint form with an appropriate inner product.

An exact solution for the general case is obtained, but numerical procedures must be used to determine the natural frequencies and mode shapes. The numerical procedure is difficult to apply, especially in determining higher frequencies. For the special case of identical beams, an exact expression for the natural frequencies is obtained in terms of the natural frequencies of a corresponding set of unstretched beams and the eigenvalues of the coupling matrix.

A Rayleigh–Ritz method is developed in which the mode shape vectors for the corresponding set of unstretched beams are used as basis functions. A beam with a free end is a special case because the boundary conditions satisfied by the stretched beam at the free end are different than the boundary conditions satisfied by the corresponding unstretched beams.

The modes for the unstretched beams may be classified as intramodal and intermodal where intermodal refers to different mode shapes over the span of the beam and intramodal refers to the different discrete mode shapes across the beams for the same mode shape across the span. This categorization can also be applied to the mode shapes for the stretched beams in the special case where all beams are identical.

The Rayleigh–Ritz method is used to develop mode shapes and natural frequencies for sets of non-identical beams. Its application shows that as the stretching parameter increases the mode shapes which are solely intramodal for a set of unstretehd beams develop components from other modes.

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