Journal of Sound and Vibration 327 (2009) 322–332

Contents lists available at ScienceDirect

Journal of Sound and Vibration

journal homepage: www.elsevier.com/locate/jsvi

Synchronization via active control of identical and non-identical Φ^6 chaotic oscillators with external excitation

A.N. Njah *

Nonlinear Dynamics Research Group, Department of Physics, University of Agriculture Abeokuta, P.M.B. 2240, Abeokuta, Nigeria

ARTICLE INFO

Article history:
Received 30 December 2008
Received in revised form
8 June 2009
Accepted 15 July 2009
Handling Editor: M.P. Cartmell
Available online 18 August 2009

ABSTRACT

Using the active control technique with Lyapunov stability theory and the Routh–Hurwitz criteria, control functions are designed to achieve complete synchronization between two identical Φ^6 Van der Pol oscillators (Φ^6-VDPOs), two identical Φ^6 Duffing oscillators (Φ^6-DOs), and two non-identical Φ^6 oscillators comprising Φ^6-VDPO and Φ^6-DO for the triple-well configuration of the Φ^6 potential. The coefficient matrix of the error dynamics between each pair of synchronized systems is chosen such that the number of active control functions reduces from two to one, thereby, significantly reducing controller complexity in the design. The designed controllers enable the state variables of the response system to synchronize with those of the drive system in both the identical and non-identical cases. The results are validated using numerical simulations.

1. Introduction

In the last two decades extensive studies have been done on the properties of nonlinear dynamical systems. One of the most important properties of nonlinear dynamical systems is that of synchronization. For two n-dimensional chaotic systems coupled in the drive–response configuration, the drive system $[\dot{x}=f(x,y)]$ and the response system $[\dot{y}=g(x,y)]$, where $x(t)=(x_1,x_2,\ldots,x_n)$ and $y(t)=(y_1,y_2,\ldots,y_n)$ are phase space or state variables, and $f=(f_1,f_2,\ldots,f_n)$ and $g=(g_1,g_2,\ldots,g_n)$ are the corresponding nonlinear functions, synchronization in a direct sense implies $|y_i(t)-x_i(t)|_{(i=1,2,\ldots,n)}\to 0$ as $t\to\infty$. When this occurs the two systems are said to be completely synchronized [1–3]. Generally chaos synchronization can be considered as the design problem of a feedback law for full observer using the known information of a plant, so as to ensure that the controlled receiver synchronizes with the transmitter [4]. Complete synchronization (CS) of two chaotic systems was first achieved by Pecora and Carroll in 1990 using replacement method [1]. Thereafter enormous research activities have been carried out in chaos control and synchronization by many researchers from different disciplines. These research activities have, over the years, established other types of synchronization such as sequential, phase, anticipated, measure, generalized, lag and projective synchronizations, as well as other effective methods of chaos control and synchronization such as linear feedback, adaptive synchronization, backstepping nonlinear control, sliding mode control, and active control [5,6] (and references therein). Outstanding among the various methods of synchronization is the active control method introduced by Bai and Lonngren [7,8]. The active control method is outstanding because its application is ubiquitous. It can be used to synchronize identical chaotic systems

* Tel.: +234 803 5640371.
E-mail address: njahabdul@yahoo.com

0022-460X/$ - see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.jsv.2009.07.015

evolving from different initial conditions [7–10] (just to cite but a few), non-identical systems [6,11–16], and to anti-synchronize both identical and non-identical systems [17–19].

The ubiquitous application of the active control technique has encouraged researchers to introduce active control techniques based on different stability criteria. For instance, Lei et al. [20] introduced the active control based on the Lyapunov stability theory and the Routh–Hurwitz criteria, which has the advantage of possible implementation, and it has been used to synchronize a few chaotic systems [20–22]. However, the active control technique always gives rise to as many control functions as is the dimension of the systems being synchronized, thereby, making the controllers complex and unsuitable for practical implementation. In this paper the active control technique based on Lyapunov direct method and the Routh–Hurwitz criteria is used to design controllers for the synchronization of two-dimensional (2-D) chaotic systems in such a way that the number of active control functions reduced from two to one, thereby, reducing significantly the controller complexity and hence cost, which makes practical implementation feasible.

Synchronization of two chaotic systems has potential applications in physical systems, lasers, plasma, circuits, chemical reactor, ecological systems, biomedical systems, cardio respiratory interaction, brain activity of Parkinsonian patient, paddlefish electro sensitive cell, solar activity, and secure communication [6,14] (and references therein). In secure communication the drive and response systems serve as the transmitter and receiver respectively. The message is embedded in the chaotic dynamics of the state variables of the transmitter via a mathematical function and is recovered at the receiver via the inverse of the mathematical function when the drive and the response systems are synchronized [23–26]. Secure communication is guaranteed if the dynamics of the drive system (transmitter) is complex [23–26]. In [23,24,26] Φ^2, Φ^4 and Φ^6 chaotic oscillators, respectively, have been synchronized via adaptive feedback approaches and applied to secure communication. The Φ^6 chaotic oscillators have more complex dynamics than their corresponding Φ^4 and Φ^2 chaotic oscillators [27–29] and hence offer more security of masked information during transmission. The complexity is even higher when the Φ^6 chaotic oscillators are non-identical due to model mismatches. Synchronization of non-identical Φ^6 chaotic oscillators is, therefore, interesting due to its potential application in highly secure communications. Moreover, the synchronization behaviour of Φ^6 chaotic oscillators, which have just been recently introduced, has not been well investigated for the triple-well potential configuration in which their dynamics are more complex.

Therefore, the aim of this paper is to design single active control functions based on Lyapunov stability theory and Routh–Hurwitz criteria that can guarantee complete synchronization of identical and non-identical Φ^6 chaotic oscillators with external excitation in the triple-well configuration of the Φ^6 potential. That is, controller complexity will be significantly reduced by carrying out the design in such a way that only one active control function emerges, instead of the normally two active control functions expected of a 2-D chaotic system.

The rest of the paper is organized as follows: the following section describes the Van der Pol and Duffing oscillators, Sections 3 and 4, respectively, deal with the synchronization between two identical Φ^6 Van der Pol oscillators and two identical Φ^6 Duffing oscillators evolving from different initial conditions; Section 5 deals with the synchronization between Φ^6 Van der Pol and Φ^6 Duffing oscillators, while Section 6 concludes the paper.

2. Description of the models

The general form of the Van der Pol and Duffing oscillators models with external excitation are, respectively, given by the second-order non-autonomous differential equations (1) and (2) as follows:

$$\ddot{x} - \mu(1 - x^2)\dot{x} + \frac{\mathrm{d}V(x)}{\mathrm{d}x} = f_1 \cos(\omega_1 t) \tag{1}$$

$$\ddot{x} + \lambda\dot{x} + \frac{\mathrm{d}V(x)}{\mathrm{d}x} = f_2 \cos(\omega_2 t) \tag{2}$$

where x is the state variable, dots over x denote derivative with respect to time t, $\mu > 0$ and λ are damping parameters, f_i and w_i, $i = 1, 2$ are, respectively, the amplitudes and angular frequencies of the external excitations, $V(x)$ is the potential. The Van der Pol oscillator model describes periodically self-excited oscillators in Physics, Engineering, Electronics, Biology, Neurology and many other disciplines [27], while the Duffing oscillator model describes various physical, electrical, mechanical, and engineering devices [28,29], for different potentials $V(x)$. The potentials $V(x)$ are approximated by finite Taylor series for the Φ^2, Φ^4 and Φ^6 chaotic oscillators as in (3), (4) and (5) respectively.

$$V(x) = \tfrac{1}{2}\alpha x^2 \tag{3}$$

$$V(x) = \tfrac{1}{2}\alpha x^2 + \tfrac{1}{4}\beta x^4 \tag{4}$$

$$V(x) = \tfrac{1}{2}\alpha x^2 + \tfrac{1}{4}\beta x^4 + \tfrac{1}{6}\delta x^6 \tag{5}$$

where α, β and δ are constant parameters of the potential. The potential (3) is single-well if $\alpha > 0$. The potential (4) is single-well if $(\alpha > 0, \beta > 0)$, double-well if $(\alpha < 0, \beta > 0)$ or double-hump if $(\alpha > 0, \beta < 0)$. The potential (5) is double-well if

$(\alpha<0,\beta>0,\delta>0)$, double-hump (or unbounded single-well) if $(\alpha>0,\beta<0,\delta<0)$, triple-well (or bounded double-hump) if $(\alpha>0,\beta<0,\delta>0)$ triple-hump (or unbounded double-well) if $(\alpha<0,\beta>0,\delta<0)$.

When the Φ^2 potential (3) is substituted into (1) the result gives the Van der Pol oscillator.

$$\ddot{x}-\mu(1-x^2)\dot{x}+\alpha x=f_1\cos\omega_1 t \tag{6}$$

However, substituting the Φ^2 (3) into (2) gives a linear equation which does not exhibit chaotic dynamics. Now, substituting the Φ^4 potential (4) into (1) and (2) we, respectively, have the Duffing–Van der Pol oscillator (7) and the Duffing oscillator (8) as follows:

$$\ddot{x}-\mu(1-x^2)\dot{x}+\alpha_1 x+\beta_1 x^3=f_1\cos\omega_1 t \tag{7}$$

$$\ddot{x}+\lambda\dot{x}+\alpha_2 x+\beta_2 x^3=f_2\cos\omega_2 t \tag{8}$$

where the subscripts 1 and 2 on the constant parameters indicate that the parameters, in general, take different values for the chaotic dynamics of the Van der Pol and Duffing oscillators respectively.

Finally, substituting the Φ^6 potential (5) into (1) and (2) we, respectively, obtain the Φ^6 Van der Pol (or extended Duffing–Van der Pol) oscillator (9) and the Φ^6 Duffing (or extended Duffing) oscillator (10) as follows:

$$\ddot{x}-\mu(1-x^2)\dot{x}+\alpha_1 x+\beta_1 x^3+\delta_1 x^5=f_1\cos\omega_1 t \tag{9}$$

$$\ddot{x}+\lambda\dot{x}+\alpha_2 x+\beta_2 x^3+\delta_2 x^5=f_2\cos\omega_2 t \tag{10}$$

The dynamics, control and synchronization of Eqs. (6)–(10), under different excitations and couplings have been extensively studied for the Van der Pol oscillator (6) [30–33] (and references therein), the Duffing–Van der Pol oscillator (7) [13,23,24,34–40] (and references therein), and the Duffing oscillator (8) [41–43] (and references therein), but have not been extensively studied for the Φ^6 Van der Pol oscillator (9) and Φ^6 Duffing oscillator (10). A few studies on the Φ^6 Van der Pol oscillator with external excitation (9) are found in [26] where the authors used adaptive feedback approach to synchronize two identical Φ^6 Van der Pol oscillators and applied to secure communications, [27] where the authors established the condition for homoclinic bifurcation and fractal basin boundaries, [44] where the authors derived conditions for chaotic motion and fractal basin boundaries in the presence of bounded noise and established threshold of bounded noise amplitude for onset of chaos, and [45] where the authors used Φ^6 Van der Pol and Φ^6 Duffing oscillators to illustrate the effectiveness of a novel feedback control scheme with an uncertainty estimator. While a few studies on the Φ^6 Duffing oscillator with external excitation (10) are found in [28] where the authors applied linear feedback and parametric control approaches and established conditions for inhibition of chaotic escape, and [29] where the authors derived the criteria for the occurrence of fractal basin boundaries for homoclinic and heteroclinic orbits (for both single and triple wells) and obtained harmonic, subharmonic and superharmonic oscillatory states.

A few studies have also been done on the parametrically and externally excited Φ^6 Van der Pol oscillator (Φ^6-VDPO) [46] and parametrically excited Φ^6 Duffing oscillator (Φ^6-DO) [47]. Most of the studies on Φ^6 chaotic oscillators, so far, are on analysis of the dynamics of the systems. Studies on their control and synchronization are limited [26,28,45].

In general, the dynamics of the Van der Pol oscillator models include the exhibition of strange attractors, Hopf and Neimark–Sacker bifurcations, Smale horseshoe chaos, multi-stability, homoclinic and heteroclinic bifurcations, fractal basin boundaries [26,28,30,31,33–36]. The dynamics of the Duffing oscillator models include hysteresis, multi-stability,

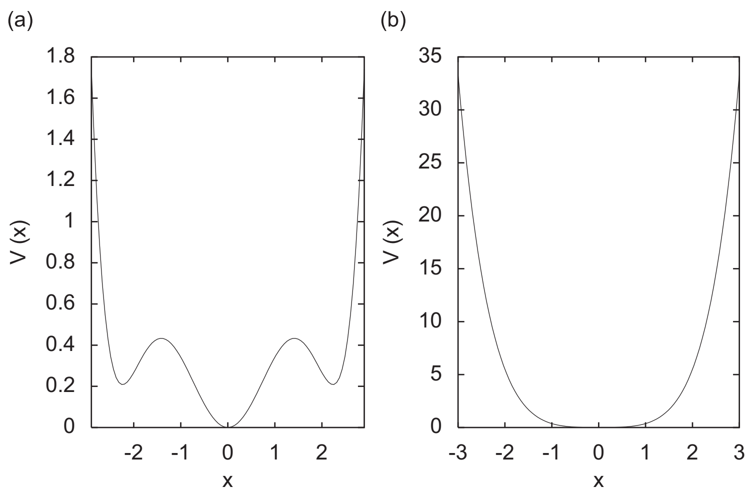

Fig. 1. The Φ^6 potential for parameter values (a) $\alpha=1.0,\ \beta=-0.7,\ \delta=0.1$ (triple-well), (b) $\alpha=0.46,\ \beta=1.0,\ \delta=0.1$ (single-well).

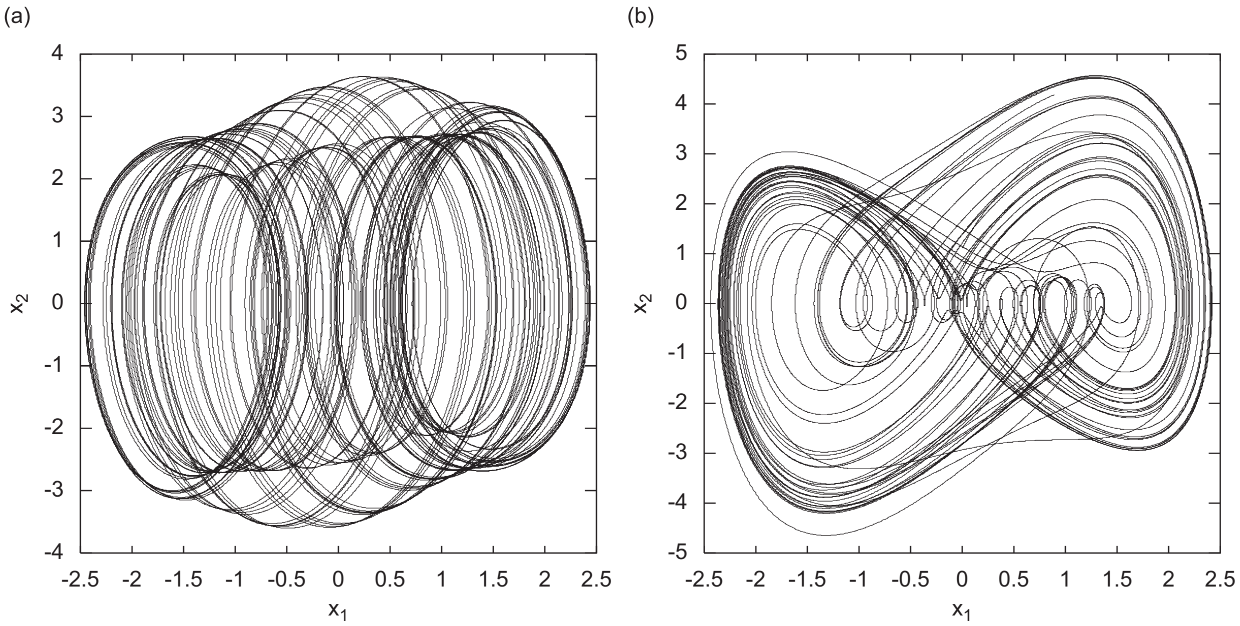

Fig. 2. Phase portrait of the chaotic attractor for the externally excited Φ^6 Van der Pol oscillator with parameter values: (a) $\mu = 0.4,\ \alpha_1 = 1.0,\ \beta_1 = -0.7,\ \delta_1 = 0.1,\ f_1 = 9,\ w_1 = 3.14$ (triple-well), (b) $\mu = 0.4,\ \alpha_1 = 0.46,\ \beta_1 = 1.0,\ \delta_1 = 0.1,\ f_1 = 4.5,\ w_1 = 0.86$ (single-well).

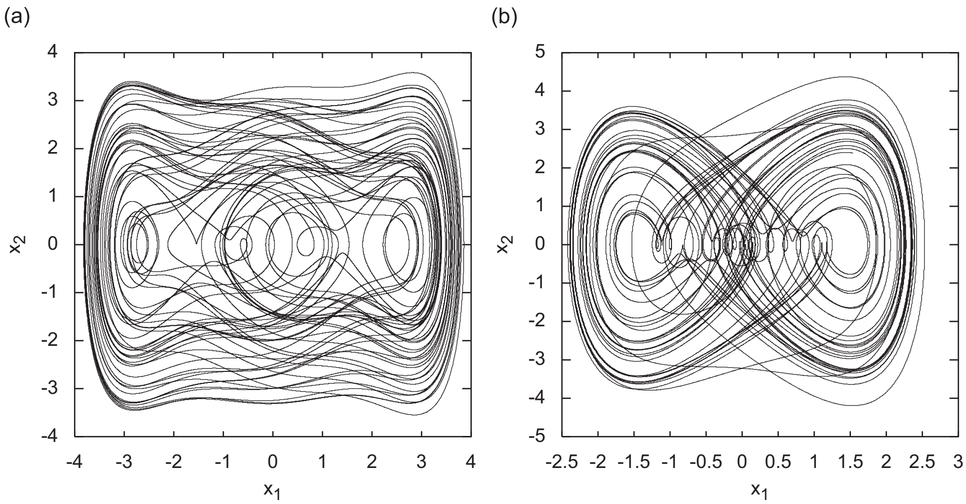

Fig. 3. Phase portrait of the chaotic attractor for the externally excited Φ^6 Duffing oscillator with parameter values (a) $\lambda = 0.01,\ \alpha_2 = 1.0,\ \beta_2 = -0.495,\ \delta_2 = 0.05,\ f_2 = 0.78,\ w_2 = 0.55$ (triple-well), (b) $\lambda = 0.4,\ \alpha_2 = 0.46,\ \beta_2 = 1.0,\ \delta_2 = 0.1,\ f_2 = 4.5,\ w_2 = 0.86$ (single-well).

period-doubling bifurcation, intermittent transitions to chaos, fractal basin boundaries [28] (and references therein). In general, the systems with Φ^6 potential have more complex dynamics than those with Φ^4 and Φ^2 potentials [27–29]. We, hereby, investigate the synchronization behaviour of the Φ^6-VDPO (9) and Φ^6-DO (10) in the triple-well configuration of the Φ^6 potential. In the literature, use has been made of the Φ^6 potential parameter values sets ($\alpha = 1$, $\beta = -0.7$ or $-0.5, \delta = 0.1$ or 0.05) [27,28] and ($\alpha = 0.46, \beta = 1.0, \delta = 0.1$) [26,45] which give the potential configurations in Figs. 1(a) and (b) respectively. The Φ^6-VDPO (9) has been shown to exhibit chaotic dynamics for the parameter values sets ($\mu = 0.4, \alpha_1 = 1.0, \beta_1 = -0.7, \delta_1 = 0.1, \omega_1 = 3.14, f_1 = 9$) and ($\mu = 0.4, \alpha_1 = 0.46, \beta_1 = 1.0, \delta_1 = 0.1, \omega_1 = 86$, $f_1 = 4.5$) [26,27] as shown in the phase space chaotic attractors in Figs. 2(a) and (b) respectively, which correspond to the potential configurations in Figs. 1(a) and (b) respectively. Whereas the Φ^6-DO (10) exhibits chaotic dynamics for the parameter values sets ($\lambda = 0.1, \alpha_2 = 1.0, \beta_2 = -0.5, \delta_1 = 0.05, \omega_2 = 0.5, f_2 = 0.3$) and ($\lambda = 0.4, \alpha_2 = 0.46, \beta_2 = 1.0$, $\delta_2 = 0.1, \omega_2 = 86, f_2 = 4.5$) [28,45] as shown in the phase space chaotic attractors in Figs. 3(a) and (b) respectively, which again correspond to the potential configurations in Figs. 1(a) and (b) respectively.

3. Synchronization of Φ^6 Van der Pol oscillators

3.1. Formulation of the active controllers

Eq. (9) can be written as

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \mu(1 - x_1^2)x_2 - \alpha_1 x_1 - \beta_1 x_1^3 - \delta_1 x_1^5 + f_1 \cos\omega_1 t \tag{11}$$

where $x = x_1$ and $\dot{x} = x_2$. Let system (11) be the drive system and system (12) be the response system:

$$\dot{y}_1 = y_2 + u_1(t)$$

$$\dot{y}_2 = \mu(1 - y_1^2)y_2 - \alpha_1 y_1 - \beta_1 y_1^3 - \delta_1 y_1^5 + f_1 \cos\omega_1 t + u_2(t) \tag{12}$$

where $u_1(t)$ and $u_2(t)$ are control functions to be determined. Subtracting (11) from (12) and using the notations $e_1 = y_1 - x_1$ and $e_2 = y_2 - x_2$ we have

$$\dot{e}_1 = e_2 + u_1(t)$$

$$\dot{e}_2 = \mu e_2 - \mu(y_1^2 y_2 - x_1^2 x_2) - \alpha_1 e_1 - \beta_1(y_1^3 - x_1^3) - \delta_1(y_1^5 - x_1^5) + u_2(t) \tag{13}$$

We now re-define the control functions such as to eliminate terms in (13) which cannot be expressed as linear terms in $e_1(t)$ and $e_2(t)$ as follows:

$$u_1(t) = v_1(t)$$

$$u_2(t) = \mu(y_1^2 y_2 - x_1^2 x_2) + \beta_1(y_1^3 - x_1^3) + \delta_1(y_1^5 - x_1^5) + v_2(t) \tag{14}$$

Substituting (14) into (13) we have

$$\dot{e}_1 = e_2 + v_1(t)$$

$$\dot{e}_2 = \mu e_2 - \alpha_1 e_1 + v_2(t) \tag{15}$$

Eq. (15) is the error dynamics, which can be interpreted as a control problem where the system to be controlled is a linear system with control inputs $v_1(t) = v_1(e_1(t), e_2(t))$ and $v_2(t) = v_2(e_1(t), e_2(t))$. As long as these feedbacks stabilize the system, $|e_i(t)_{,i=1,2}| \to 0$ as $t \to \infty$. This implies that the two systems (11) and (12) evolving from different initial conditions are synchronized. As functions of $e_1(t)$ and $e_2(t)$ we choose $v_1(t)$ and $v_2(t)$ as follows:

$$\begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} = \mathbf{D}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \tag{16}$$

where $\mathbf{D} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a 2×2 constant feedback matrix to be determined. Hence the error system (15) can be written as

$$\begin{pmatrix} \dot{e}_1(t) \\ \dot{e}_2(t) \end{pmatrix} = \mathbf{C}\begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix} \tag{17}$$

where $\mathbf{C} = \begin{pmatrix} a & 1+b \\ c-\alpha & \mu+d \end{pmatrix}$ is the coefficient matrix. According to the Lyapunov stability theory and the Routh–Hurwitz criteria, if

$$a + d + \mu < 0$$

$$(c - \alpha_1)(1 + b) - a(\mu + d) < 0 \tag{18}$$

then the eigenvalues of the coefficient matrix of error system (15) must be real negative or complex with negative real parts and, hence, stable synchronized dynamics between systems (11) and (12) is guaranteed. Let

$$a + d + \mu = -E$$

$$(c - \alpha_1)(1 + b) - a(\mu + d) = -E \tag{19}$$

where $E > 0$ is a real number which is usually set equal to 1. There are several ways of choosing the constant elements a, b, c, d of matrix $\mathbf{D}$ in order to satisfy (18). We optimize the way this choice is made so that not only the synchronization time is short (see Section 4.2) but also the controller complexity is reduced. This can be achieved by letting $a = b = 0$ in (19) to obtain the matrix

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ \alpha_1 - E & -\mu - E \end{pmatrix} \tag{20}$$

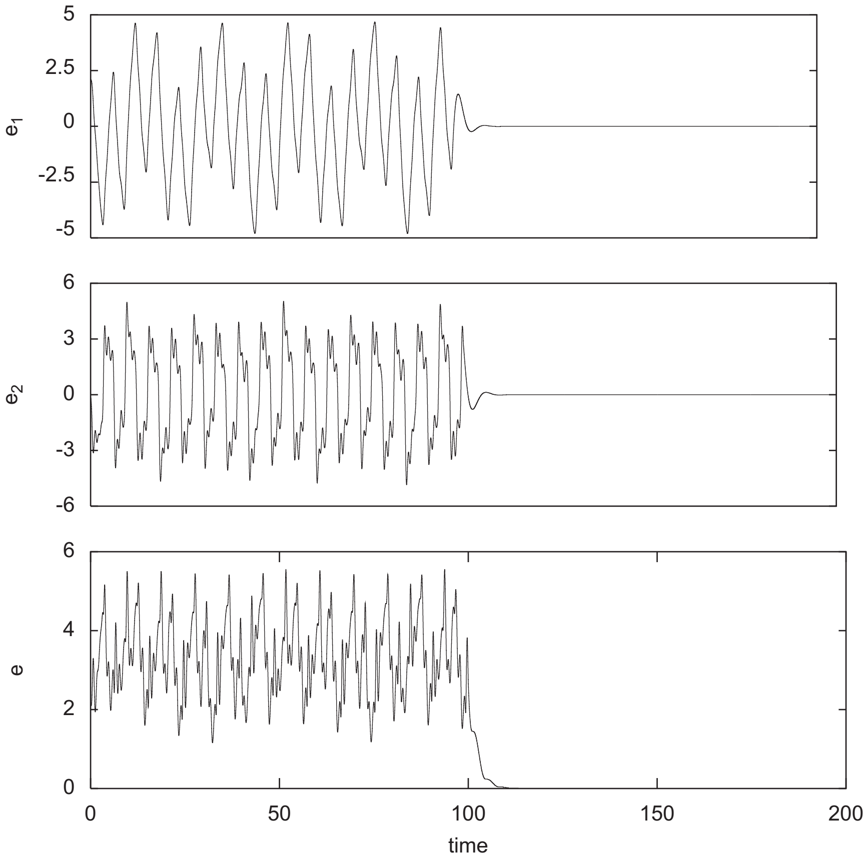

Fig. 4. Error dynamics between two Φ^6 Van der Pol oscillators with the controller deactivated for $0<t<100$ and activated for $t \geq 100$.

which satisfies (18) and gives $u_1(t) = 0$, thereby, leading to a single active control function $u_2(t)$ as follows:

$$u_2(t) = \mu(y_1^2 y_2 - x_1^2 x_2) + \beta_1(y_1^3 - x_1^3) + \delta_1(y_1^5 - x_1^5) + (\alpha_1 - E)e_1 - (\mu + E)e_2 \tag{21}$$

3.2. Numerical simulation

Using the fourth-order Runge–Kutta algorithm with initial conditions $(x_1, x_2) = (0.1, 0.2)$, $(y_1, y_2) = (2.2, 0.05)$, a time step of 0.001, $E = 1$ (note, however, that $E = 3.5$ leads to optimally short synchronization time as explained in Section 4.2), and parameter values as in Fig. 2(a) to ensure chaotic dynamics of the state variables, systems (11) and (12), with the controller as defined in (21), were numerically solved. The results obtained show that the error states oscillate chaotically when the controller is switch off, and when the controller is switched on at $t = 100$, Fig. 4, the error states converge to zero, thereby, guaranteeing synchronization between systems (11) and (12). This is also confirmed by the synchronization quality e defined as

$$e = \sqrt{e_1^2 + e_2^2} \tag{22}$$

4. Synchronization of Φ^6 Duffing oscillators

4.1. Formulation of the active controllers

Eq. (10) can be written as

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\lambda x_2 - \alpha_2 x_1 - \beta_2 x_1^3 - \delta_2 x_1^5 + f_2 \cos \omega_2 t \tag{23}$$

where $x = x_1$ and $\dot{x} = x_2$. Let system (23) be the drive system and system (24) be the response system.

$$\dot{y}_1 = y_2 + u_1(t)$$

$$\dot{y}_2 = -\lambda y_2 - \alpha_2 y_1 - \beta_2 y_1^3 - \delta_2 y_1^5 + f_2 \cos\omega_2 t + u_2(t) \tag{24}$$

where $u_1(t)$ and $u_2(t)$ are control function to be determined. Subtracting (23) from (24) and using the notations $e_1 = y_1 - x_1$ and $e_2 = y_2 - x_2$ we have the error dynamics

$$\dot{e}_1 = e_2 + u_1(t)$$

$$\dot{e}_2 = -\lambda e_2 - \alpha_2 e_1 - \beta_2(y_1^3 - x_1^3) - \delta_2(y_1^5 - x_1^5) + u_2(t) \tag{25}$$

We now re-define the control functions such as to eliminate terms in (25) which cannot be expressed as linear terms in $e_1(t)$ and $e_2(t)$ as follows:

$$u_1(t) = v_1(t)$$

$$u_2(t) = \beta_2(y_1^3 - x_1^3) + \delta_2(y_1^5 - x_1^5) + v_2(t) \tag{26}$$

Substituting (26) into (25) we have

$$\dot{e}_1 = e_2 + v_1(t)$$

$$\dot{e}_2 = -\lambda e_2 - \alpha_2 e_1 + v_2(t) \tag{27}$$

Eq. (27) corresponds to (15) for the Φ^6 Van der Pol oscillator. Following the same procedure from (15) to (17) in Section 3.1 we obtain the coefficient matrix **C** as

$$\mathbf{C} = \begin{pmatrix} a & 1+b \\ c-\alpha_2 & d-\lambda \end{pmatrix} \tag{28}$$

According to the Lyapunov stability theory and the Routh–Hurwitz criteria, if

$$a + d - \lambda < 0$$

$$(c - \alpha_2)(1+b) - a(d-\lambda) < 0 \tag{29}$$

then the eigenvalues of the coefficient matrix of error system (27) must be real negative or complex with negative real parts and, hence, stable synchronized dynamics between systems (23) and (24) is guaranteed. Let

$$a + d - \lambda = -E$$

$$(c - \alpha_2)(1+b) - a(d-\lambda) = -E \tag{30}$$

where $E > 0$ is a real number as earlier explained. From (30) we again choose $a = b = 0$ for the same reason as Section 3.1 to obtain **D** as

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ \alpha_2 - E & \lambda - E \end{pmatrix} \tag{31}$$

which satisfies (29) and gives $u_1(t) = 0$, thereby, leading to a single active control function $u_2(t)$ as follows:

$$u_2(t) = \beta_2(y_1^3 - x_1^3) + \delta_2(y_1^5 - x_1^5) + (\alpha_2 - E)e_1 + (\lambda - E)e_2 \tag{32}$$

4.2. Numerical simulation

Using the fourth-order Runge–Kutta algorithm with initial conditions $(x_1, x_2) = (0, 1.5)$, $(y_1, y_2) = (0.5, 1.0)$, a time step of 0.002, $E = 1$ (note, however, that $E = 4$ leads to optimally short synchronization time as explained below), and parameter values as in Fig. 3(a) to ensure chaotic dynamics of the state variables, system (23) and (24), with the controller as defined in (32), were numerically solved. The results obtained show that the error states oscillate chaotically when the controller is switched off, and when the controller is switched on at $t = 100$, Fig. 5, the error states converge to zero, thereby, guaranteeing synchronization between systems (23) and (24). This is also confirmed by the synchronization quality e defined in (22).

The effect of the parameter E on the synchronization process was investigated by calculating the synchronization time t_s, which is defined as the time taken by the synchronization quality (22) to decrease to 10^{-6} [33], for different values of E. The relation between E and t_s is shown in Fig. 6 for the cases of the Φ^6-VDPO and Φ^6-DO with the controllers of Eqs. (21) and (32). Clearly the synchronization time decreases as E increases to a near constant value. It is also observed from the

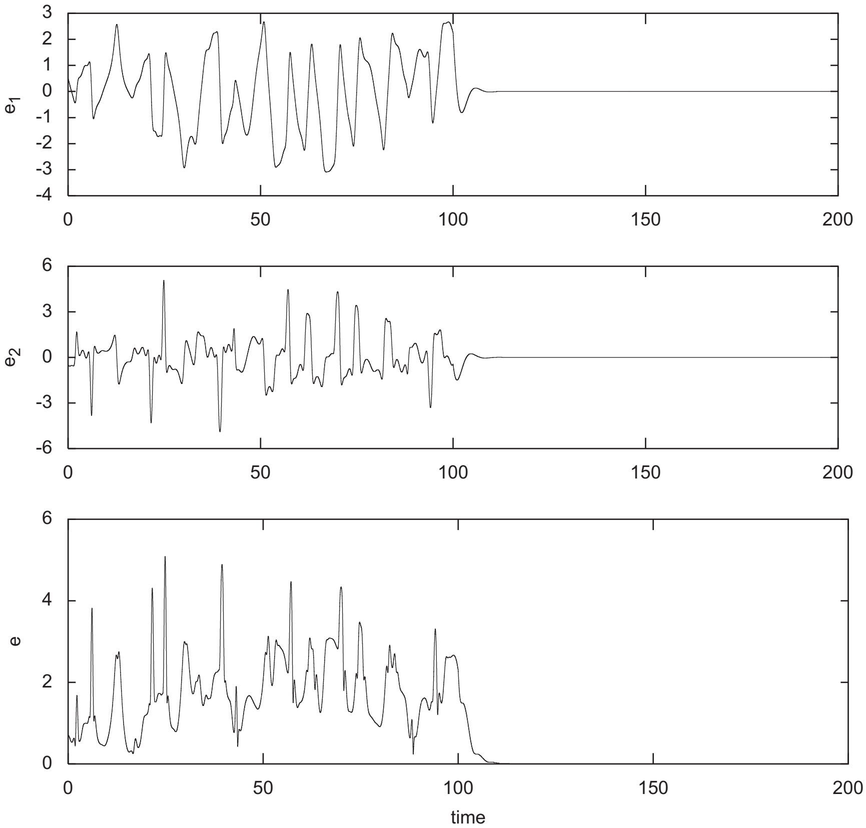

Fig. 5. Error dynamics between two Φ^6 Duffing oscillators with the controller deactivated for $0<t<100$ and activated for $t \geq 100$.

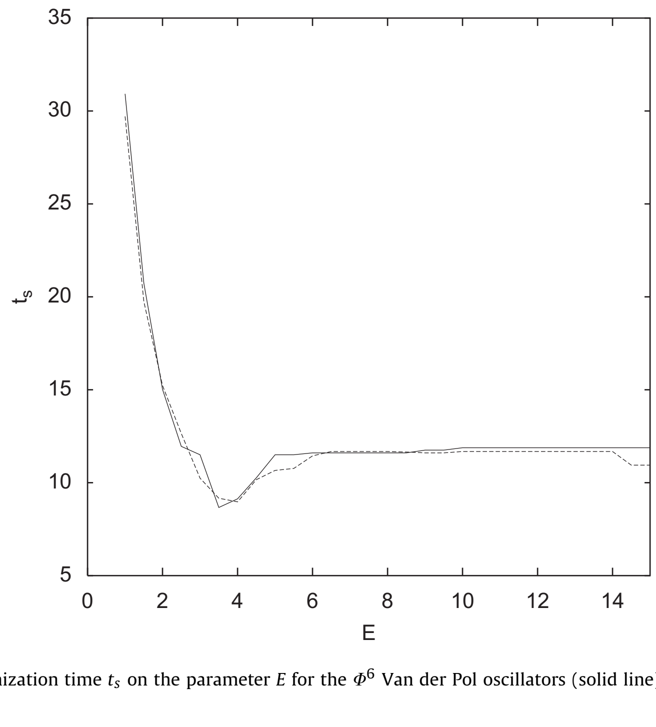

Fig. 6. Dependence of the synchronization time t_s on the parameter E for the Φ^6 Van der Pol oscillators (solid line) and Φ^6 Duffing oscillators (dashed line).

computation that t_s is optimally short for certain values of E, e.g. $E=3.5$ and 4 for the Φ^6-VDPO and Φ^6-DO with the controllers of Eqs. (21) and (32) respectively.

5. Synchronization between Φ^6 Van der Pol and Φ^6 Duffing oscillators

5.1. Formulation of the active controller

Here we choose the Φ^6 Van der Pol oscillator (11) as the drive system and the Φ^6 Duffing oscillator (24) as the response system. To achieve synchronization between these two systems we proceed as in Sections 3.1 and 4.1, that is, we subtract (11) from (24) and apply the relation.

$$y_1 = x_1 + e_1 \quad \text{and} \quad y_2 = x_2 + e_2 \tag{33}$$

to obtain the error dynamics

$$\dot{e}_1 = e_2 + u_1(t)$$

$$\begin{aligned}\dot{e}_2 = & -\lambda e_2 - (\lambda+\mu)x_2 + \mu x_1^2 x_2 + (\alpha_1-\alpha_2)x_1 - \alpha_2 e_1 - \beta_2 y_1^3 - \beta_1 x_1^3 \\ & - \delta_2 y_1^5 + \delta_1 x_1^5 + f_2\cos\omega_2 t - f_1\cos\omega_1 t + u_2(t)\end{aligned} \tag{34}$$

We again re-define the control functions such as to eliminate terms in (34) which cannot be expressed as linear terms in $e_1(t)$ and $e_2(t)$ as follows:

$$u_1(t) = v_1(t)$$

$$u_2(t) = (\lambda+\mu)x_2 - \mu x_1^2 x_2 - (\alpha_1-\alpha_2)x_1 + \beta_2 y_1^3 - \beta_1 x_1^3 + \delta_2 y_1^5 - \delta_1 x_1^5 - f_2\cos\omega_2 t + f_1\cos\omega_1 t + v_2(t) \tag{35}$$

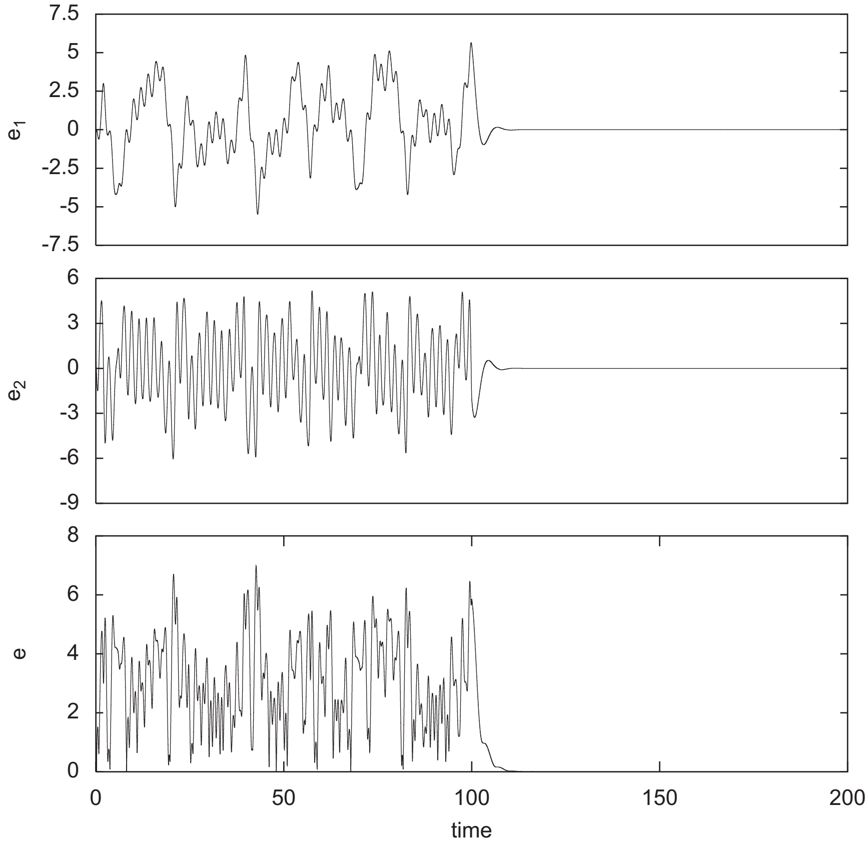

Fig. 7. Error dynamics between Φ^6 Van der Pol and Duffing oscillators with the controller deactivated for $0<t<100$ and activated for $t \geq 100$.

Substituting (35) into (34) we obtain the error dynamics as

$$\dot{e}_1 = e_2 + v_1(t)$$

$$\dot{e}_2 = -\lambda e_2 - \alpha_2 e_1 + v_2(t) \tag{36}$$

Again, following the same procedure from (15) to (17) in Section 3.1 we obtain the coefficient matrix **C** as

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ \alpha_2 - E & \lambda - E \end{pmatrix} \tag{37}$$

which gives $u_1(t) = 0$ and, hence, leads to a single active control function $u_2(t)$ as

$$u_2(t) = (\lambda + \mu)x_2 - \mu x_1^2 x_2 - (\alpha_1 - \alpha_2)x_1 + \beta_2 y_1^3 - \beta_1 x_1^3 + \delta_2 y_1^5 - \delta_1 x_1^5 \\ - f_2 \cos \omega_2 t + f_1 \cos \omega_1 t + (\alpha_2 - E)e_1 + (\lambda - E)e_2 \tag{38}$$

5.2. Numerical simulations

Using the fourth-order Runge–Kutta algorithm with initial conditions $(x_1, x_2) = (0.1, 0.2)$, $(y_1, y_2) = (0,1.5)$, a time step of 0.001, $E = 1$, and parameter values as in Figs. 2(a) and 3(a) to ensure chaotic dynamics of the state variables, systems (11) and (24), with the controller as defined in (38), were numerically solved. The results obtained show that the error states oscillate chaotically when the controller is switched off, and when the controller is switched on at $t = 100$, Fig. 7, the error states converge to zero, thereby, guaranteeing synchronization between systems (11) and (24). Therefore the Φ^6 Duffing oscillator (24) tracks the Φ^6 Van der Pol oscillator (11). This is also confirmed by the synchronization quality e defined in (22). Here again a rough calculation revealed that the relation between the synchronization time and E follows the pattern shown in Fig. 6.

6. Conclusion

In this paper the active control technique based on the Lyapunov stability theory and the Routh–Hurwitz criteria has been used to design control functions for the synchronization of 2-D chaotic systems in such a way that the number of control functions decreased from two to one. As examples of 2-D systems we chose the recently introduced Φ^6 Van der Pol and Φ^6 Duffing oscillators with triple-well potential, which exhibit highly complex dynamics and whose synchronization behaviour has not been well investigated. Single control functions were designed for the purpose of achieving complete synchronization between two identical Φ^6 Van der Pol oscillators, two identical Φ^6 Duffing oscillators, and between two non-identical Φ^6 chaotic oscillators comprising the Van der Pol and Duffing oscillators. The results of the numerical simulations carried out to verify the effectiveness of the designed controllers showed that the single control functions are quite effective in enabling the variables of the response system to completely synchronize with those of the drive system in both the identical and non-identical cases. A parameter E that can be tuned to optimize the synchronization time was also introduced in the design. The single controller design significantly reduced the complexity and hence the cost of the controller, thereby, making it suitable for practical implementation. The synchronization of complex chaotic systems using simple controllers has potential application in secure communications.

References

[1] L.M. Pecora, T.L. Carroll, Synchronization in chaotic systems, *Physical Review Letters* 64 (1990) 821–824.
[2] J. Kurths, S. Boccaletti, C. Grebogi, Y.-C. Lai, Focus issue: control and synchronization in chaotic systems, *Chaos* 13 (2003) 126–127.
[3] M. Zhang, S. Wang, X. Gong, G.W. Wei, C.-H. Lai, Complete synchronization and generalized synchronization of one-way coupled time-delay systems, *Physical Review E* 68 (2003) 036208.
[4] H. Nimeijer, M.Y. Mareels Ivan, An observer looks at synchronization, *IEEE Transactions on Circuit and System I* 44 (1997) 882–890.
[5] J. Yang, C. Li, Generalized projective synchronization of unified chaotic system, *Chaos, Solitons and Fractals* 26 (2005) 1119–1124.
[6] U.E. Vincent, Synchronization of Rikitake chaotic attractor via active control, *Physics Letters A* 343 (2005) 133–138.
[7] E.W. Bai, K.E. Lonngren, Synchronization of two Lorenz systems using active control, *Chaos, Solitons and Fractals* 8 (1997) 51–58.
[8] E.W. Bai, K.E. Lonngren, Sequential synchronization of two Lorenz systems using active control, *Chaos, Solitons and Fractals* 11 (2000) 1041–1044.
[9] A.N. Njah, Synchronization of forced damped pendulum via active control, *Journal of the Nigerian Association of Mathematical Physics* 10 (2006) 143–148.
[10] A. Ucar, K.E. Lonngren, E.W. Bai, Synchronization of the unified chaotic system via active control, *Chaos, Solitons and Fractals* 27 (2006) 1292–1297.
[11] M.-C. Ho, Y.-C. Hung, Synchronization of two different systems by using generalized active control, *Physics Letters A* 301 (2002) 424–428.
[12] M.T. Yassen, Chaotic synchronization between two different chaotic systems using active control, *Chaos, Solitons and Fractals* 23 (2005) 131–140.
[13] A.N. Njah, U.E. Vincent, Chaos synchronization between single and double wells Duffing–Van der Pol oscillators using active control, *Chaos, Solitons and Fractals* 37 (2008) 1356–1361.
[14] B.A. Idowu, U.E. Vincent, A.N. Njah, Synchronization of chaos in non-identical parametrically excited systems, *Chaos, Solitons and Fractals* 39 (2009) 2322–2331.
[15] U.E. Vincent, Synchronization of identical and non-identical 4-D chaotic systems via active control, *Chaos, Solitons and Fractals* 37 (2008) 1065–1075.
[16] H.-K. Chen, Synchronization of two different systems: a new system and each of the dynamical systems Lorenz, Chen and Lü, *Chaos, Solitons and Fractals* 25 (2005) 1049–1056.

[17] A.N. Njah, U.E. Vincent, Synchronization and anti-synchronization of chaos in an extended Bonhöffer-van der Pol oscillator using active control, *Journal of Sound and Vibration* 319 (2009) 41–49.
[18] M.-C. Ho, Y.-C. Hung, C.H. Chou, Phase and anti-phase synchronization of two chaotic systems by using active control, *Physics Letters A* 296 (2002) 43–48.
[19] A.A. Emadzadeh, M. Haeri, Anti-synchronization of two different chaotic systems via active control, *Transactions on Engineering, Computing and Technology (ENFORMATIKA)* 6 (2005) 62–65.
[20] Y. Lei, W. Xu, J. Shen, F. Fang, Global synchronization of two parametrically excited systems using active control, *Chaos, Solitons and Fractals* 28 (2006) 428–436.
[21] Y. Lei, W. Xu, W. Xie, Synchronization of two chaotic four-dimensional systems using active control, *Chaos, Solitons and Fractals* 32 (2007) 1823–1829.
[22] A.N. Njah, Synchronization and anti-synchronization of double-hump Duffing oscillators, *Nigerian Journal of Physics* 20 (2008) 206–211.
[23] G.J. Fodjouong, H.B. Fotsin, P. Waofo, Synchronizing modified Van der Pol–Duffing oscillators with offset terms using observer design, *Physica Scripta* 75 (2007) 638–644.
[24] H. Fotsin, S. Bowong, J. Daafouz, Adaptive synchronization of two chaotic systems consisting of modified Van der Pol–Duffing and Chua oscillators, *Chaos, Solitons and Fractals* 26 (2005) 215–229.
[25] H.-K. Chen, T.-N. Lin, J. -H Chen, The stability of chaos synchronization of the Japanese attractor, *Japan Journal Applied Physics* 42 (2003) 7603–7610.
[26] F.M. Moukam Kakmeni, S. Bowong, C. Tchawoua, Nonlinear adaptive synchronization of a class of chaotic systems, *Physics Letters A* 355 (2006) 47–54.
[27] M. Siewe Siewe, F.M. Moukam Kakmeni, C. Tchawoua, Resonant oscillation and homoclinic bifurcation in Φ^6 Van der Pol oscillator, *Chaos, Solitons and Fractals* 21 (2004) 841–853.
[28] R. Tchoukuegno, B.R. Nana Nbendjo, P. Woafo, Resonant oscillations and fractal basin boundaries of a particle in a Φ^6 potential, *Physica A* 304 (2002) 362–368.
[29] R. Tchoukuegno, B.R. Nana Nbendjo, P. Woafo, Linear feedback and parametric controls of vibrations and chaotic escape in a Φ^6 potential, *International Journal of Non-linear Mechanics* 38 (2003) 531–541.
[30] J. Tang, Z. Chen, Amplitude control of limit circle in Van der Pol system, *International Journal of Bifurcation and Chaos* 16 (2006) 487–495.
[31] Q. Bi, Dynamic analysis of two coupled parametrically excited Van der Pol oscillators, *International Journal of Non-linear Mechanics* 39 (2004) 33–54.
[32] S. Wirkus, R. Rand, The dynamics of two coupled Van der Pol oscillators with delay coupling, *Nonlinear Dynamics* 30 (2002) 205–221.
[33] H. Fotsin, S. Bowong, Chaos, Adaptive control and synchronization of chaotic systems consisting of Van der Pol oscillator coupled to linear oscillator, *Chaos, Solitons and Fractals* 27 (2006) 822–835.
[34] J.C. Ji, C.H. Hansen, Stability and dynamics of a controlled Van der Pol–Duffing oscillator, *Chaos, Solitons and Fractals* 28 (2006) 555–570.
[35] X. Li, J.C. Ji, C.H. Hansen, C. Tan, The response of a Duffing–Van der Pol oscillator under delayed feedback control, *Journal of Sound and Vibrations* 291 (2006) 644–655.
[36] F.M. Moukam Kakmeni, S. Bowong, C. Tchawoua, E. Kaptouom, Strange attractors and chaos control in a Duffing–Van der Pol oscillator, *Journal of Sound and Vibrations* 277 (2004) 783–799.
[37] S. Rajasekar, K. Murali, Resonance behaviour and jump phenomenon in a two coupled Duffing–Van der Pol oscillators, *Chaos, Solitons and Fractals* 19 (2004) 925–934.
[38] J.C. Ji, Nonresonant Hopf bifurcations of a controlled Van der Pol–Duffing oscillator, *Journal of Sound and Vibrations* 297 (2006) 183–199.
[39] H.G. Enjieu Kadji, R. Yamapi, General synchronization dynamics of coupled Van der Pol–Duffing oscillators, *Physica A* 370 (2006) 316–328.
[40] D.S. Goldobin, A. Pikovsky, Synchronization and desynchronization of self-sustained oscillators by common noise, *Physical Review E* 71 (2005) 045201(R).
[41] B.R. Nana Nbendjo, R. Tchoukuegno, P. Woafo, Active control with delay of vibration and chaos in a double-well Duffing oscillator, *Chaos, Solitons and Fractals* 18 (2003) 345–353.
[42] Y. Kim, S.Y. Lee, S.-Y. Kim, Experimental observation of dynamic stabilization in double-well Duffing oscillator, *Physics Letters A* 275 (2000) 254–259.
[43] U. Lepik, H. Hein, On response of nonlinear oscillators with random frequency of excitation, *Journal of Sound and Vibrations* 288 (2005) 275–292.
[44] X. Yang, W. Xu, Z. Sun, Effect of bounded noise on the chaotic motion of a Duffing–Van der Pol oscillator in a Φ^6 potential, *Chaos, Solitons and Fractals* 27 (2006) 778–788.
[45] S. Bowong, F.M. Moukam Kakmeni, J.L. Dimi, R. Koina, Synchronizing chaotic dynamics with uncertainties using a predictable synchronization delay design, *Communication in Nonlinear Science and Numerical Simulation* 11 (2006) 973–987.
[46] M. Siewe Siewe, F.M. Moukam Kakmeni, C. Tchawoua, P. Woafo, Bifurcation and chaos in triple-well Φ^6-Van der Pol oscillator driven by external and parametric excitations, *Physica A* 357 (2005) 383–396.
[47] R. Tchoukuegno, P. Woafo, Dynamics and active control of motion of a particle in a Φ^6 potential with a parametric forcing, *Physica D* 167 (2002) 86–100.