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Parameter reconstruction of vibration systems from partial eigeninformation

Bo Dong [a,1], Matthew M. Lin [b,2], Moody T. Chu [b,*,3]

[a] Department of Applied Mathematics, Dalian University of Technology, Dalian, Liaoning 116024, China
[b] Department of Mathematics, North Carolina State University, Raleigh, NC 27695-8205, USA

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ABSTRACT

Quadratic matrix polynomials are fundamental to vibration analysis. Because of the predetermined interconnectivity among the constituent elements and the mandatory nonnegativity of the physical parameters, most given vibration systems will impose some inherent structure on the coefficients of the corresponding quadratic matrix polynomials. In the inverse problem of reconstructing a vibration system from its observed or desirable dynamical behavior, respecting the intrinsic structure becomes important and challenging both theoretically and practically. The issue of whether a structured inverse eigenvalue problem is solvable is problem dependent and has to be addressed structure by structure. In an earlier work, physical systems that can be modeled under the paradigm of a serially linked mass–spring system have been considered via specifically formulated inequality systems. In this paper, the framework is generalized to arbitrary generally linked systems. In particular, given any configuration of interconnectivity in a mass–spring system, this paper presents a mechanism that systematically and automatically generates a corresponding inequality system. A numerical approach is proposed to determine whether the inverse problem is solvable and, if it is so, computes the coefficient matrices while providing an estimate of the residual error. The most important feature of this approach is that it is problem independent, that is, the approach is general and robust for any kind of physical configuration. The ideas discussed in this paper have been implemented into a software package by which some numerical experiments are reported.

1. Introduction

Vibration is ubiquitous. Many vibrating phenomena are modeled by or can be reduced to a second-order ordinary differential system

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} = \mathbf{f}(t), \tag{1}$$

where $\mathbf{y}(t) \in \mathbb{R}^d$ varies in time t, $\mathbf{M}, \mathbf{C}, \mathbf{K} \in \mathbb{R}^{d\times d}$ are constant matrices, and d stands for the number of degrees of freedom. It is well known that if $\mathbf{y}(t) = \mathbf{v}e^{\lambda t}$ represents a fundamental solution of (1), then the scalar λ and the vector $\mathbf{v}$ must solve the

* Corresponding author.
E-mail addresses: dongbodlut@gmail.com (B. Dong), mlin@ncsu.edu (M.M. Lin), chu@math.ncsu.edu (M.T. Chu).

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quadratic eigenvalue problem (QEP)

$$(\lambda^2\mathbf{M}+\lambda\mathbf{C}+\mathbf{K})\mathbf{v}=0. \tag{2}$$

Given a physical application such as those arising from applied mechanics, circuit analysis, electrical oscillation, vibro-acoustics, or finite element models of some PDEs, often the physical parameters are embedded in the coefficient matrices ($\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$). In a forward setting, the bearing of the dynamical system (1) are to be interpreted via the eigenvalues and eigenvectors of the QEP whose coefficient matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ have already been determined from the specified physical parameters. In contrast, the quadratic inverse eigenvalue problem (QIEP) intends to validate, determine or estimate the parameters of the system according to its observed or expected behavior.

In recent years, there has been considerable research interest in the inverse eigenvalue problems. See, for example, the review article [1], the books by Chu and Golub [2], Friswell and Mottershead [3], Gladwell [4], and the extensive references collected therein. Among the various settings for inverse eigenvalue problems, the QIEPs are perhaps the most important in practice and challenging in theory [5]. Generally speaking, a QIEP can be formulated as follows [6]:

> (QIEP). Construct a nontrivial quadratic pencil $Q(\lambda)=\lambda^2\mathbf{M}+\lambda\mathbf{C}+\mathbf{K}$ so that its matrix coefficients ($\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$) are of a specified structure and $Q(\lambda)$ has a specified set $\{(\lambda_i,\mathbf{v}_i)\}_{i=1}^{k}$ as its eigenpairs.

The key words in the above definition of a QIEP are the phrase "of a specified structure". What makes the QIEP hard are the following factors:

- Often only partial eigeninformation is available. In vibration industries, engineers have in hand techniques to retrieve eigeninformation. For large scale systems, however, not all eigeninformation is retrievable. For high complexity systems, not all eigeninformation retrieved is reliable. In the field of model updating, sometimes certain eigenvectors are required to satisfy some specific conditions [7,8]. In all, only a few measured eigenvectors and eigenvalues are available [3,9–12]. It is more sensible and practically implementable to employ only the available, reliable, yet limited eigenpair information in the reconstruction or updating process.
- Often the coefficient matrices are structured. Depending on the physical applications, the coefficient matrices ($\mathbf{M}$,$\mathbf{C}$,$\mathbf{K}$) often inherit some common structures such as symmetry or positive definiteness. The most challenging structure, nonetheless, comes from the interconnectivity of elements in the original physical configuration. The connectivity constraint mandates certain zero patterns or algebraic relationships among the entries of ($\mathbf{M}$,$\mathbf{C}$,$\mathbf{K}$). What is even more compounding is that these patterns and relationships vary from problem to problem. For feasibility, it is necessary that the QIEP takes this connectivity into account. This is the first main thrust of this paper.
- Often the parameters to be recovered must be nonnegative. The physical parameters, such as mass, stiffness, voltage, resistance and so on, are embedded in the coefficient matrices ($\mathbf{M}$,$\mathbf{C}$,$\mathbf{K}$) in a fixed but often mixed way. Merely solving the QIEP subject to the structural constraints is not enough. We must insist that the recovered parameters remain nonnegative for physical realization. This is the second main thrust of this paper.

Research results advanced thus far for the QIEPs cannot address all the issues mentioned above. A partial list of existent works on this subject and their limitations are briefed below. In [13], Ram and Elhay studied the QIEP with symmetric tridiagonal coefficient matrices and two sets of eigenvalues. The structure involved in the QIEP studied by Starek and Inman [14] is for nonproportional underdamped systems. Lancaster and Prells [15] proposed a way to construct symmetric coefficient matrices ($\mathbf{M}$,$\mathbf{C}$,$\mathbf{K}$) subject to the constraint that $\mathbf{C}$ is positive semi-definite, but they required that the complete set of eigenvalues and eigenvectors be given and that the eigenvalues be all simple and nonreal. Later, Lancaster [16] considered the QIEP where only the spectrum is given and semisimple, in which case special eigenvectors are found first to construct symmetric coefficient matrices ($\mathbf{M}$,$\mathbf{C}$,$\mathbf{K}$). In [5], Chu et al. put forward a parametric characterization for the QIEP with partially prescribed eigenvectors where the reconstructed $\mathbf{M}$ is guaranteed to be positive definite, $\mathbf{C}$ symmetric and $\mathbf{K}$ positive semi-definite. The result was further generalized in [17]. More recently, a representation of symmetric ($\mathbf{M}$,$\mathbf{C}$,$\mathbf{K}$) in terms of eigenvalues and eigenvectors is discussed in [18], which is real-valued version of the well known GLR theory [19] and provides a theoretical basis for the QIEP. See also [20–22] for the related work on applying the notions of feedback control to reassign the eigenstructure.

We must stress that in all the above-mentioned articles and in most others in the literature, very few have paid attention to the constraints of either connectivity or nonnegativity. Such a negligence in the literature certainly is not due to its lack of interest, but rather because of the difficulties associated with these constraints. Since the underlying structure is application dependent, it seems that we can tackle the structured QIEP only problem by problem.

Only recently, Chu et al. [6] considered for the first time the special structure of a linearly linked system where, given at most two eigenpairs, the structured QIEP was solved numerically for symmetric and tridiagonal coefficient matrices that respect both the connectivity and the nonnegativity. Still, it appears very difficult to generalize the mathematics exerted in deriving the special inequality systems in [6] for numerical computation to other types of physical configurations. Our main contribution in this regard thus is to develop a software package that, based on a user's description of the physical configuration and available eigenpair information, automates the generation of a linear inequality system which is necessary for respecting both connectivity and nonnegativity. Our numerical approach greatly advances our ability to solve

more complicated problems than those in [6] efficiently in practice. Since the measured eigeninformation is not always precise, we allow some leeway in the accuracy. Our algorithm can check the consistency of the inequality system within a user-supplied tolerance. If the inequality system is solvable within the tolerance, we compute a solution to the structured QIEP and return the residual error for the user's decision of accepting or rejecting the constructed model.

Because the structure of the QIEP varies according to applications, even merely formulating the problem is a laborious task itself. Having in hand a tool that alleviates the burden of problem dependency and offers a universal approach to arbitrary physical systems should be highly desirable for practitioners. We believe our contribution is innovative and should be of significance to the field. Our package is still in beta test, but since our approach appears to have a broad range of applications, it might be appropriate to communicate two goals via this presentation: one is to explain our ideas of developing a general-purpose problem-independent formulation for the structured QIEP, and the other is to provide the mathematical basis of our error control strategy.

This paper is organized as follows. In Section 2 we outline a general process converting a given configuration of connectivity into a linear equality system. The formation depends on the physical law that governs how the elements interact. We focus on the mass–spring system governed by Hooke's law for the sake of exposition, but our setting is general enough to include many other physical systems. The point to make is that we are able to handle models with arbitrary generally linked connectivity, effectively generalizing the notion proposed in [6]. In Section 3 we derive a linear inequality system which, arising from rank deficiency and varying from structure to structure, is a necessary condition for a nonnegative solution to a QIEP with prescribed connectivity. By this inequality system, we transform a structured QIEP into a maximin problem for the sake of efficiency in numerical computation. If the prescribed eigeninformation is inexact, which often is the case in practice, the expected rank deficiency might not occur and the corresponding inequality system would have only a trivial solution. To remedy this, we propose in Section 4 a strategy of truncation that generates an approximate but consistent subsystem. We offer a posterior estimate on the resulting residual error which could be used to assess the quality of the reconstructed model. All of these ideas can be implemented into an algorithm which we briefly outline in Section 5. Finally, equipped with our software, we are able to explore some interesting perturbation results numerically in Section 6.

2. Linear equality system

Since we are interested only in real matrices, we may assume without loss of generality that the prescribed eigenpairs $\{(\lambda_i, \mathbf{v}_i)\}_{i=1}^{k}$ are closed under complex-conjugation. Denote the prescribed eigenpairs in the matrix form $(\mathbf{\Lambda}, \mathbf{X})$, where $\mathbf{\Lambda} \in \mathbb{R}^{k\times k}$ is block diagonal with at most 2×2 blocks along the diagonal wherever a complex-conjugate pair of eigenvalues appears in the prescribed spectrum, and $\mathbf{X} \in \mathbb{R}^{d\times k}$ represents the "eigenvector matrix" in the sense that each pair of column vectors associated with a 2×2 block in $\mathbf{\Lambda}$ retains the real and the imaginary parts, respectively, of the original complex eigenvector. In this way, we may identify the given eigenpairs by the pair of matrices $(\mathbf{\Lambda}, \mathbf{X}) \in \mathbb{R}^{k\times k} \times \mathbb{R}^{d\times k}$. For our parameter reconstruction, it is instructive to first outline the basic idea of converting a QIEP to a linear equality system which is fundamental to our algorithm.

At first glance, the QIEP amounts to solving the algebraic equation

$$\mathbf{M}\mathbf{X}\mathbf{\Lambda}^2 + \mathbf{C}\mathbf{X}\mathbf{\Lambda} + \mathbf{K}\mathbf{X} = 0, \tag{3}$$

which is linear in the unknowns $(\mathbf{M},\mathbf{C},\mathbf{K})$. But there are two extra constraints to be imposed. Firstly, the matrices $(\mathbf{M},\mathbf{C},\mathbf{K})$ depend linearly on some nonnegative physical parameters $\mathbf{x}$. Secondly, the dependence on $\mathbf{x}$ is characterized by the inter connectivity among elements in the system which, appearing in specific zero patterns or algebraic relationships among the parameters, defines a much more refined texture of the coefficient matrices $(\mathbf{M},\mathbf{C},\mathbf{K})$ than the general requirements such as merely symmetry and positive semi-definiteness. The following two examples should evince our point.

Example 1. Consider a four-degrees-of-freedom vibration system whose masses, dampers and springs are connected as in Fig. 1 with marked system parameters $\mathbf{m}$, $\mathbf{c}$ and $\mathbf{k}$, respectively. Assuming that the restoring force follows Hooke's law and that the damping is negatively proportional to the velocity, the corresponding coefficient matrices $(\mathbf{M},\mathbf{C},\mathbf{K})$ for the dynamical system (1) should be structured as follows:

$$\mathbf{M} = \begin{pmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} c_1+c_2 & 0 & -c_2 & 0 \\ 0 & 0 & 0 & 0 \\ -c_2 & 0 & c_2+c_3 & -c_3 \\ 0 & 0 & -c_3 & c_3 \end{pmatrix},$$

$$\mathbf{K} = \begin{pmatrix} k_1+k_2+k_5 & -k_2 & -k_5 & 0 \\ -k_2 & k_2+k_3 & -k_3 & 0 \\ -k_5 & -k_3 & k_3+k_4+k_5 & -k_4 \\ 0 & 0 & -k_4 & k_4 \end{pmatrix}. \tag{4}$$

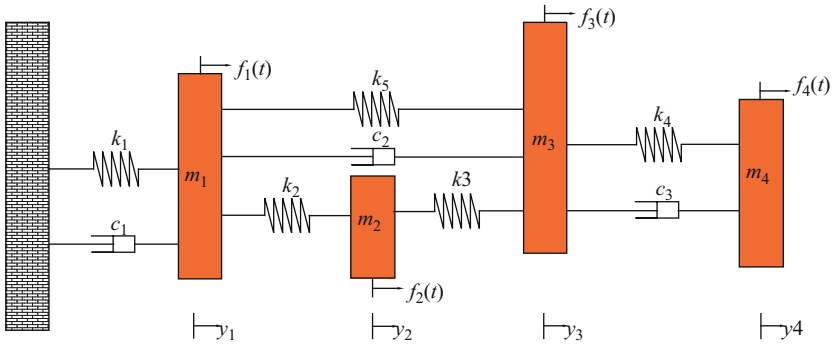

Fig. 1. A four-degrees-of-freedom mass–spring system.

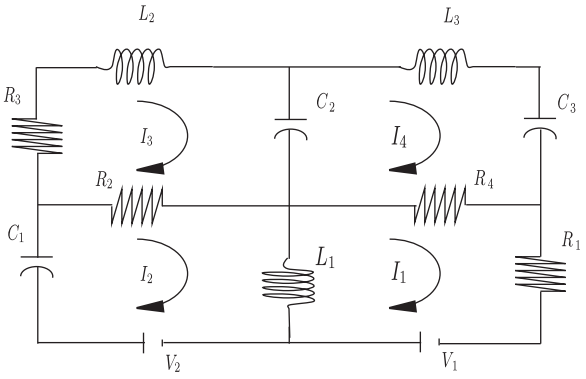

Fig. 2. An RLC electronic network.

It is readily provable that these matrices are symmetric and positive semi-definite. However, be aware of the zero patterns and implicit algebraic relationships among rows in these matrices. Apparently, to reconstruct the specific mass–spring system in Fig. 1, merely solving symmetric and positive definite matrices (**M**,**C**,**K**) is not enough. The (**M**,**C**,**K**) must respect the structure in (4) and the reconstructed parameters (**m**,**c**,**k**) must be nonnegative.

Example 2. Consider a resonant circuit consisting of three inductors, four resistors and three capacitors which are connected in the way depicted in Fig. 2. After applying Ohm's law and Kirchhoff's law to the circuit, we obtain the coefficient matrices as in (5) for the governing equation of the current in the circuit. In contrast to the previous structure in (4), the structure associated with a general electric circuit may not be definite or even symmetric.

$$\mathbf{M}=\begin{pmatrix} -L_1 & L_1 & 0 & 0\\ L_1 & -L_1 & 0 & 0\\ 0 & 0 & L_2 & 0\\ 0 & 0 & 0 & L_3 \end{pmatrix},\quad \mathbf{C}=\begin{pmatrix} 0 & R_2 & -R_2 & 0\\ R_1+R_4 & 0 & 0 & -R_4\\ 0 & -R_2 & R_2+R_3 & 0\\ -R_4 & 0 & 0 & R_4 \end{pmatrix},\quad \mathbf{K}=\begin{pmatrix} 0 & \frac{1}{C_1} & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & \frac{1}{C_2} & -\frac{1}{C_2}\\ 0 & 0 & -\frac{1}{C_2} & \frac{1}{C_2}+\frac{1}{C_3} \end{pmatrix}. \tag{5}$$

Obviously, if the elements are connected differently or if the governing physical law is changed, then the corresponding structure in (**M**,**C**,**K**) needs to be changed accordingly. This makes a general approach for QIEPs much harder to come by, and that is precisely where our contribution in this paper becomes useful.

Let $\mathbf{x}:=(x_1,\ldots,x_n)^{\top}$ denote the vector of n parameters. This could be, for example, the parameters of mass **m**, damping **c** and stiffness **k** in a mass–spring system. By linearity, we can rewrite (3) as a homogeneous system

$$\mathbf{A}\mathbf{x}=0, \tag{6}$$

where **A** is a fixed $dk\times n$ matrix. Our goal is to find a nonnegative solution for the linear equality system (6). The mathematics thus far does not seem difficult, but there remain two computational challenges—how to generate **A** automatically from arbitrarily prescribed connectivity configuration and how to deal with inexact eigeninformation.

In principle, the system (3) can be written in the form (6) via the notion of Kronecker product, but that process would not have exploited the sparsity pattern inherent in the connectivity effectively. Instead, we propose a mathematically equivalent mechanism that builds the matrix **A** in two stages. It is not possible to cover comprehensively the entire scope of applications, but the discussion in the sequel can be modified if the underlying system is changed.

2.1. Identify inherent structure

Our goal is to develop a scheme that can automatically identify the structure of the coefficient matrices (**M**,**C**,**K**) according to the underlying physical principles, e.g., Hooke's law, Ohm's law, Kirchhoff's laws, and so on. Following [6], we shall focus on the vibration of a mass–spring system only. Indeed, through appropriate interpretation, many physical phenomena can be modeled via a mass–spring system.

Assuming that the restoring force follows Hooke's law and that the damping is negatively proportional to the velocity, we demonstrate the construction of the coefficient matrices (**M**,**C**,**K**) which then will be converted to the matrix **A**. Keep in mind that a different configuration of the connectivity leads to a different matrix structure. We want a general-purpose algorithm that can generate the structure for all cases. Toward that end, we exploit the following rules which are well developed in the field of structural mechanics [23].

Theorem 2.1. *For a mass–spring system with d degrees of freedom where the motion is limited to one dimension, let* **m**, **c**, *and* **k** *denote the vectors of masses, damping and stiffness coefficients, respectively. The matrices* (**M**,**C**,**K**) *inherit the following structure*:

1. *The mass matrix* **M** *is a diagonal matrix with masses* $(m_1,\dots,m_d)$ *along its diagonal.*
2. *The damping matrix* **C** *is symmetric and positive semi-definite. If there are* ℓ *dampers, identified by* $i_1,\dots,i_\ell$, *between the p-th mass and the q-th mass, then the entries* $\mathbf{C}_{pq}$ *and* $\mathbf{C}_{qp}$ *of the damping matrix are given by* $-\sum_{s=1}^{\ell} c_{i_s}$, *where* c_{i_s} *is the damping coefficient of damper* i_s. *Otherwise,* $\mathbf{C}_{pq}=\mathbf{C}_{qp}=0$. *If the p-th mass is connected to the dampers* $j_1,\dots,j_\ell$, *then* $\mathbf{C}_{pp}=\sum_{s=1}^{\ell} c_{j_s}$.
3. *The stiffness matrix* **K** *is symmetric and positive semi-definite. If there are* ℓ *springs, identified by* $i_1,\dots,i_\ell$, *between the p-th mass and the q-th mass, then the entries* $\mathbf{K}_{pq}$ *and* $\mathbf{K}_{qp}$ *of the stiffness matrix are given by* $-\sum_{s=1}^{\ell} k_{i_s}$, *where* k_{i_s} *is the stiffness coefficient of spring* i_s. *Otherwise* $\mathbf{K}_{pq}=\mathbf{K}_{qp}=0$. *If the p-th mass is connected to the springs* $j_1,\dots,j_\ell$, *then* $\mathbf{K}_{pp}=\sum_{s=1}^{\ell} k_{j_s}$.

Readers might find it informative to compare these rules with the matrices (**M**,**C**,**K**) constructed earlier in Example 1. The point to make is that rules such as those in Theorem 2.1 are fully implementable into computer codes. We thus establish a library containing distinct modules each of which collects the different rules of constructing (**M**,**C**,**K**) such as those in Theorem 2.1 according to the different physical laws. Such a task will not be endeavored in this paper, but new rule sets are being gradually added to our software package module by module. Once the user classifies which physical law is to be used in the model, the library is able to automatically identify the corresponding structure for (**M**,**C**,**K**) based on the specified connectivity configuration. In our software we also allow users to enter their structure manually into our software package through a text interface as the lowest level of input mechanism. Details are available in our package.

2.2. Construct matrix **A**

Once the inherent structure of (**M**,**C**,**K**) is known, it remains to rewrite the linear equation (3) in the form $\mathbf{Ax}=0$. This sounds mathematically trivial since we only need to collect like terms together. Our point, again, is to avoid the tedious task of bookkeeping case by case, especially when the involvement is complicated. We prefer to automated the generation of **A** from any specified structure of (**M**,**C**,**K**) and eigenpairs $(\mathbf{\Lambda},\mathbf{X})$. This task is software realizable.

Partition $\mathbf{Ax}=0$ as

$$(\mathbf{A_m},\mathbf{A_c},\mathbf{A_k})\begin{pmatrix}\mathbf{m}\\ \mathbf{c}\\ \mathbf{k}\end{pmatrix}=0.$$

It suffices to explain how the first term $\mathbf{MX\Lambda}^2$ in (3) can be converted into the form $\mathbf{A_m m}$ from a given structure of **M**. The matrix $\mathbf{A_m}$ is constructed in the order that the block of its first k rows is equivalent to the first row of **M** times $\mathbf{X\Lambda}^2$, the next block of k rows to the second row of **M** times $\mathbf{X\Lambda}^2$, and so on. The size of $\mathbf{A_m}$ should be $dk\times\dim(\mathbf{m})$. The other two matrices $\mathbf{A_c}$ and $\mathbf{A_k}$ can be constructed in a similar way with the respective structure of **C** and **K** in mind.

The matrix $\mathbf{A_m}$ is built row by row. It is illustrative to use Example 2 to demonstrate our idea of construction. Denote columns of $\mathbf{X\Lambda}^2$ by $\mathbf{z}_1,\dots,\mathbf{z}_k$, each of which is a vector in $\mathbb{R}^4$. After identifying $L_i=m_i$ for $i=1,2,3$, each entry in the rows of **M** is coded as a triplet $\langle\alpha_1,\alpha_2,\alpha_3\rangle$ which stands for the coefficients in the expression $\alpha_1m_1+\alpha_2m_2+\alpha_3m_3$. The first row $[-m_1,m_1,0,0]$ of **M**, for example, is converted to $[\langle-1,0,0\rangle,\langle1,0,0\rangle,\langle0,0,0\rangle,\langle0,0,0\rangle]$. The block of the first k rows of $\mathbf{A_m}$ is given by the product

$$\begin{pmatrix}\mathbf{z}_1^\top\\ \vdots\\ \mathbf{z}_k^\top\end{pmatrix}\begin{pmatrix}-1&0&0\\ 1&0&0\\ 0&0&0\\ 0&0&0\end{pmatrix}=\left(\underbrace{\begin{pmatrix}-1&1&0&0\\ 0&0&0&0\\ 0&0&0&0\end{pmatrix}\mathbf{X\Lambda}^2}_{\mathfrak{M}_1}\right)^\top.$$

Continuing this process, the second row of $\mathbf{M}$ is converted to an operator matrix $\mathfrak{M}_2$ of size 3×4 and the next k rows of $\mathbf{A_m}$ is generated by $(\mathfrak{M}_2\mathbf{X}\mathbf{\Lambda}^2)^\top$.

In general, given any linearly structured matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ in terms of the physical parameters ($\mathbf{m}$, $\mathbf{c}$, $\mathbf{k}$), each row gives rise to corresponding operator matrices $\mathfrak{M}_i$, $\mathfrak{C}_i$ and $\mathfrak{K}_i$ of sizes $\dim(\mathbf{m}) \times d$, $\dim(\mathbf{c}) \times d$ and $\dim(\mathbf{k}) \times d$, respectively. The matrix $\mathbf{A}$ then is given by

$$\mathbf{A} = \begin{pmatrix} (\mathfrak{M}_1\mathbf{X}\mathbf{\Lambda}^2)^\top & (\mathfrak{C}_1\mathbf{X}\mathbf{\Lambda})^\top & (\mathfrak{K}_1\mathbf{X})^\top \\ & \vdots & \\ (\mathfrak{M}_d\mathbf{X}\mathbf{\Lambda}^2)^\top & (\mathfrak{C}_d\mathbf{X}\mathbf{\Lambda})^\top & (\mathfrak{K}_d\mathbf{X})^\top \end{pmatrix}. \tag{7}$$

It should be pointed out that the above expression is for explanatory purpose only. The operator matrices need not be generated explicitly. All we need are a few indices pointing to the locations where the connectivity is active. Consequently, the matrix multiplications involved in (7) are not as expensive as the expression seems to have suggested. The software we are developing has two input modes—either it picks out these indices from reading the text interface or, better yet, it generates the indices directly from the connectivity configuration and the underlying physical law.

To demonstrate the last point of input mode, we summarize below the rules by which the operator matrices $\mathfrak{M}_i, \mathfrak{C}_i, \mathfrak{K}_i$ for a mass–spring system can be generated directly from the connectivity configuration of the system without making any reference to ($\mathbf{M}$,$\mathbf{C}$,$\mathbf{K}$).

Theorem 2.2. *Given a mass–spring system with d degrees of freedom where the motion is limited to one dimension, let* $\dim(\mathbf{m})$, $\dim(\mathbf{c})$ *and* $\dim(\mathbf{k})$ *denote the number of masses, dampers and springs, respectively. Then for* $1 \le p, q \le d$, *the operator matrices are given as follows*:

1. *Each* $\mathfrak{M}_p$ *is of size* $\dim(\mathbf{m}) \times d$. *Its entries are all zero except* 1 *at the* (p,p) *position.*
2. *Each* $\mathfrak{C}_p$ *is of size* $\dim(\mathbf{c}) \times d$. *Its entries are all zero with the following exceptions*:
   - *If there are* ℓ *dampers, identified by* $i_1, \ldots, i_\ell$, *linking the p-th mass to the q-th mass, then the q-th column of the matrix* $\mathfrak{C}_p$ *has* -1 *at its* $i_1, \ldots, i_\ell$ *entries. Likewise, the p-th column of* $\mathfrak{C}_q$ *has* -1 *at its* $i_1, \ldots, i_\ell$ *entries.*
   - *If the p-th mass is connected to the dampers* $j_1, \ldots, j_\ell$, *then the p-th column of* $\mathfrak{C}_p$ *has* 1 *at its* $j_1, \ldots, j_\ell$ *entries.*
3. *Each* $\mathfrak{K}_p$ *is of size* $\dim(\mathbf{k}) \times d$. *Its entries are all zero with the following exceptions*:
   - *If there are* ℓ *springs, identified by* $i_1, \ldots, i_\ell$, *linking the p-th mass to the q-th mass, then the q-th column of* $\mathfrak{K}_p$ *has* -1 *at its* $i_1, \ldots, i_\ell$ *entries. Likewise, the p-th column of* $\mathfrak{K}_q$ *has* -1 *at its* $i_1, \ldots, i_\ell$ *entries.*
   - *If the p-th mass is connected to the springs* $j_1, \ldots, j_\ell$, *then the p-th column of* $\mathfrak{K}_p$ *has* 1 *at its* $j_1, \ldots, j_\ell$ *entries.*

3. Linear inequality system

Suppose that the prescribed eigenpairs $(\mathbf{\Lambda}, \mathbf{X})$ are exact for some quadratic matrix polynomials. Then the system (6) has a nontrivial solution, implying that the matrix $\mathbf{A}$ must be rank deficient. Let s denote the rank of the matrix $\mathbf{A}$. Write the compact *QR* decomposition of the matrix $\mathbf{A}$ as

$$\mathbf{A} = \mathbf{QRP}, \tag{8}$$

where $\mathbf{Q} \in \mathbb{R}^{dk \times s}$ satisfies $\mathbf{Q}^\top\mathbf{Q} = \mathbf{I}_s$ and $\mathbf{P} \in \mathbb{R}^{n \times n}$ is a permutation matrix such that

$$\mathbf{R} = [\mathbf{S}, \mathbf{T}] \in \mathbb{R}^{s \times n} \tag{9}$$

is upper "trapezoidal" with $\mathbf{S} \in \mathbb{R}^{s \times s}$ being upper triangular, nonsingular, and having non-increasing absolute values along its diagonal. Partition

$$\overline{\mathbf{x}} := \mathbf{Px} = (\overline{\mathbf{x}}_1^\top, \overline{\mathbf{x}}_2^\top)^\top$$

with $\overline{\mathbf{x}}_1 \in \mathbb{R}^s$. Trivially we see the following equivalence:

$$\mathbf{Ax} = 0 \iff \overline{\mathbf{x}}_1 = -\mathbf{S}^{-1}\mathbf{T}\overline{\mathbf{x}}_2. \tag{10}$$

We have thus recast the structured QIEP as an inequality system.

Theorem 3.1. *Given eigenpairs* $(\mathbf{\Lambda}, \mathbf{X})$ *and a fixed connectivity configuration, define matrices* $\mathbf{S}$ *and* $\mathbf{T}$ *for the matrix* $\mathbf{A}$ *as above. Then finding the nonnegative parameters* $\mathbf{x} \in \mathbb{R}^n$ *for the structured QIEP is equivalent to solving the inequality system*

$$\overline{\mathbf{x}}_2 \ge 0, \quad \mathbf{S}^{-1}\mathbf{T}\overline{\mathbf{x}}_2 \le 0 \tag{11}$$

for $\overline{\mathbf{x}}_2 \in \mathbb{R}^{n-s}$.

Observe that any solution to (11), if exists, can be scaled by positive constants. It suffices to restrict entries of $\overline{\mathbf{x}}_2$ to the interval $[0, 1]$ whereas we should ignore the trivial solution $\overline{\mathbf{x}}_2 = 0$. Solving inequality system of convex functions is a

classical problem. Extensive discussions can be found in the literature, including the well known Farkas lemma [24], the von Neumann theorem [25] and the Ky Fan theorem [26]. For our application, we find that the maximin formulation is particularly easy and effective to use. That is, we solve the inequality system (11) by considering the maximin problem:

$$\max_{0\le\bar{\mathbf{x}}_2\le 1} \min(-\mathbf{S}^{-1}\mathbf{T}\bar{\mathbf{x}}_2), \tag{12}$$

where the min function is taken over all entries of the vector $-\mathbf{S}^{-1}\mathbf{T}\bar{\mathbf{x}}_2$. As the objective function $\min(-\mathbf{S}^{-1}\mathbf{T}\bar{\mathbf{x}}_2)$ is concave over a convex feasible set, we generally expect that the solution to (12) is unique and global. If the maximum objective value turns out to be negative, then the inequality system (11) has no nonnegative solution and the parameter reconstruction for the vibration system fails. In other words, the solvability is determined by a nonnegative objective value in (12).

It has to be stressed that the rank deficiency of the matrix $\mathbf{A}$ is due to the (unrealistic) assumption that the prescribed eigenpairs are exact for some (unknown) quadratic matrix polynomials. Without the rank deficiency, the inequality system might not be even formulated, and that is precisely the trouble since the prescribed eigeninformation in practice is most likely inexact. We propose a way to overcome this difficulty in the next section.

4. Handling inexact eigenpairs

Current vibration testing devices are not perfect. The prescribed eigenpairs $(\mathbf{\Lambda},\mathbf{X})$ can easily be inexact, causing the corresponding matrix $\mathbf{A}$ in (6) to be of full rank or have an obscured numerical rank. Without preprocessing $\mathbf{A}$, the solutions to $\mathbf{Ax}=0$ would be either zero or far off the target. For engineering applications, quite often an approximate solution that is physically feasible, such as satisfying the nonnegative constraint, is acceptable. Our goal in this section is to find approximate nonnegative solutions to the system $\mathbf{Ax}=0$, when $\mathbf{A}$ is inexact, and estimate the error.

Perhaps the most straightforward formulation for finding an approximate solution to (6) is to consider the quadratic programming (QP) problem:

$$\begin{aligned} &\text{minimize} && \|\mathbf{Ax}\|_2^2 \\ &\text{subject to} && \mathbf{x}\ge 1. \end{aligned} \tag{13}$$

This is a convex programming problem. To avoid the infinitesimal scaling, we require $\mathbf{x}\ge 1$ instead of $\mathbf{x}\ge 0$ as the constraint. The presence of any zero physical parameter will cause the scaling to fail. This case, fortunately, is not generic and should have been ruled out in the first place. On the other hand, a substantially large scattering among elements of $\mathbf{x}$ might be an indication that the smallest element is numerically zero. By keeping the computed parameters $\mathbf{x}^*$ positive, the rationale in forming (13) is that the return of a small residual $\|\mathbf{Ax}^*\|_2$ might have heuristically and approximately reconstructed the system.

Solving (13) involves the full length n of $\mathbf{x}$, whereas (12) involves only $n-s$ variables. Inspired by the discussion in Section 2, we propose an adaptive method to approximately solve the system $\mathbf{Ax}=0$ whereas we also offer a posterior estimate for the residual of the constructed model which can be used to accept or reject the constructed model. To begin with, we perform the full *QR* decomposition with column pivoting of the matrix $\mathbf{A}$ to obtain the following relationship:

$$\mathbf{Ax}=0 \iff \mathbf{R}\bar{\mathbf{x}}=0,$$

where $\mathbf{R}\in\mathbb{R}^{dk\times n}$ is an upper trapezoidal matrix whose diagonal elements are arranged in accordance with non-increasing absolute values by the permutation matrix $\mathbf{P}$ and $\bar{\mathbf{x}}:=\mathbf{Px}$. Due to inexact eigeninformation, the matrix $\mathbf{A}$ generally is of full rank and thus $\mathbf{x}=0$ is the only solution, which is of little value. To obtain some meaningful nontrivial solutions, we must replace $\mathbf{R}$ by some low rank approximations. There are two ways to determine the low rank.

Let ε be a predetermined threshold. Define $\mathbf{R}_\varepsilon$ to be the submatrix of $\mathbf{R}$ consisting of rows of $\mathbf{R}$ whose diagonal elements are greater than ε. Suppose $\mathbf{R}_\varepsilon$ is made of t rows. We consider a nonnegative solution to $\mathbf{R}_\varepsilon\bar{\mathbf{x}}=0$, if it exists, as an approximate solution to $\mathbf{R}\bar{\mathbf{x}}=0$. The choice of ε is often made on a trial and error basis. The value of ε should be large enough to bring forth substantial truncation so that $t<n$, otherwise, $\mathbf{R}_\varepsilon\bar{\mathbf{x}}=0$ still has only trivial solution, but ε also should be small enough to avoid losing too much information about $\mathbf{A}$. Assuming $t<n$, write $\mathbf{R}_\varepsilon=[\mathbf{S}_\varepsilon,\mathbf{T}_\varepsilon]$, where $\mathbf{S}_\varepsilon\in\mathbb{R}^{t\times t}$ is invertible, and partition $\bar{\mathbf{x}}=(\bar{\mathbf{x}}_1^\top,\bar{\mathbf{x}}_2^\top)^\top$ accordingly. From the equivalence

$$\mathbf{R}_\varepsilon\bar{\mathbf{x}}=0 \iff \bar{\mathbf{x}}_1=-\mathbf{S}_\varepsilon^{-1}\mathbf{T}_\varepsilon\bar{\mathbf{x}}_2,$$

we intend to solve the inequality system

$$\bar{\mathbf{x}}_2\ge 0,\quad -\mathbf{S}_\varepsilon^{-1}\mathbf{T}_\varepsilon\bar{\mathbf{x}}_2\le 0 \tag{14}$$

for some $\bar{\mathbf{x}}_2\in\mathbb{R}^{n-t}$. From this point forward, we can resort to the maximin ideas outlined in Section 3 to solve (14).

Suppose that $\bar{\mathbf{x}}_\varepsilon^*$ is a nonnegative solution to the truncated linear system $\mathbf{R}_\varepsilon\bar{\mathbf{x}}=0$. The vector $\mathbf{x}_\varepsilon^*:=\mathbf{P}^\top\bar{\mathbf{x}}_\varepsilon^*$ is regarded as an approximate nonnegative solution to the original system $\mathbf{Ax}=0$. The following theorem justifies the notion of this truncation.

Theorem 4.1. *Given a matrix* $\mathbf{A} \in \mathbb{R}^{dk\times n}$, *let* $\mathbf{A} = \mathbf{QRP}$ *denote its full QR decomposition, where* $\mathbf{R}$ *is an upper trapezoidal matrix with its diagonal elements arranged in descending absolute value by the permutation matrix* $\mathbf{P}$. *Given a positive number* ε, *let* $\mathbf{R}_\varepsilon$ *denote the truncated submatrix of* $\mathbf{R}$ *consisting of rows of* $\mathbf{R}$ *whose diagonal elements are greater than* ε. *Suppose* $\overline{\mathbf{x}}_\varepsilon$ *is a solution to the system* $\mathbf{R}_\varepsilon\overline{\mathbf{x}} = 0$ (*for our application, we are interested in a nonnegative solution* $\overline{\mathbf{x}}_\varepsilon^*$). *Define* $\mathbf{x}_\varepsilon := \mathbf{P}^\top\overline{\mathbf{x}}_\varepsilon$. *Then*

$$\frac{\|\mathbf{A}\mathbf{x}_\varepsilon\|_2}{\|\mathbf{x}_\varepsilon\|_2} = \mathcal{O}(\varepsilon).$$

Proof. For any integer $1 \le j \le dk$, let $\mathbf{R}_{[j]}$ denote the lower right submatrix of $\mathbf{R}$ by deleting its first $j-1$ rows and columns. The column pivoting ensures that the sorted *QR* decomposition has the property that $|r_{jj}|$ is greater than or equals to the 2-norm of any columns in $\mathbf{R}_{[j]}$. Consequently, it is true that

$$\|\mathbf{R}_{[j]}\|_2 \le \sqrt{n-j+1}\|\mathbf{R}_{[j]}\|_1 \le \sqrt{n-j+1}\sqrt{dk-j+1}|r_{jj}|.$$

Since $\overline{\mathbf{x}}_\varepsilon$ is a solution of $\mathbf{R}_\varepsilon\overline{\mathbf{x}} = 0$, we have

$$\begin{aligned}\|\mathbf{A}\mathbf{x}_\varepsilon\|_2 = \|\mathbf{R}\overline{\mathbf{x}}_\varepsilon\|_2 &= \|\mathbf{R}\overline{\mathbf{x}}_\varepsilon - \mathbf{R}_\varepsilon\overline{\mathbf{x}}_\varepsilon + \mathbf{R}_\varepsilon\overline{\mathbf{x}}_\varepsilon\|_2\\ &\le \|\mathbf{R}-\mathbf{R}_\varepsilon\|_2\cdot\|\overline{\mathbf{x}}_\varepsilon\|_2 + \|\mathbf{R}_\varepsilon\overline{\mathbf{x}}_\varepsilon\|_2 < \sqrt{n-i}\sqrt{dk-i}\|\mathbf{x}_\varepsilon\|_2\varepsilon.\end{aligned}$$

The last inequality follows from the fact that, for a given ε, the only possible nonzero elements in $\mathbf{R}-\mathbf{R}_\varepsilon$ would be those in $\mathbf{R}_{[i+1]}$ where i is the smallest integer such that $|r_{i+1,i+1}|<\varepsilon$ and the fact that $\|\overline{\mathbf{x}}_\varepsilon\|_2 = \|\mathbf{x}_\varepsilon\|_2$. □

In most cases, we have no idea about how to truncate the matrix $\mathbf{R}$. Too much truncation means less precise solution. Too little truncation means no feasible solution. The following automatic, self-examining truncation strategy therefore might be more appealing.

Suppose that the matrix $\mathbf{R}$ has numerical rank s. For $1 \le i \le s$, let $\mathbf{R}^{[i]}$ denote the submatrix of thefirst i rows of $\mathbf{R}$. For each i, write $\mathbf{R}^{[i]} = [\mathbf{S}_i, \mathbf{T}_i]$ with $\mathbf{S}_i \in \mathbb{R}^{i\times i}$ and partition $\overline{\mathbf{x}} = (\overline{\mathbf{x}}_1^\top, \overline{\mathbf{x}}_2^\top)^\top$ with $\overline{\mathbf{x}}_1 \in \mathbb{R}^i$. Starting with $i = s$ and gradually decreasing to $i = 1$, we solve the maximin problem

$$\max_{0\le\overline{\mathbf{x}}_2\le 1}\min(-\mathbf{S}_i^{-1}\mathbf{T}_i\overline{\mathbf{x}}_2) \tag{15}$$

over $\overline{\mathbf{x}}_2 \in \mathbb{R}^{n-i}$ successively. The process is terminated at the first (largest) i when we obtain a nonnegative objective value. The rationale is to keep the integrity of $\mathbf{R}$ as much as possible while seeking consistency. Taking advantage of the fact that the maximin problem returns a global solution, we throw away a portion of $\mathbf{R}$ only when it absolutely cannot compromise with the consistency.

The latter approach differs from the former approach in that it avoids truncating the matrix $\mathbf{R}$ too late. Suppose the dynamic rank approach stops at i, then for any $\varepsilon < r_{ii}$ the system (14) is inconsistent. On the other hand, with $\varepsilon = r_{ii}$, we have $\mathbf{R}_\varepsilon = \mathbf{R}^{[i]}$. Obviously, for any $\varepsilon > r_{ii}$, the truncated system $\mathbf{R}_\varepsilon\overline{\mathbf{x}} = 0$ still has a nonnegative solution, except that it results in a larger residual.

A few comments are worth mentioning in the application of Theorem 4.1. Firstly, our algorithm checks to determine $\mathbf{R}_\varepsilon$ so that $\mathbf{R}_\varepsilon\overline{\mathbf{x}} = 0$ has a nonnegative solution. In order to achieve consistency, the rank of $\mathbf{R}_\varepsilon$ might need to be fairly low, that is, the value of ε might be relatively large. In fact, it is possible that consistency cannot be achieved for any ε. Secondly, once such a nonnegative solution is found to exist, the ratio of norms of the residual vector relative to the nonnegative solution is independent of scaling. The scaling is only for computational convenience. It has no effect on the relative error. Thirdly, our reconstructed parameters guarantee not only the connectivity constraint being satisfied, but also the estimate $\|\mathbf{M}^*\mathbf{X}\mathbf{\Lambda}^2 + \mathbf{C}^*\mathbf{X}\mathbf{\Lambda} + \mathbf{K}^*\mathbf{X}\|_F = \|\mathbf{A}\mathbf{x}^*\|_2$ being of order $\mathcal{O}(\varepsilon)$ relative to $\|\mathbf{x}^*\|_2$. This information can be used to determine whether the reconstruction is acceptable or not.

5. Algorithm for QIEP

Based on the ideas discussed thus far, we have developed a software package named OPT4QIEP to reconstruct the parameters involved the quadratic matrix polynomials. The package is available upon request. For simplicity, we only highlight the design of the numerical way for solving a structured QIEP in the following prototype algorithm:

Algorithm 5.1. *QIEP Algorithm. Input values*:

- *The desirable connectivity configuration.*
- *Partial eigeninformation in the form of eigenpairs closed under complex conjugation.*

Algorithm description:

- *Step* 1. *Identify the inherent structure of the coefficient matrices* $\mathbf{M}$,$\mathbf{C}$,$\mathbf{K}$.
- *Step* 2. *Prepare* $\mathbf{X}\mathbf{\Lambda}^2$ *and* $\mathbf{X}\mathbf{\Lambda}$.

- *Step* 3. *Transform Eq.* (3) *into* $\mathbf{Ax}=0$ *with variable* $\mathbf{x}$ *denoting parameters to be reconstructed.*
- *Step* 4. *Apply either the QP or the maximin techniques to search for an approximate nonnegative solution.*

Output:

- *If the QP approach has a feasible solution, or the maximin approach returns a nonnegative objective value, the structured QIEP is deemed solvable within a specified tolerance on its residual.*
- *Otherwise, the structured QIEP has no solution.*

We stress that Algorithm 5.1 should be able to take in arbitrary connectivity configuration and eigeninformation. Furthermore, within a prescribed tolerance, the numerical approach should have both high "specificity" in determining that a QIEP is not solvable and high "sensitivity" in predicting a problem is solvable.

6. Numerical experiment

Our software can handle much more complicated systems, but for demonstration purposes, we shall restrict ourselves to the connectivity configured in Fig. 1. Even so, we have observed some interesting points that are worth reporting.

To generate test data, we randomly generate positive values as the "real" physical parameters,

$$\mathbf{m}=[0.42052,0.95581,0.94875,0.65968]^{\top},$$

$$\mathbf{c}=[0.55187,1.00000,0.91316]^{\top},$$

$$\mathbf{k}=[0.71675,0.90909,0.73377,0.38006,0.32200]^{\top},$$

where we have scaled the numbers so that the largest parameter is normalized to unity. The coefficient matrices in the corresponding quadratic pencil $\lambda^2\mathbf{M}+\lambda\mathbf{C}+\mathbf{K}$ should follow the structure specified in (4). The resulting QEP has four pairs of complex conjugate eigenvalues and eigenvectors. We use only the following two eigenpairs (and their complex conjugates) as the "exact" eigeninformation $(\mathbf{\Lambda},\mathbf{X})$ in our test:

$$\lambda_1=-0.2560344023+1.5586653651\mathrm{i},\quad \lambda_2=-0.0775078020+0.3777316241\mathrm{i},$$

$$\mathbf{v}_1=\begin{pmatrix}-0.4849049878+1.2935348932\mathrm{i}\\ -1.5045182718-1.2912238078\mathrm{i}\\ 0.5225204188+1.0465626189\mathrm{i}\\ 1\end{pmatrix},\quad \mathbf{v}_2=\begin{pmatrix}0.4687637938+0.1026944256\mathrm{i}\\ 0.6690611955+0.1297948996\mathrm{i}\\ 0.8080014019+0.0892317746\mathrm{i}\\ 1\end{pmatrix},$$

where the last entry of each eigenvector has been normalized to unity. It is important to note that we have purposefully set exactly five digits for the physical parameters. On the other hand, we have displayed only the first 10 digits for the exact eigenpairs which really should be accurate to the machine precision. Our intentions are to employ our software to explore the effect of various scenarios of inexact eigeninformation on the reconstruction.

Scenario 1: Suppose we are given only $(\lambda_1,\mathbf{v}_1)$ (and its complex conjugate) as the prescribed eigenpair. Note that this eigenpair agrees with the "truly exact" eigenpair only up to the tenth digit. Recall that our algorithm endeavors to construct internally a linear inequality system with nonnegative solutions. With this slightly perturbed eigeninformation, our OPT4QIEP does return a nonnegative solution which, after rounded to five digits, has computed parameters,

$$\mathbf{m}^*=[0.42052,0.95581,0.94875,0.65968]^{\top},$$

$$\mathbf{c}^*=[0.55187,1.00000,0.91316]^{\top},$$

$$\mathbf{k}^*=[0.71675,0.90909,0.73377,0.38006,0.32200]^{\top}.$$

In other words, these computed parameters agree with the original physical parameters up to the fifth digit. The residual $\|\mathbf{M}^*\mathbf{X}\mathbf{\Lambda}^2+\mathbf{C}^*\mathbf{X}\mathbf{\Lambda}+\mathbf{K}^*\mathbf{X}\|_F$ with the computed physical parameters is around 10^{-11}, justifying a successful calculation.

The coincidence of this computed physical parameters to the original physical parameters is somewhat surprising because inverse problems generally are ill-posed, that is, the solutions typically are not unique. In our calculation we have required the additional constraint of nonnegativity. Our inequality setting also enjoys the feature that a minimizer, if it exists, is necessarily a global minimizer. Whether these conditions attributable to the coincidence as has been observed is a theoretical question of interest in its own right and deserves further investigation.

Scenario 2: We have mentioned earlier that any number of specified eigenpairs can be used in the reconstruction. Suppose now two eigenpairs $(\lambda_1,\mathbf{v}_1)$ and $(\lambda_2,\mathbf{v}_2)$ (and their conjugates) are used simultaneously. Given what has happened in Scenario 1, we are not surprised to obtain similar agreement between the computed parameters and the original parameters with residual $\|\mathbf{M}^*\mathbf{X}\mathbf{\Lambda}^2+\mathbf{C}^*\mathbf{X}\mathbf{\Lambda}+\mathbf{K}^*\mathbf{X}\|_F\approx 10^{-11}$.

The difference between Scenarios 1 and 2 is that this time $\mathbf{\Lambda}$ and $\mathbf{X}$ are of sizes 4×4. The "extra" eigeninformation with approximately the same accuracy as the previous eigeninformation increases the computational overhead (for computing $\mathbf{A}$), but does not seem to have an impact on the final answer.

Scenario 3: In this test, we perturb the "exact" eigenpairs to the extent that the "prescribed" eigenpairs are given by

$$\widetilde{\lambda}_1 = -0.25603 + 1.55867\mathrm{i}, \quad \widetilde{\lambda}_2 = -0.07751 + 0.37773\mathrm{i},$$

$$\widetilde{\mathbf{v}}_1 = \begin{pmatrix} -0.48490 + 1.29353\mathrm{i} \\ -1.50452 - 1.29122\mathrm{i} \\ 0.52252 + 1.04656\mathrm{i} \\ 1 \end{pmatrix}, \quad \widetilde{\mathbf{v}}_2 = \begin{pmatrix} 0.46876 + 0.10269\mathrm{i} \\ 0.66906 + 0.12979\mathrm{i} \\ 0.80800 + 0.08923\mathrm{i} \\ 1 \end{pmatrix}.$$

In other words, the "accuracy" of the prescribed eigenpairs is only at the fifth decimal. Note that we typically would have no assessment about the accuracy of a prescribed eigeninformation. Thus it is not clear at all that any quadratic matrix polynomial structured as in (4) could have $(\widetilde{\lambda}_1, \widetilde{\mathbf{v}}_1)$ and $(\widetilde{\lambda}_2, \widetilde{\mathbf{v}}_2)$ as its eigenpairs. This is precisely the difficulty of the structured QIEP, not to mention that we are seeking further nonnegative solutions.

Upon applying our OPT4QIEP, however, we do find a nonnegative solution

$$\mathbf{m}^* = [0.42053, 0.95585, 0.94878, 0.65971]^\top,$$

$$\mathbf{c}^* = [0.55188, 1.00000, 0.91319]^\top,$$

$$\mathbf{k}^* = [0.71677, 0.90911, 0.73383, 0.38009, 0.32200]^\top,$$

by which the residual $\|\mathbf{M}^*\widetilde{\mathbf{X}}\widetilde{\mathbf{\Lambda}}^2 + \mathbf{C}^*\widetilde{\mathbf{X}}\widetilde{\mathbf{\Lambda}} + \mathbf{K}^*\widetilde{\mathbf{X}}\|_F$ of the corresponding reconstructed quadratic pencil is of the order 10^{-6}. This example is a numerical justification of our theory in Theorem 4.1.

Scenario 4: To further dramatize the scenario and to demonstrate the situation where our code is able to indicate that a QIEP is not solvable when the prescribed eigenpairs are infeasible, consider the randomly generated eigenpairs

$$\widehat{\lambda}_1 = 0.6068 + 0.8913\mathrm{i}, \quad \widehat{\lambda}_2 = 0.4860 + 0.7621\mathrm{i},$$

$$\widehat{\mathbf{v}}_2 = \begin{pmatrix} 0.0185 + 0.4057\mathrm{i} \\ 0.8214 + 0.9355\mathrm{i} \\ 0.4447 + 0.9169\mathrm{i} \\ 1 \end{pmatrix}, \quad \widehat{\mathbf{v}}_2 = \begin{pmatrix} 0.7919 + 0.8936\mathrm{i} \\ 0.9218 + 0.0579\mathrm{i} \\ 0.7382 + 0.3529\mathrm{i} \\ 1 \end{pmatrix}.$$

We use only four digits this time for the reason that, in contrast to the above scenarios, we generally do not have any clue at all in practice about the feasibility of prescribed eigenpairs. Thus, not only that we are not sure whether a structured QIEP is solvable with $(\widehat{\lambda}_1, \widehat{\mathbf{v}}_1)$ and $(\widehat{\lambda}_2, \widehat{\mathbf{v}}_2)$ as its eigenpairs, neither can we foretell whether an approximate truncated system exists. What we are certain about is that, if a solution or a nearby solution ever exists, our method will find it (high sensitivity) and that, if the QIEP is not solvable, our method will indicate either a large residual or no solution (high specificity).

Applying our software to the above two randomly generated eigenpairs, we do find a nonnegative "approximate" solution, but the residuals of the corresponding quadratic matrix polynomials are 1.2885 and 0.7667, respectively, at each eigenpair. It should be an obvious indication that the reconstructed model is unacceptable.

We conclude the discussion with this final remark. It seems reasonable to expect that the eigenstructure of structured problems should be inherently "structured". Given a specified structure for the coefficient matrices ($\mathbf{M}$,$\mathbf{C}$,$\mathbf{K}$), it seems rational to expect that the corresponding eigenvalues and eigenvectors cannot be arbitrary. Scenario 4 exemplifies such a case. It remains an open question to determine the eigenstructure of a structured matrix polynomial. Our algorithm might serve as a numerical tool to help investigation in this direction.

7. Conclusion

Just as the quadratic eigenvalue problems are critical to scores of important applications, the quadratic inverse eigenvalue problems are equally vital in many different fields of disciplines. The inverse problems become particularly challenging when the structure due to the interconnectivity of the system and the nonnegativity of the parameters must be respected. Thus far, no theory can address these difficulties.

The main contribution of this paper is to offer a general purpose and robust numerical approach that, via a returned residual estimate, has high specificity in determining that a QIEP is not solvable and high sensitivity in predicting a problem is solvable. The method imposes no restriction on how much eigeninformation should be given and has the ability of handling almost all kinds of connectivity configurations.

The crux of our computation is the mechanism of establishing a proper inequality system and solving the system via optimization techniques. The setting allows us to find the nearest possible global solution up to scaling and thus provides us with the desirable robustness.

We have built a package of software that can automatically handle the above-mentioned task from any input of connectivity configuration and eigeninformation by users. The beta release of this package is available upon request. We are continuing the expansion of its library to include various settings according to the underlying physical laws.

There is still much room for improvement in our approach. For instance, a *QR* factorization for the matrix A of size $dk \times n$ is needed in our present scheme, which makes the method less practical for large problems. Also, the decision of the low rank approximation to the matrix R even with our built-in dynamic rank mechanism is heuristic, which may require users' intervention to balance between competing interests in cases where the diagonal elements of R go to zero very gradually.

Despite its imperfection, we hope that our generic and automatic approach to the difficult but important quadratic inverse eigenvalue problem would serve as the first step leading to a general algorithm with significant potential.

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