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An efficient algebraic method for computing eigensolution sensitivity of asymmetric damped systems

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ABSTRACT

In this paper, an efficient algebraic method for the computation of eigensolution derivatives of asymmetric damped systems with distinct eigenvalues is presented. By introducing an additional and new normalization condition, we construct two extended systems of linear equations with nonsingular coefficient matrices which are transpose to each other. We can compute the derivatives of the eigenvalues and their associated right and left eigenvectors by solving the two systems, respectively. In this way, the CPU computation time and the storage space are considerably reduced. Finally, a numerical example is included to demonstrate the validity of the proposed method.

1. Introduction

Because of the increasing importance of the derivatives of eigenvalues and eigenvectors in structural design optimization, damage detection, structural dynamic modification and other applications [1], many methods have been developed to compute the eigensolution sensitivities with respect to system parameters. The computation of the derivatives of eigenvalues of a mechanical system is well documented, but the calculation of derivatives of the corresponding eigenvectors requires the solution of singular problems. Further difficulties appear in computation of eigensolution sensitivities for asymmetric damped systems where the left and right eigenvectors are different and complex, and require an additional normalization condition for the relative scaling of eigenvectors.

The common approaches for the calculation of derivatives of eigenvectors can be divided into the modal method, Nelson's method and the algebraic methods. The modal method employs a modal superposition idea. In 1968, Fox and Kapoor [2] first gave exact expressions of the first-order derivative of eigensolutions for symmetric undamped systems by using the modal expansion technique. Plaut and Huseyin [3] extended this method to asymmetric damped systems using $2N$-space formulation where N is the system dimension. Adhikari and Friswell [4] applied the modal method to asymmetric nonconservative systems using N-space formulation. Note that the accuracy of the modal method is dependent on the number of modes used in calculation. To guarantee the accurate sensitivities, the modal method needs all of the modes. However, it is often difficult to obtain all the modes. Generally, only the partial modes are computed and are used as the basis vectors of eigenvector derivatives. A significant error will yield if a large number of modes are truncated. Wang [5] improved the modal truncation method by using a residual static mode to approximate the contribution due to unavailable high-frequency modes. Zeng [6] presented modified modal methods such as the multiple modal acceleration methods with

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shifted-poles for the complex eigenvectors in symmetric viscous damping systems. Furthermore, by combining the modal method in [4] and the modal acceleration with shifted-poles in [6], Moon et al. [7] presented the modified modal methods for asymmetric damped systems. In 1976, Nelson [8] proposed a method to calculate the first-order derivatives of eigenvectors with distinct eigenvalues for the general real eigensystems. In contrast to the modal method, the Nelson's method requires only the eigenvalue and eigenvector under consideration. Unfortunately, the method cannot directly be used for the case of repeated eigenvalues. Ojalvo [9], Mills-Curran [10], Dailey [11] and Wu et al. [12] developed Nelson's method for solving the first-order derivatives of eigensolutions of structures with repeated eigenvalues for symmetric undamped systems. Shaw and Jayasuriya [13] extended the method in Refs. [9–11] for computing the derivatives of eigensolutions in the case of repeated eigenvalues with repeated first-order derivatives. For symmetric and asymmetric damped systems, Friswell and Adhikari [14] and Guedria et al. [15] generalized Nelson's method [8] to calculate the first-order and second-order eigenvector derivatives, respectively. However, the methods in Refs. [14,15] have been developed for the systems with distinct eigenvalues only. A review on calculating the eigensolution derivatives of a general real eigensystems with distinct eigenvalues was given by Murthy and Haftka [16]. Tang et al. [17,18] investigated the eigensolution derivatives with repeated eigenvalues for general asymmetric eigensystems. Lee et al. [19] derived an algebraic method to compute the first-order derivatives of eigensolutions for symmetric undamped systems with distinct eigenvalues. Later, Lee et al. [20] and Choi et al. [21] generalized the algebraic methods to symmetric damped systems. The methods in Refs. [19–21] preserved the symmetry of the original eigensystem. Garg [22], Rudisill [23], Rudisill and Chu [24] investigated the algebraic methods for general asymmetric eigensystems, however, as pointed out by Murthy and Haftka [16], the normalization conditions adopted in Refs. [22–24] were not always valid. Recently, Guedria et al. [25] presented an algebraic method for computing simultaneously the first-order derivatives of the eigenvalues and their associated right and left eigenvectors for asymmetric damped systems by solving a system of $2N+1$ linear equations. The methods in Refs. [19–25] have been developed for the systems with distinct eigenvalues only. Lee et al. [26,27] and Choi et al. [28] further generalized their algebraic methods to the case of repeated eigenvalues, but these generalizations were not correct since a mistake was made in the derivation of equations on derivatives of the normalization condition [29].

In this paper, we present an efficient algebraic method for the computation of eigensolution derivatives of asymmetric damped systems with distinct eigenvalues. By introducing an additional and new normalization condition, we construct two extended systems of $N+1$ linear equations with nonsingular coefficient matrices which are transpose to each other. We can then compute the derivatives of the eigenvalues and their associated right and left eigenvectors by solving the two systems, respectively. The proposed method requires only the eigenvalues and eigenvectors under consideration, gives exact solution and guarantees numerical stability. Compared with the method in Ref. [25], the CPU computation time and the storage space are significantly reduced, thus it is more efficient. In addition, the method can be extended to compute higher order eigensolution derivatives with less computing effort. A numerical example is used to illustrate the validity of the proposed method.

2. Theoretical background

The equations of motion describing the free vibration of a linear, damped discrete system with N degrees of freedom are

$$\mathbf{M}(p)\ddot{\mathbf{q}}(t)+\mathbf{C}(p)\dot{\mathbf{q}}(t)+\mathbf{K}(p)\mathbf{q}(t)=\mathbf{0} \tag{1}$$

where $\mathbf{M}(p)$, $\mathbf{C}(p)$ and $\mathbf{K}(p)\in R^{N\times N}$ are the mass, damping and stiffness matrices, respectively, whose elements depend continuously on the real parameter p, and are asymmetric matrices, and $\mathbf{M}(p)$ is nonsingular, $\mathbf{q}(t)\in R^N$ is the vector of generalized coordinates and $t\in R^+$ denotes time.

The eigenvalues associated with Eq. (1) are the roots of the characteristic polynomial

$$\det[\lambda^2(p)\mathbf{M}(p)+\lambda(p)\mathbf{C}(p)+\mathbf{K}(p)]=0 \tag{2}$$

The order of the polynomial is $2N$ and the roots appear in complex conjugate pairs for an underdamped system

$$\lambda_1(p),\lambda_2(p),\ldots,\lambda_N(p),\lambda_1^*(p),\lambda_2^*(p),\ldots,\lambda_N^*(p) \tag{3}$$

where * denotes complex conjugate. These eigenvalues are assumed to be distinct.

The right eigenvalue problem associated with Eq. (1) can be represented by the λ-matrix problem [30]

$$[\lambda_i^2(p)\mathbf{M}(p)+\lambda_i(p)\mathbf{C}(p)+\mathbf{K}(p)]\boldsymbol{\varphi}_i(p)=\mathbf{0},\quad i=1,2,\ldots,N \tag{4}$$

where $\lambda_i(p)\in C$ is the ith eigenvalue and $\boldsymbol{\varphi}_i(p)\in C^N$ is the ith right eigenvector. Similarly, the left eigenvalue problem can be written as

$$[\lambda_i^2(p)\mathbf{M}(p)+\lambda_i(p)\mathbf{C}(p)+\mathbf{K}(p)]^{\mathrm{T}}\boldsymbol{\psi}_i(p)=\mathbf{0},\quad i=1,2,\ldots,N \tag{5}$$

where $\boldsymbol{\psi}_i(p)\in C^N$ is the ith left eigenvector.

We can easily obtain the right and left eigenvectors from a first-order formulation, for example, the state-space method [31] or Duncan forms [32]. Introducing state variables $\mathbf{u}(t)=[\mathbf{q}^{\mathrm{T}}(t),\dot{\mathbf{q}}^{\mathrm{T}}(t)]^{\mathrm{T}}$, Eq. (1) is transformed into the first-order

(Duncan) form as

$$\mathbf{A}(p)\mathbf{u}(t) = \mathbf{B}(p)\dot{\mathbf{u}}(t) \tag{6}$$

where $\mathbf{A}(p)$, $\mathbf{B}(p) \in R^{2N\times 2N}$ are the first-order system matrices given by

$$\mathbf{A}(p) = \begin{bmatrix} -\mathbf{K}(p) & \mathbf{0} \\ \mathbf{0} & \mathbf{M}(p) \end{bmatrix}, \quad \mathbf{B}(p) = \begin{bmatrix} \mathbf{C}(p) & \mathbf{M}(p) \\ \mathbf{M}(p) & \mathbf{0} \end{bmatrix} \tag{7}$$

Then, the right and left eigenvalue problems associated with Eq. (6) can be represented by

$$\mathbf{A}(p)\mathbf{x}_i(p) = \lambda_i(p)\mathbf{B}(p)\mathbf{x}_i(p) \tag{8a}$$

$$\mathbf{A}^{\mathrm{T}}(p)\mathbf{y}_i(p) = \lambda_i(p)\mathbf{B}^{\mathrm{T}}(p)\mathbf{y}_i(p), \quad i = 1,2,\ldots,2N \tag{8b}$$

where $\lambda_i(p) \in C$ is the ith eigenvalue and $\mathbf{x}_i(p)$, $\mathbf{y}_i(p) \in C^{2N}$ are the ith right and left eigenvectors of the first-order system, respectively, which are related to the ith right and left eigenvectors of the second-order system by

$$\mathbf{x}_i(p) = \begin{Bmatrix} \boldsymbol{\varphi}_i(p) \\ \lambda_i(p)\boldsymbol{\varphi}_i(p) \end{Bmatrix}, \quad \mathbf{y}_i(p) = \begin{Bmatrix} \boldsymbol{\psi}_i(p) \\ \lambda_i(p)\boldsymbol{\psi}_i(p) \end{Bmatrix} \tag{9}$$

For distinct eigenvalues it is easy to show that the right and left eigenvectors satisfy the following biorthogonal relationship:

$$\mathbf{y}_i^{\mathrm{T}}(p)\mathbf{A}(p)\mathbf{x}_j(p) = 0 \quad \text{and} \quad \mathbf{y}_i^{\mathrm{T}}(p)\mathbf{B}(p)\mathbf{x}_j(p) = 0, \quad \forall i \neq j \tag{10}$$

As $\mathbf{M}(p)$ is nonsingular, then the right and left eigenvectors may be normalized to satisfy

$$\mathbf{y}_i^{\mathrm{T}}(p)\mathbf{B}(p)\mathbf{x}_i(p) = 1, \quad i = 1,2,\ldots,2N \tag{11}$$

Substituting $\mathbf{x}_i(p)$ and $\mathbf{y}_i(p)$ in Eq. (9) into Eq. (11) yields the following relationship in terms of the eigensolutions of the second-order system:

$$\boldsymbol{\psi}_i^{\mathrm{T}}(p)[2\lambda_i(p)\mathbf{M}(p) + \mathbf{C}(p)]\boldsymbol{\varphi}_i(p) = 1, \quad i = 1,2,\ldots,N \tag{12}$$

The normalization in Eq. (12) is insufficient and the eigenvectors are not unique to the extent of a nonzero complex factor. This point can be demonstrated by multiplying the left eigenvector by any nonzero scalar and dividing the right eigenvector by the same scalar. So an additional normalization condition should be imposed to yield unique eigenvectors.

The paper is concerned with the derivatives of eigenvalues and associated right and left eigenvectors at $p = p_0$, and hereafter "(p_0)" is omitted for variables evaluated at $p = p_0$. Let $\boldsymbol{\varphi}_i$ be an arbitrarily chosen right eigenvector associated with the eigenvalue λ_i at $p = p_0$. The associated left eigenvector $\boldsymbol{\psi}_i$ can uniquely be determined by using Eq. (12) with $p = p_0$. For $p = p_0$ and p near p_0, we propose the following additional normalization condition for the right eigenvector $\boldsymbol{\varphi}_i(p)$:

$$\mathbf{l}_i^{\mathrm{T}}\boldsymbol{\varphi}_i(p) = 1, \quad i = 1,2,\ldots,N \tag{13}$$

where $\mathbf{l}_i$ is given by

$$\mathbf{l}_i = (2\lambda_i\mathbf{M} + \mathbf{C})^{\mathrm{T}}\boldsymbol{\psi}_i \tag{14}$$

It should be noted that $\mathbf{l}_i$ in Eq. (14) is a constant complex vector evaluated at $p = p_0$ and does not vary with the design parameter p. Once the right eigenvector $\boldsymbol{\varphi}_i$ at $p = p_0$ is selected, the eigenvalue $\lambda_i(p)$ and associated right eigenvector $\boldsymbol{\varphi}_i(p)$ for p near p_0 can uniquely be determined by utilizing Eqs. (4) and (13). This point can be shown as follows. Consider the system of equations (4) and (13). For $p = p_0$, $(\boldsymbol{\varphi}_i, \lambda_i)$ is a solution to this system and the associated Jacobian matrix is

$$J(p_0) = \begin{bmatrix} \lambda_i^2\mathbf{M} + \lambda_i\mathbf{C} + \mathbf{K} & (2\lambda_i\mathbf{M} + \mathbf{C})\boldsymbol{\varphi}_i \\ \boldsymbol{\psi}_i^{\mathrm{T}}(2\lambda_i\mathbf{M} + \mathbf{C}) & 0 \end{bmatrix} \tag{15}$$

Assume $J(p_0)\boldsymbol{\Phi} = \mathbf{0}$, for some $\boldsymbol{\Phi} = [\mathbf{q}^{\mathrm{T}}\ r]^{\mathrm{T}} \in C^{N+1}$. That is

$$(\lambda_i^2\mathbf{M} + \lambda_i\mathbf{C} + \mathbf{K})\mathbf{q} + r(2\lambda_i\mathbf{M} + \mathbf{C})\boldsymbol{\varphi}_i = \mathbf{0} \tag{16a}$$

$$\boldsymbol{\psi}_i^{\mathrm{T}}(2\lambda_i\mathbf{M} + \mathbf{C})\mathbf{q} = 0 \tag{16b}$$

Premultiplying Eq. (16a) by $\boldsymbol{\psi}_i^{\mathrm{T}}$ and using Eqs. (5) and (12) with $p = p_0$ leads to $r = 0$. Substituting the result into Eq. (16a) yields

$$(\lambda_i^2\mathbf{M} + \lambda_i\mathbf{C} + \mathbf{K})\mathbf{q} = \mathbf{0} \tag{17}$$

Using Eq. (4) with $p = p_0$, the solution to Eq. (17) can be expressed by

$$\mathbf{q} = c\boldsymbol{\varphi}_i \tag{18}$$

where c is a complex number. Substituting Eq. (18) into Eq. (16b) and using Eq. (12) with $p = p_0$ yields $c = 0$, and thus $\mathbf{q} = \mathbf{0}$. Therefore, equation $J(p_0)\mathbf{\Phi} = 0$ has solution $\mathbf{\Phi} = 0$ only and the Jacobian matrix in Eq. (15) is nonsingular. Now the implicit function theorem indicates that Eqs. (4) and (13) have unique solution $\lambda_i(p)$ and $\boldsymbol{\varphi}_i(p)$ for each p near p_0. Thus the adopted additional and new normalization condition in Eq. (13) for the right eigenvector is valid. Using the $\boldsymbol{\varphi}_i(p)$ above and utilizing Eqs. (5) and (12), we can uniquely determine the left eigenvector $\boldsymbol{\psi}_i(p)$ associated with $\lambda_i(p)$ for each p near p_0. We can then calculate the derivatives of eigenvalues and associated right and left eigenvectors at $p = p_0$. Note that these derivatives are computed by choosing an arbitrarily right eigenvector associated the eigenvalue λ_i at $p = p_0$, which is one of advantages of the proposed method.

In next section, we will present an efficient algebraic method for computing the first and second-order eigensolution derivatives in N-space.

3. The proposed method

For convenience, we denote the first-order derivative with respect to parameter p by a prime.

Differentiating Eqs. (4) and (13) with respect to the parameter p at $p = p_0$ and rearranging them yields, respectively,

$$(\lambda_i^2\mathbf{M} + \lambda_i\mathbf{C} + \mathbf{K})\boldsymbol{\varphi}_i' + (2\lambda_i\mathbf{M} + \mathbf{C})\boldsymbol{\varphi}_i\lambda_i' = -(\lambda_i^2\mathbf{M}' + \lambda_i\mathbf{C}' + \mathbf{K}')\boldsymbol{\varphi}_i \tag{19}$$

and

$$\mathbf{l}_i^{\mathrm{T}}\boldsymbol{\varphi}_i' = 0 \tag{20}$$

Utilizing Eq. (14) and combining Eqs. (19) and (20) leads to the following system:

$$\begin{bmatrix} \lambda_i^2\mathbf{M} + \lambda_i\mathbf{C} + \mathbf{K} & (2\lambda_i\mathbf{M} + \mathbf{C})\boldsymbol{\varphi}_i \\ \boldsymbol{\psi}_i^{\mathrm{T}}(2\lambda_i\mathbf{M} + \mathbf{C}) & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\varphi}_i' \\ \lambda_i' \end{bmatrix} = \begin{bmatrix} -(\lambda_i^2\mathbf{M}' + \lambda_i\mathbf{C}' + \mathbf{K}')\boldsymbol{\varphi}_i \\ 0 \end{bmatrix} \tag{21}$$

As shown in Section 2, the coefficient matrix in Eq. (21) is nonsingular. Thus we can solve Eq. (21) to obtain the eigenvalues derivatives λ_i' and associated right eigenvector derivatives $\boldsymbol{\phi}_i'$.

Similarly, differentiating Eqs. (5) and (12) with respect to the parameter p at $p = p_0$, respectively, and rearranging them results in

$$(\lambda_i^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K})^{\mathrm{T}}\boldsymbol{\psi}_i' = -(2\lambda_i\mathbf{M} + \mathbf{C})^{\mathrm{T}}\boldsymbol{\psi}_i\lambda_i' - (\lambda_i^2\mathbf{M}' + \lambda_i\mathbf{C}' + \mathbf{K}')^{\mathrm{T}}\boldsymbol{\psi}_i \tag{22}$$

and

$$\boldsymbol{\psi}_i'^{\mathrm{T}}(2\lambda_i\mathbf{M} + \mathbf{C})\boldsymbol{\varphi}_i = -\boldsymbol{\psi}_i^{\mathrm{T}}(2\lambda_i\mathbf{M} + \mathbf{C})\boldsymbol{\varphi}_i' - 2(\boldsymbol{\psi}_i^{\mathrm{T}}\mathbf{M}\boldsymbol{\varphi}_i)\lambda_i' - \boldsymbol{\psi}_i^{\mathrm{T}}(2\lambda_i\mathbf{M}' + \mathbf{C}')\boldsymbol{\varphi}_i \tag{23}$$

Note that the term $\boldsymbol{\psi}_i'^{\mathrm{T}}(2\lambda_i\mathbf{M} + \mathbf{C})\boldsymbol{\varphi}_i$ is a scalar, thus we have

$$\boldsymbol{\psi}_i'^{\mathrm{T}}(2\lambda_i\mathbf{M} + \mathbf{C})\boldsymbol{\varphi}_i = \boldsymbol{\varphi}_i^{\mathrm{T}}(2\lambda_i\mathbf{M} + \mathbf{C})^{\mathrm{T}}\boldsymbol{\psi}_i' \tag{24}$$

So, Eq. (23) can be rewritten as

$$\boldsymbol{\varphi}_i^{\mathrm{T}}(2\lambda_i\mathbf{M} + \mathbf{C})^{\mathrm{T}}\boldsymbol{\psi}_i' = -\boldsymbol{\psi}_i^{\mathrm{T}}(2\lambda_i\mathbf{M} + \mathbf{C})\boldsymbol{\varphi}_i' - 2(\boldsymbol{\psi}_i^{\mathrm{T}}\mathbf{M}\boldsymbol{\varphi}_i)\lambda_i' - \boldsymbol{\psi}_i^{\mathrm{T}}(2\lambda_i\mathbf{M}' + \mathbf{C}')\boldsymbol{\varphi}_i \tag{25}$$

We introduce the following extended system of equations with unknowns $\boldsymbol{\psi}_i'$ and $\mu \in C$:

$$\begin{bmatrix} (\lambda_i^2\mathbf{M} + \lambda_i\mathbf{C} + \mathbf{K})^{\mathrm{T}} & (2\lambda_i\mathbf{M} + \mathbf{C})^{\mathrm{T}}\boldsymbol{\psi}_i \\ \boldsymbol{\varphi}_i^{\mathrm{T}}(2\lambda_i\mathbf{M} + \mathbf{C})^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\psi}_i' \\ \mu \end{bmatrix} = \begin{bmatrix} \mathbf{G} \\ r \end{bmatrix} \tag{26}$$

where

$$\mathbf{G} := -(2\lambda_i\mathbf{M} + \mathbf{C})^{\mathrm{T}}\boldsymbol{\psi}_i\lambda_i' - (\lambda_i^2\mathbf{M}' + \lambda_i\mathbf{C}' + \mathbf{K}')^{\mathrm{T}}\boldsymbol{\psi}_i \tag{27a}$$

$$r := -\boldsymbol{\psi}_i^{\mathrm{T}}(2\lambda_i\mathbf{M} + \mathbf{C})\boldsymbol{\varphi}_i' - 2(\boldsymbol{\psi}_i^{\mathrm{T}}\mathbf{M}\boldsymbol{\varphi}_i)\lambda_i' - \boldsymbol{\psi}_i^{\mathrm{T}}(2\lambda_i\mathbf{M}' + \mathbf{C}')\boldsymbol{\varphi}_i \tag{27b}$$

Eq. (26) can be expanded into

$$(\lambda_i^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K})^{\mathrm{T}}\boldsymbol{\psi}_i' + (2\lambda_i\mathbf{M} + \mathbf{C})^{\mathrm{T}}\boldsymbol{\psi}_i\mu = \mathbf{G} \tag{28a}$$

$$\boldsymbol{\varphi}_i^{\mathrm{T}}(2\lambda_i\mathbf{M} + \mathbf{C})^{\mathrm{T}}\boldsymbol{\psi}_i' = r \tag{28b}$$

Premultiplying Eq. (28a) by $\boldsymbol{\varphi}_i^{\mathrm{T}}$ and using Eqs. (4), (12) and (19) with $p = p_0$ leads to

$$\mu = 0 \tag{29}$$

from which Eq. (28a) is reduced to Eq. (22), and Eq. (28b) is just Eq. (25). This means that the part $\boldsymbol{\psi}_i'$ of the solution to Eq. (26) satisfies Eqs. (22) and (25) simultaneously and it is the derivative of the associated left eigenvector calculated at $p = p_0$.

It should be noted that the coefficient matrices in systems (21) and (26) are not symmetric, but the coefficient matrix of Eq. (26) is the transpose of that of Eq. (21), thus it is also nonsingular. If λ_i' and $\boldsymbol{\varphi}_i'$ are solved from Eq. (21), then the $\mathbf{G}$ and r on the right-hand side of Eq. (26) are known. Further, we can solve Eq. (26) without performing a new decomposition of its coefficient matrix to achieve the left eigenvector derivative $\boldsymbol{\psi}_i'$.

So far we obtain the first-order derivatives λ_i', $\boldsymbol{\varphi}_i'$ and $\boldsymbol{\psi}_i'$ of eigensolutions.

Furthermore, the proposed method can be extended to calculate higher order derivatives of eigensolutions with less computing effort. For example, the second-order derivatives of eigensolutions may be found by using the similar procedure above. Differentiating Eqs. (4), (5), (12) and (13) twice with respect to the parameter p at $p = p_0$, respectively, yields the following two systems:

$$\begin{bmatrix} \lambda_i^2\mathbf{M} + \lambda_i\mathbf{C} + \mathbf{K} & (2\lambda_i\mathbf{M} + \mathbf{C})\boldsymbol{\varphi}_i \\ \boldsymbol{\psi}_i^{\mathrm{T}}(2\lambda_i\mathbf{M} + \mathbf{C}) & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\varphi}_i'' \\ \lambda_i'' \end{bmatrix} = \begin{bmatrix} \mathbf{T} \\ 0 \end{bmatrix} \tag{30}$$

and

$$\begin{bmatrix} (\lambda_i^2\mathbf{M} + \lambda_i\mathbf{C} + \mathbf{K})^{\mathrm{T}} & (2\lambda_i\mathbf{M} + \mathbf{C})^{\mathrm{T}}\boldsymbol{\psi}_i \\ \boldsymbol{\varphi}_i^{\mathrm{T}}(2\lambda_i\mathbf{M} + \mathbf{C})^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\psi}_i'' \\ v \end{bmatrix} = \begin{bmatrix} \mathbf{H} \\ s \end{bmatrix} \tag{31}$$

where

$$\begin{aligned} \mathbf{T} := & -(\lambda_i^2\mathbf{M}'' + \lambda_i\mathbf{C}'' + \mathbf{K}'')\boldsymbol{\varphi}_i - 2(\lambda_i^2\mathbf{M}' + \lambda_i\mathbf{C}' + \mathbf{K}')\boldsymbol{\varphi}_i' - 2\lambda_i'(2\lambda_i\mathbf{M} + \mathbf{C})\boldsymbol{\varphi}_i' \\ & - 2\lambda_i'(2\lambda_i\mathbf{M}' + \mathbf{C}')\boldsymbol{\varphi}_i - 2(\lambda_i')^2\mathbf{M}\boldsymbol{\varphi}_i \end{aligned} \tag{32a}$$

$$\begin{aligned} \mathbf{H} := & -(\lambda_i^2\mathbf{M}'' + \lambda_i\mathbf{C}'' + \mathbf{K}'')^{\mathrm{T}}\boldsymbol{\psi}_i - 2(\lambda_i^2\mathbf{M}' + \lambda_i\mathbf{C}' + \mathbf{K}')^{\mathrm{T}}\boldsymbol{\psi}_i' - 2\lambda_i'(2\lambda_i\mathbf{M} + \mathbf{C})^{\mathrm{T}}\boldsymbol{\psi}_i' \\ & - 2\lambda_i'(2\lambda_i\mathbf{M}' + \mathbf{C}')^{\mathrm{T}}\boldsymbol{\psi}_i - 2(\lambda_i')^2\mathbf{M}^{\mathrm{T}}\boldsymbol{\psi}_i - \lambda_i''(2\lambda_i\mathbf{M} + \mathbf{C})^{\mathrm{T}}\boldsymbol{\psi}_i \end{aligned} \tag{32b}$$

$$\begin{aligned} s := & -\boldsymbol{\psi}_i^{\mathrm{T}}(2\lambda_i\mathbf{M}'' + \mathbf{C}'')\boldsymbol{\varphi}_i - 2\boldsymbol{\psi}_i'^{\mathrm{T}}(2\lambda_i\mathbf{M}' + \mathbf{C}')\boldsymbol{\varphi}_i - 2\boldsymbol{\psi}_i^{\mathrm{T}}(2\lambda_i\mathbf{M}' + \mathbf{C}')\boldsymbol{\varphi}_i' \\ & - 2\boldsymbol{\psi}_i'^{\mathrm{T}}(2\lambda_i\mathbf{M} + \mathbf{C})\boldsymbol{\varphi}_i' - 2\lambda_i''\boldsymbol{\psi}_i^{\mathrm{T}}\mathbf{M}\boldsymbol{\varphi}_i - \boldsymbol{\psi}_i^{\mathrm{T}}(2\lambda_i\mathbf{M} + \mathbf{C})\boldsymbol{\varphi}_i'' \\ & - 4\lambda_i'\boldsymbol{\psi}_i^{\mathrm{T}}\mathbf{M}'\boldsymbol{\varphi}_i - 4\lambda_i'\boldsymbol{\psi}_i'^{\mathrm{T}}\mathbf{M}\boldsymbol{\varphi}_i - 4\lambda_i'\boldsymbol{\psi}_i^{\mathrm{T}}\mathbf{M}\boldsymbol{\varphi}_i' \end{aligned} \tag{32c}$$

Based on the obtained λ_i', $\boldsymbol{\varphi}_i'$ and $\boldsymbol{\psi}_i'$, the second-order derivatives λ_i'', $\boldsymbol{\varphi}_i''$ and $\boldsymbol{\psi}_i''$ may be evaluated from Eqs. (30) and (31). Noting that the coefficient matrices in Eqs. (30) and (31) are identical with those in Eqs. (21) and (26), respectively, we need not factor them once more. The aforementioned procedure may be continued to any higher order eigensolution derivatives.

Finally, it is noted that since the adopted normalization condition is different from the existing other ones, the obtained eigenvector derivatives are different from those obtained by the existing other methods even if for the same eigenvector at $p = p_0$, but the eigenvalue derivatives are same, as the eigenvalue derivatives are independent of the selection of normalization conditions.

4. Discussion

The preceding sections are mainly focused on the asymmetric damped systems. It should be noted that the proposed method is also applicable to symmetric damped systems. For the latter, the right and left eigenvectors are identical, so we only need to solve the eigenpair derivatives λ_i' and $\boldsymbol{\varphi}_i'$, and one of the both normalization conditions (12) with $\boldsymbol{\psi}_i(p) = \boldsymbol{\varphi}_i(p)$ and (13) with $\boldsymbol{\psi}_i = \boldsymbol{\varphi}_i$ is sufficient for the computation of the derivatives.

For symmetric damped systems, the eigenpair derivatives may be obtained by solving Eq. (21) with $\boldsymbol{\psi}_i = \boldsymbol{\varphi}_i$. Here we have adopted the normalization condition (13) with $\boldsymbol{\psi}_i = \boldsymbol{\varphi}_i$

$$\boldsymbol{\varphi}_i^{\mathrm{T}}(2\lambda_i\mathbf{M} + \mathbf{C})\boldsymbol{\varphi}_i(p) = 1 \tag{33}$$

It is obvious that at $p = p_0$ the eigenvector $\boldsymbol{\varphi}_i$ is not arbitrary but satisfies Eq. (33) with $p = p_0$ which is different from the case of asymmetric damped systems.

In addition, we may also obtain the eigenvector derivatives $\boldsymbol{\varphi}_i'$ by using Eq. (26). Utilizing the symmetry and rearranging Eq. (26) yields

$$\begin{bmatrix} \lambda_i^2\mathbf{M} + \lambda_i\mathbf{C} + \mathbf{K} & (2\lambda_i\mathbf{M} + \mathbf{C})\boldsymbol{\varphi}_i \\ \boldsymbol{\varphi}_i^{\mathrm{T}}(2\lambda_i\mathbf{M} + \mathbf{C}) & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\varphi}_i' \\ \mu \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{G}} \\ \tilde{r} \end{bmatrix} \tag{34}$$

where

$$\tilde{\mathbf{G}} := -(2\lambda_i\mathbf{M}+\mathbf{C})\boldsymbol{\varphi}_i\lambda_i' - (\lambda_i^2\mathbf{M}'+\lambda_i\mathbf{C}'+\mathbf{K}')\boldsymbol{\varphi}_i \tag{35a}$$

$$\tilde{r} := -(\boldsymbol{\varphi}_i^{\mathrm{T}}\mathbf{M}\boldsymbol{\varphi}_i)\lambda_i' - 0.5\boldsymbol{\varphi}_i^{\mathrm{T}}(2\lambda_i\mathbf{M}'+\mathbf{C}')\boldsymbol{\varphi}_i \tag{35b}$$

Here we have adopted the usual normalization condition (12) with $\boldsymbol{\psi}_i(p)=\boldsymbol{\varphi}_i(p)$, i.e.

$$\boldsymbol{\varphi}_i^{\mathrm{T}}(p)[2\lambda_i(p)\mathbf{M}(p)+\mathbf{C}(p)]\boldsymbol{\varphi}_i(p)=1 \tag{36}$$

Then the eigenvector derivatives $\boldsymbol{\varphi}_i'$ can be obtained by solving Eq. (34) in which the eigenvalue derivatives λ_i' is given by

$$\lambda_i' = -\boldsymbol{\varphi}_i^{\mathrm{T}}(\lambda_i^2\mathbf{M}'+\lambda_i\mathbf{C}'+\mathbf{K}')\boldsymbol{\varphi}_i \tag{37}$$

The method has been described in Ref. [20].

Noting $\mu=0$ in the solution to Eq. (34), we may rewrite the equation into the following equivalent form:

$$\begin{bmatrix} \lambda_i^2\mathbf{M}+\lambda_i\mathbf{C}+\mathbf{K} & (2\lambda_i\mathbf{M}+\mathbf{C})\boldsymbol{\varphi}_i \\ \boldsymbol{\varphi}_i^{\mathrm{T}}(2\lambda_i\mathbf{M}+\mathbf{C}) & \boldsymbol{\varphi}_i^{\mathrm{T}}\mathbf{M}\boldsymbol{\varphi}_i \end{bmatrix}\begin{bmatrix} \boldsymbol{\varphi}_i' \\ \lambda_i' \end{bmatrix} = \begin{bmatrix} -(\lambda_i^2\mathbf{M}'+\lambda_i\mathbf{C}'+\mathbf{K}')\boldsymbol{\varphi}_i \\ -0.5\boldsymbol{\varphi}_i^{\mathrm{T}}(2\lambda_i\mathbf{M}'+\mathbf{C}')\boldsymbol{\varphi}_i \end{bmatrix} \tag{38}$$

Thus we can directly get the eigensolution derivatives λ_i' and $\boldsymbol{\varphi}_i'$ by solving Eq. (38). Note that Eq. (38) is identical to that developed in Ref. [21].

In a word, the proposed method for the asymmetric damped systems can also applicable to symmetric damped systems discussed in Refs. [20,21].

5. A numerical example

The validity of the proposed method will be demonstrated by the following example.

Example. A finite element rotor dynamic model will be used to demonstrate the efficiency of the method developed in this paper. A schematic drawing of the rotor on flexible supports is shown in Fig. 1. The rotor system consists of a flexible variable section shaft supported at the first and fourth nodes by two bearings and two disks located at the third and fifth nodes. The rotor shaft is modeled using Timoshenko beam189 model which is a quadratic (3-node) beam element in 3-D, and in which the gyroscopic and the inertia of rotation effects are taken into account, and the two bearings are modeled using springs and dashpots. So the rotor model is an asymmetric damped system. As shown in Fig. 2, the diameters of the variable section shaft are $d_1=0.07$ m, $d_2=0.1$ m, $d_3=0.12$ m and $d_4=0.08$ m, respectively, the lengths are $L_1=0.6$ m and $L_2=0.2$ m, Young's module $E=2.1\times10^{11}$ Pa, density $\rho=7850\,\mathrm{kg/m^3}$ and Poisson's ratio $\nu=0.3$. The disk 1 and disk 2 have masses $m_1=20$ kg and $m_2=50$ kg, moments of inertia $J_{1,0}=1\,\mathrm{kg\,m^2}$ and $J_{2,0}=2\,\mathrm{kg\,m^2}$, polar moments of inertia $J_{1,p}=2\,\mathrm{kg\,m^2}$ and $J_{2,p}=4\,\mathrm{kg\,m^2}$, respectively. Both disks have the same material as the shaft. The spring stiffness coefficients and the damping coefficients have the following numerical values: $K_{y1}=K_{y2}=10^7$ N/m, $K_{z1}=K_{z2}=10^8$ N/m, $C_{y1}=C_{y2}=10^3$ N s/m and $C_{z1}=C_{z2}=2\times10^3$ N s/m. The coupling stiffness and damping coefficients of the bearings are assumed to be negligible, and the damping in the system is only due to the bearings.

The rotor shaft is discretized into 10 shaft elements, each of which has three nodes (two end nodes and one midside node), and all the shaft elements have the same length $L=0.2$ m. When we model the disk 1 at the node 3, two "special" beam188 elements, which are connecting the shaft and disk element, will be automatically created with disk element, at

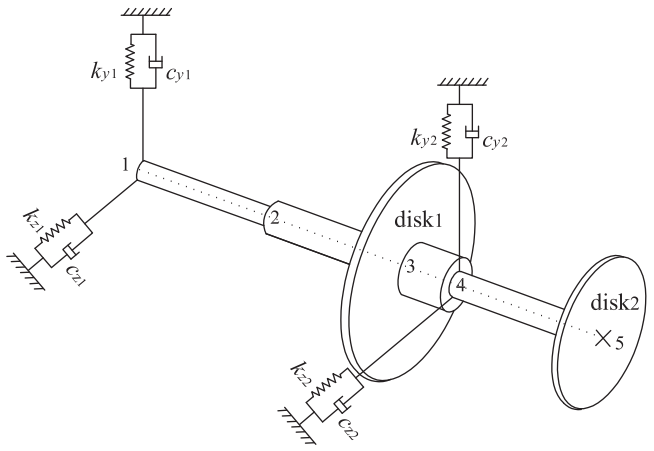

Fig. 1. A schematic drawing of the rotor with two disks.

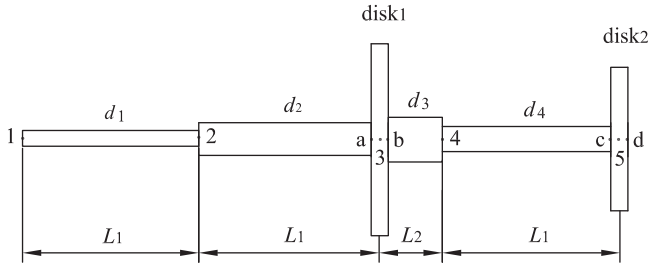

Fig. 2. The geometric dimension of the rotor.

Table 1
The first five eigenvalues and their first order derivatives.

Eigenvalues λ_i (i = 1, 2, 3, 4, 5)	Eigenvalues derivatives λ_i' $(i = 1, 2, 3, 4, 5)$
$-8.4343 \times 10^{-2}+1.1255 \times 10^{+2}i$	$-4.7727 \times 10^{-1}+1.4512 \times 10^{+2}i$
$-1.2584 \times 10^{+0}+2.2968 \times 10^{+2}i$	$-6.2029 \times 10^{+0}+8.6917 \times 10^{+1}i$
$-2.4071 \times 10^{+0}+3.4216 \times 10^{+2}i$	$1.1652 \times 10^{+1}-1.5570 \times 10^{+2}i$
$-7.4957 \times 10^{-1}+4.7114 \times 10^{+2}i$	$1.8970 \times 10^{+0}-7.5338 \times 10^{+0}i$
$-2.28166 \times 10^{+0}+6.2992 \times 10^{+2}i$	$-4.4362 \times 10^{+0}+5.8252 \times 10^{+1}i$

Table 2
The first right eigenvector and its first order derivative.

DOF	First right eigenvector $\boldsymbol{\varphi}_1$	First right eigenvector derivative $\boldsymbol{\varphi}_1'$
1	$2.9140 \times 10^{-4}-2.9714 \times 10^{-4}i$	$6.0456 \times 10^{-4}-6.1991 \times 10^{-4}i$
2	$-2.8769 \times 10^{-5}-2.8738 \times 10^{-5}i$	$-5.6383 \times 10^{-5}-5.6737 \times 10^{-5}i$
3	$1.9286 \times 10^{-3}+1.9354 \times 10^{-3}i$	$2.9647 \times 10^{-3}+3.0136 \times 10^{-3}i$
4	$1.4401 \times 10^{-3}-1.4209 \times 10^{-3}i$	$1.9054 \times 10^{-3}-1.8096 \times 10^{-3}i$
⋮	⋮	⋮
97	$-3.3594 \times 10^{-3}+3.3727 \times 10^{-3}i$	$-3.01063 \times 10^{-4}+3.5185 \times 10^{-4}i$
98	$2.4844 \times 10^{-3}+2.4927 \times 10^{-3}i$	$-2.0015 \times 10^{-3}-1.9626 \times 10^{-3}i$
99	$-5.8862 \times 10^{-3}-5.9056 \times 10^{-3}i$	$5.5933 \times 10^{-4}+4.5476 \times 10^{-4}i$
100	$-6.4940 \times 10^{-3}+6.4886 \times 10^{-3}i$	$1.03329 \times 10^{-3}+9.8237 \times 10^{-4}i$

the same time, two nodes a and b are also automatically created at the locations of half disk thickness at both sides of node 3. At the node 5, the case is same and nodes c and d are automatically created, see Fig. 2. Thus the system has $25(3 \times 10-9+2 \times 2)$ nodes. Each node has four degrees of freedom: the transverse displacements and the rotations in both the XY and XZ planes. Therefore, the model has 100 degrees of freedom, and the degrees of freedom are ordered as follows:

$$q = [y_1, z_1, \theta_{y1}, \theta_{z1}, \ldots, y_{25}, z_{25}, \theta_{y25}, \theta_{z25}]^{\mathrm{T}}$$

The gyroscopic effect of both disks has also been included in the model, and the rotor speed is chosen to be 7000 rev/min.

The system is modeled using the Rotordynamics module for ANSYS, from which we extract the system matrices and then compute the sensitivities using our own program. The computation is completed on a PC: Pentium 4, 2.6 GHz CPU, 512 MB RAM. To demonstrate the calculation of the eigensolution derivatives, we choose the design parameter p to be the shaft diameter d_3. Table 1 shows the first five eigenvalues and their first-order derivatives obtained by using the proposed method.

Tables 2 and 3 show some components of the first right and left eigenvectors and their first-order derivatives computed by using Eqs. (21) and (26), respectively. Here, we have used the following conditions adopted in Refs. [4,14,15]:

$$\{\boldsymbol{\varphi}_1\}_j = \{\boldsymbol{\psi}_1\}_j \quad \text{and} \quad |\{\boldsymbol{\varphi}_1\}_j||\{\boldsymbol{\psi}_1\}_j| = \max_k(|\{\boldsymbol{\varphi}_1\}_k||\{\boldsymbol{\psi}_1\}_k|)$$

by Eq. (12) with $p = p_0$ to determine the right and left eigenvectors $\boldsymbol{\varphi}_1$ and $\boldsymbol{\psi}_1$ to be differentiated. We may also select an arbitrary right eigenvector [the corresponding left eigenvector is then determined by Eq. (12) with $p = p_0$] to compute the derivative regardless of the conditions above.

The computation times for both Guedria et al.'s method in Ref. [25] and the proposed method are compared. Fig. 3 shows the computation times for both methods with respect to the number of the computed eigensolution derivatives. In Fig. 3, the computation of the eigenpair derivatives has been repeated 10 times and the average computation times are

Table 3
The first left eigenvector and its first order derivative.

DOF	First left eigenvector ψ_1	First left eigenvector derivative ψ_1'
1	$2.9140\times10^{-4}-2.9714\times10^{-4}i$	$3.9870\times10^{-4}-4.1062\times10^{-4}i$
2	$2.8769\times10^{-5}+2.8738\times10^{-5}i$	$3.6121\times10^{-5}+3.6435\times10^{-5}i$
3	$-1.9286\times10^{-3}-1.9354\times10^{-3}i$	$-1.6063\times10^{-3}-1.6464\times10^{-3}i$
4	$1.4401\times10^{-3}-1.4209\times10^{-3}i$	$8.8808\times10^{-4}-8.0878\times10^{-4}i$
⋮	⋮	⋮
97	$-3.3594\times10^{-3}+3.3727\times10^{-3}i$	$2.0721\times10^{-3}-2.0236\times10^{-3}i$
98	$-2.4844\times10^{-3}-2.4927\times10^{-3}i$	$3.7512\times10^{-3}+3.7235\times10^{-3}i$
99	$5.8862\times10^{-3}+5.9056\times10^{-3}i$	$-4.7050\times10^{-3}-4.6266\times10^{-3}i$
100	$-6.4940\times10^{-3}+6.48836\times10^{-3}i$	$3.5544\times10^{-3}-3.5876\times10^{-3}i$

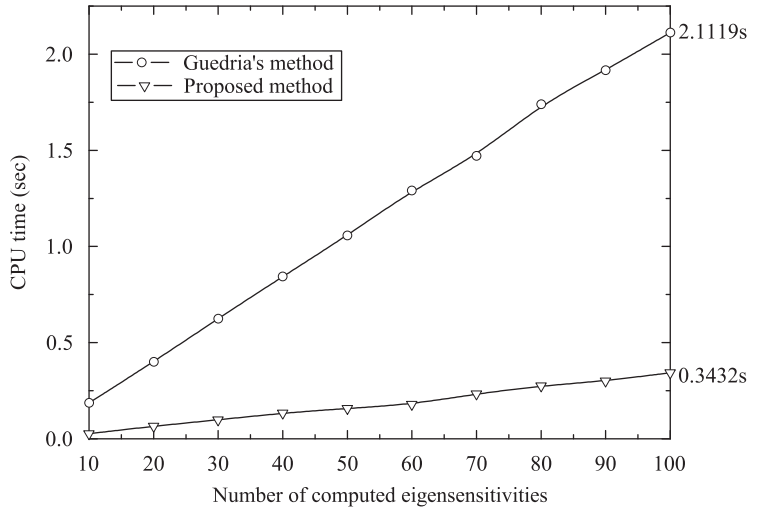

Fig. 3. Comparison of CPU times for Guedria et al.'s method and the proposed method.

adopted. As we can see in Fig. 3, the CPU time to obtain all the eigensolution derivatives is 2.1119 s for Guedria et al.'s method and 0.3432 s for the proposed method. Thus Guedria et al.'s method requires a computation time roughly 6.154 times that of the proposed method. For a system with N degrees of freedom, Guedria et al.'s method requires solving a system of $2N+1$ linear equations by performing one LU decomposition of its coefficient matrix, the proposed method needs to solve two systems of $N+1$ linear equations by performing only one LU decomposition of the corresponding coefficient matrix. Therefore, the proposed method is more efficient than Guedria et al.'s method.

6. Conclusions

An efficient algebraic method for the computation of eigensolution derivatives for asymmetric damped systems with distinct eigenvalues has been proposed. By solving two systems of linear equations, we can obtain the derivatives of the eigenvalues and their associated right and left eigenvectors directly. One of the coefficient matrices of two systems of linear equations is the transpose of another and has small dimension. The proposed method maintains N-space without use of state space equation, requires only the knowledge of the right and left eigenvectors under consideration, gives exact solution and guarantees numerical stability. Thus the proposed method saves the CPU computation time and the storage space. In addition, the proposed method can be extended to compute the higher order eigensolution derivatives with less computing effort. The algorithm is simple and compact and easy to be implemented on computers. A numerical example has demonstrated the validity of the proposed method.

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