Journal of Sound and Vibration 328 (2009) 291–300

Contents lists available at ScienceDirect

Journal of Sound and Vibration

journal homepage: www.elsevier.com/locate/jsvi

Semi-analytical determination of natural frequencies and mode shapes of multi-span bridge decks

A. Rezaiguia [a,*], D.F. Laefer [b,1]

[a] Mechanics & Structures Laboratory, Guelma University, P.O.B 401, Guelma, 24000, Algeria
[b] Urban Modelling Group, School of Architecture, Landscape, and Civil Engineering, University College Dublin, Belfield, Dublin 4, Ireland

A R T I C L E I N F O

Article history:
Received 13 December 2008
Received in revised form
8 August 2009
Accepted 9 August 2009
Handling Editor: C.L. Morfey
Available online 27 August 2009

A B S T R A C T

The determination of the natural frequencies and mode shapes of structures requires an analytical, semi-analytical or numerical method. This paper presents a new semi-analytical approach to determine natural frequencies and mode shapes of a multi-span, continuous, orthotropic bridge deck. The suggested approach is based on the modal method, which differs from other approaches in its decomposition of the admissible functions defining the mode shapes. Implementation of this technique is simple and enables avoidance of cumbersome mathematical calculations. In this paper, application of the semi-analytic approach to a three-span, orthotropic roadway bridge deck is compared in the first 16 modes of previously published fully analytical results and to a finite element method analysis. The simplified implementation matches within 2 percent in all cases, with the additional benefit of including intermodal coupling. The approach can be extended to similar bridges with more than three spans.

1. Introduction

In bridge design, dynamic analysis offers a more complicated but potentially more critical assessment. As a part of this, the analysis of free vibrations of roadway bridge decks is the first essential step to study forced vibrations due to passing vehicles. Such analysis requires determination of the natural frequencies and mode shapes. In general, roadway bridge decks have a rectangular form, which may be continuous over a number of intermediate line supports in the longitudinal direction and free in the transverse direction. In previous research, bridge decks have been modeled as thin, rectangular, isotropic or orthotropic plates [1,2] to consider the dynamic effects of roadway traffic on bridges and the resulting dynamic amplification factors from which major static effects are used to check limiting states.

Several methods and techniques have been developed previously to determine natural frequencies and natural mode shapes of multi-span plates. Among the related studies, analytical methods represent a considerable portion. For example, Veletsos and Newmark [3] used Holzer's method for torsional vibration of shafts to determine natural frequencies of plates simply supported along the continuous edges. Dickinson and Warburton [4] utilized Bolotin's edge-effect method [5,6] for the study of two-span plates involving clamped, simply supported end, free edges. The modified Bolotin method, developed by Vijayakumar [7] and Elishakoff [8], was applied by Elishakoff and Sternberg [9] to determine eigenfrequencies of rectangular plates, with continuous over line supports with an arbitrary number of equal spans in one direction. More recently, the receptance method was exploited by Azimit et al. [10] in a similar application. Gorman and Garibaldi [11]

* Corresponding author. Tel.: +213 791320276; fax: +213 37207268.
E-mail addresses: a.rezaiguia@gmail.com (A. Rezaiguia), debra.laefer@ucd.ie (D.F. Laefer).
[1] Tel.: +353 1716 3226; fax: +353 1716 3297.

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2009.08.007

Nomenclature

a_{ij}	modal amplitudes
b	width of the bridge deck
D_x, D_y	flexural rigidities for the x- and y-directions, respectively
D_{xy}	flexural rigidity for the x–y plane
E_x, E_y	Young's moduli for the x- and y-directions, respectively
G_{xy}	shear modulus in bending for the x–y plane
h	thickness of the bridge deck
$h_{ij}(y)$	eigenfunctions of single span Euler–Bernoulli beam satisfying the boundary conditions of a plate for the y-direction
H	equivalent rigidity of the bridge deck
k_i, k_{1i}	eigenvalues
l	length of the bridge deck
l_i	length of the ith span of the bridge deck
t	time
$w(x, y, t)$	vertical displacement of the bridge deck
x, y, z	axis of the reference system
v_{xy}, v_{yx}	Poisson's ratios
ρ	mass density of the bridge deck
$\phi_{ij}(x,y)$	mode shapes of multi-span continuous bridge-deck
$\varphi_i(x)$	eigenfunctions of multi-span continuous Euler–Bernoulli beam
$\varphi_{ri}(x_r)$	ith mode shape in the rth span of the bridge deck for the x-direction
$\psi_j(y)$	eigenfunctions of single span Euler–Bernoulli beam
ω_{ij}	natural frequency of multi-span continuous bridge deck

applied the superposition method and the span-by-span approach to obtain an accurate analytical solution for free vibration of multi-span bridge decks. Zhou [12], Zhu and Law [13], and Marchesiello et al. [14] employed eigenfunctions of continuous multi-span beam in one direction and single-span beam in the other direction into the Rayleigh–Ritz method for determination of eigenfrequencies of a continuous multi-span rectangular bridge deck.

These analytical and semi-analytical methods are precise, but they are limited to geometrically simple plates. Therefore, numerical methods may be considered to be more powerful, alternative tools for analysis of plates with complex geometries. Among these methods, the finite element method is dominant [15–18]. Arguably, problems having regular geometry can be solved more efficiently by approximate techniques. Cheung et al. [19] used the finite-strip method, while Wu and Cheung [20] devised a method of finite elements in conjunction with Bolotin's method to analyze continuous plates in two directions. The transfer matrix method was developed by Mercer and Seavey [21] for analysis of such plates. Plates with mixed boundary conditions, however, require other techniques. For example, Keer and Sthal [22] used Fredholm integral equations to calculate the eigenfrequencies of a simply supported plate partially clamped on the edge, while the differential quadrature method proposed by Bellman et al. [23] was adopted by Laura and Gutierez [24] and Lu et al. [25].

This paper presents a new, semi-analytical approach to determine the natural frequencies and the natural mode shapes of a multi-span continuous roadway bridge deck. The bridge deck is modeled as a three-span, continuous, orthotropic, rectangular plate with intermediate line rigid supports. The suggested approach is based on the modal method, which differs from other approaches in the decomposition of the admissible functions defining the mode shapes. The implementation of this method is simple and generates very satisfactory results in comparison with previously published values.

2. Natural frequencies and mode shapes of a bridge deck

The bridge deck is modeled as a continuous, rectangular, orthotropic plate of length l, width b, uniform thickness h and mass density ρ as shown in Fig. 1. The bridge deck is simply supported at two ends ($x = 0, l$), and the other edges are free ($y = 0, b$). A linear elastic behavior is assumed, and the effects of shear deformation and rotary inertia are neglected. The intermediate line supports of the bridge are linear, rigid, and orthogonal to the free edges of the plate. Since the horizontal dimension of the bridge deck is much larger than its thickness, a thin plate assumption is used. With these assumptions,

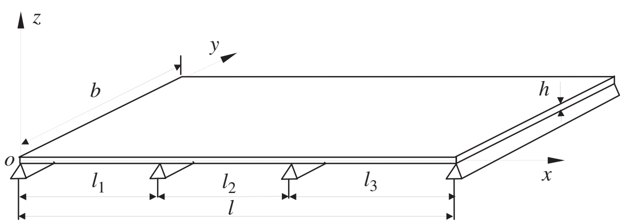

Fig. 1. Model of the continuous three span bridge deck.

the governing differential equation of free vibration of the orthotropic plate is given by

$$\rho h \frac{\partial^2 w}{\partial t^2} + D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2\, \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} = 0 \tag{1}$$

where $w(x,y,t)$ is the vertical displacement of the plate in z-direction, $D_x = E_x h^3/12(1-\nu_{xy}\nu_{yx})$, $D_y = E_y h^3/12(1-\nu_{xy}\nu_{yx})$, $D_{xy} = G_{xy} h^3/12$, and $H = \nu_{xy} D_y + 2D_{xy}$ are flexural rigidities, in which E_x and E_y are Young's moduli in the x- and y-directions respectively, G_{xy} is the shear modulus, ν_{xy} and ν_{yx} are Poisson's ratios. Using modal superposition, the vertical displacement for free vibration of the plate [3] may be expressed as

$$w(x,y,t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \phi_{ij}(x,y) \mathrm{e}^{\mathrm{J}\omega_{ij} t} \tag{2}$$

where $\phi_{ij}(x,y)$ are the mode shapes of a multi-span continuous bridge deck corresponding to the ith mode in the x-direction and jth mode in the y-direction with associated natural frequency ω_{ij}, and a_{ij} is the unknown modal amplitude, while t is time, and $\mathrm{J} = \sqrt{-1}$.

Substituting expression (2) into Eq. (1) generates Eq. (3) in the space variables x, y:

$$D_x \frac{\partial^4 \phi_{ij}}{\partial x^4} + 2H \frac{\partial^4 \phi_{ij}}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 \phi_{ij}}{\partial y^4} - \rho h \omega_{ij}^2 \phi_{ij} = 0 \tag{3}$$

Such an approach (e.g. [11]) uses a set of functions that constitute a complete set (in the sense functional analysis definition). This ensures the uniform convergence of the solutions to the classic (exact) ones, with the advantage of the approach being in the general fashion of the admissible functions.

Furthermore, several authors [12–14] use the Rayleigh–Ritz method to determine the natural frequencies of the vibration of the bridge deck. These authors decompose $\phi_{ij}(x,y)$ as the product of two admissible functions: $\varphi_i(x)$ and $\psi_j(y)$, which are eigenfunctions of the continuous multi-span Euler–Bernoulli beam and eigenfunctions of the single-span, Euler–Bernoulli beam, thereby, satisfying the boundary conditions in the x- and y-directions, respectively, and also the boundary and continuity conditions at the rigid line supports. This decomposition neglects the intermodal coupling. Moreover, several integrals must be evaluated.

To take account of the intermodal coupling, one considers $\varphi_i(x)$ as the mode shapes of a continuous, Euler–Bernoulli beam in the x-direction. While in the y-direction, mode shapes are presented by function $h_{ij}(y)$, thus satisfying the boundary conditions of a plate at the free edges $y = 0$ and b of the bridge deck. This technique is simple and makes it possible to avoid cumbersome mathematical calculation. This decomposition may be expressed as

$$\phi_{ij}(x,y) = \varphi_i(x) h_{ij}(y) \tag{4}$$

The mode shapes $\varphi_i(x)$ of a continuous three-span Euler–Bernoulli beam in x-direction are shown in Eq. (5) and further detailed in Appendix A:

$$\varphi_i(x) = \begin{cases} A_{1i}\left(\sin(k_i x) - \dfrac{\sin(k_i l_1)}{sh(k_i l_1)} sh(k_i x)\right) & \text{for } 0 \le x \le l_1 \\ A_{2i}\left(\sin(k_i(x-l_1)) - \dfrac{\sin(k_i l_2)}{sh(k_i l_2)} sh(k_i(x-l_1))\right) + B_{2i}(\cos(k_i(x-l_1)) & \\ \quad -ch(k_i(x-l_1)) + \dfrac{ch(k_i l_2) - \cos(k_i l_2)}{sh(k_i l_2)} sh(k_i(x-l_1))) & \text{for } l_1 \le x \le l_1 + l_2 \\ A_{3i}\left(\sin(k_i(l-x)) - \dfrac{\sin(k_i l_3)}{sh(k_i l_3)} sh(k_i(l-x))\right) & \text{for } l_1 + l_2 \le x \le l \end{cases} \tag{5}$$

The differential Eq. (3) must be satisfied for all values of x, but determining its resolution for every value of x is practically impossible to achieve. For this reason, it is proposed to substitute expression (4) into Eq. (3), then multiply it by $\varphi_i(x)$ and integrate the equation over the bridge length. From this one obtains

$$D_y \frac{\mathrm{d}^4 h_{ij}}{\mathrm{d}y^4} \int_0^l \varphi_i^2 \,\mathrm{d}x + 2H \frac{\mathrm{d}^2 h_{ij}}{\mathrm{d}y^2} \int_0^l \varphi_i'' \varphi_i \,\mathrm{d}x + (D_x k_i^4 - \rho h \omega_{ij}^2) h_{ij} \int_0^l \varphi_i^2 \,\mathrm{d}x = 0 \tag{6}$$

Dividing Eq. (6) by $D_y \int_0^l \varphi_i^2 \,\mathrm{d}x$, one obtains

$$\frac{\mathrm{d}^4 h_{ij}}{\mathrm{d}y^4} + \frac{2Hk_{1i}^2}{D_y} \frac{\mathrm{d}^2 h_{ij}}{\mathrm{d}y^2} + \left(\frac{D_x k_i^4 - \rho h \omega_{ij}^2}{D_y}\right) hij = 0 \tag{7}$$

with

$$k_{1i} = \sqrt{\int_0^l \phi_i'' \varphi_i \,\mathrm{d}x \Big/ \int_0^l \varphi_i^2 \,\mathrm{d}x} \tag{8}$$

Integrals that appear in expression (8) of the new frequency parameter k_{1i} are simple to calculate. Hence, the solution of Eq. (7) is given by the general form in

$$h_{ij}(y) = A_{ij} e^{s_{ij} y} \tag{9}$$

Substituting expression (9) into Eq. (7), one obtains

$$s_{ij}^4 - \frac{2Hk_{1i}^2}{D_y} s_{ij}^2 + \frac{D_x k_i^4 - \rho h \omega_{ij}^2}{D_y} = 0 \tag{10}$$

Solutions of Eq. (10) are as follows:

$$s_{1ij} = \pm \frac{1}{\sqrt{D_y}} \sqrt{Hk_{1i}^2 + \sqrt{H^2 k_{1i}^4 - D_y (D_x k_i^4 - \bar{m}\omega_{ij}^2)}} = \pm r_{1ij} \tag{11a}$$

$$s_{2ij} = \pm \mathrm{J} \frac{1}{\sqrt{D_y}} \sqrt{Hk_{1i}^2 - \sqrt{H^2 k_{1i}^4 - D_y (D_x k_i^4 - \bar{m}\omega_{ij}^2)}} = \pm \mathrm{J} r_{2ij} \tag{11b}$$

Note that the parameters r_{1ij} and r_{2ij} are not independent but are related by the pulsations ω_{ij}. In order to reduce the writing, one omits the $_{ij}$ indices in r_{1ij} and r_{2ij}. Substituting solutions (11a and 11b) into expression (9), one obtains

$$h_{ij}(y) = A_{1ij} e^{r_1 y} + A_{2ij} e^{-r_1 y} + A_{3ij} e^{\mathrm{J} r_2 y} + A_{4ij} e^{-\mathrm{J} r_2 y} \tag{12}$$

where A_{1ij}, A_{2ij}, A_{3ij} and A_{4ij} are constants of integrations. The exponential functions can be expressed by trigonometric and hyperbolic functions. The Eq. (12) can be written as

$$h_{ij}(y) = C_{ij} \sin(r_2 y) + D_{ij} \cos(r_2 y) + E_{ij} sh(r_1 y) + F_{ij} ch(r_1 y) \tag{13}$$

where C_{ij}, D_{ij}, E_{ij} and F_{ij} are new constants of integration. They are determined by the application of the boundary conditions at the free edges of the bridge: $y = 0$ and b. At these edges, the bending moment and the shear force are zero, thus

$$D_y \frac{\partial^2 w}{\partial y^2}(x,0,t) + \nu_{yx} D_x \frac{\partial^2 w}{\partial x^2}(x,0,t) = 0$$

$$D_y \frac{\partial^3 w}{\partial y^3}(x,0,t) + (\nu_{yx} D_x + 4D_{xy}) \frac{\partial^3 w}{\partial x^2 \partial y}(x,0,t) = 0$$

$$D_y \frac{\partial^2 w}{\partial y^2}(x,b,t) + \nu_{yx} D_x \frac{\partial^2 w}{\partial x^2}(x,b,t) = 0$$

$$D_y \frac{\partial^3 w}{\partial y^3}(x,b,t) + (\nu_{yx} D_x + 4D_{xy}) \frac{\partial^3 w}{\partial x^2 \partial y}(x,b,t) = 0 \tag{14}$$

Taking account of the expressions (2) and (4), the boundary conditions for Eq. (14) become:

$$D_y \frac{\mathrm{d}^2 h_{ij}}{\mathrm{d}y^2}(0) - \nu_{yx} D_x k_{1i}^2 h_{ij}(0) = 0$$

$$D_y \frac{\mathrm{d}^3 h_{ij}}{\mathrm{d}y^3}(0) - (\nu_{yx} D_x + 4D_{xy}) k_{1i}^2 \frac{\mathrm{d}h_{ij}}{\mathrm{d}y}(0) = 0$$

$$D_y \frac{\mathrm{d}^2 h_{ij}}{\mathrm{d}y^2}(b) - \nu_{yx} D_x k_{1i}^2 h_{ij}(b) = 0$$

$$D_y \frac{\mathrm{d}^3 h_{ij}}{\mathrm{d}y^3}(b) - (\nu_{yx} D_x + 4D_{xy}) k_{1i}^2 \frac{\mathrm{d}h_{ij}}{\mathrm{d}y}(b) = 0 \tag{15}$$

The application of the boundary conditions from Eq. (15) in Eq. (13), gives the following system (omitting the indices $_{ij}$ in r_{1ij}, r_{2ij}, α_{ij}, θ_{ij}, γ_{ij}, and χ_{ij}) as shown in

$$\begin{bmatrix} 0 & \alpha & 0 & \theta \\ \gamma & 0 & \chi & 0 \\ \alpha \sin(r_2 b) & \alpha \cos(r_2 b) & \theta sh(r_1 b) & \theta ch(r_1 b) \\ \gamma \cos(r_2 b) & -\gamma \sin(r_2 b) & \chi ch(r_1 b) & \chi sh(r_1 b) \end{bmatrix} \begin{Bmatrix} Cij \\ Dij \\ Eij \\ Fij \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{16}$$

with

$$\alpha = -D_y r_2^2 - \nu_{yx} D_x k_{1i}^2 \tag{17a}$$

$$\theta = D_y r_1^2 - \nu_{yx} D_x k_{1i}^2 \tag{17b}$$

$$\gamma = -D_y r_2^3 - (\nu_{yx} D_x + 4D_{xy}) r_2 k_{1i}^2 \tag{17c}$$

$$\chi = D_y r_1^3 - (\nu_{yx} D_x + 4D_{xy}) r_1 k_{1i}^2 \tag{17d}$$

For non-trivial solutions of the system (16), the frequency equation is

$$2\alpha\theta\gamma\chi(\cos(r_2 b)ch(r_1 b) - 1) + (\theta^2\gamma^2 - \alpha^2\chi^2)\sin(r_2 b)sh(r_1 b) = 0 \tag{18}$$

The parameters r_1 or r_2 can be solved from Eq. (18), while the natural frequency ω_{ij} can be obtained from expressions (11a) and (11b).

To determine the natural mode shapes of the bridge, one simplifies the system (16) by the standardization of the first component C_{ij} of the unknown vector with 1, thereby reducing the problem to four equations with three unknown. One then chooses three equations among the four available:

$$\begin{bmatrix} \alpha & 0 & \theta \\ 0 & \chi & 0 \\ \alpha\cos(r_2 b) & \theta sh(r_1 b) & \theta ch(r_1 b) \end{bmatrix} \begin{Bmatrix} Dij \\ Eij \\ Fij \end{Bmatrix} + \begin{Bmatrix} 0 \\ \gamma \\ \alpha\sin(r_2 b) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{19}$$

From which one obtains the expressions for the constants D_{ij}, E_{ij}, and F_{ij}:

$$D_{ij} = \left(\alpha\sin(r_2 b) - \frac{\gamma\theta}{\chi} sh(r_1 b)\right) \Big/ (\alpha ch(r_1 b) - \alpha\cos(r_2 b)) \tag{20a}$$

$$E_{ij} = -\frac{\gamma}{\chi} \tag{20b}$$

$$F_{ij} = \left(-\alpha\sin(r_2 b) + \frac{\gamma\theta}{\chi} sh(r_1 b)\right) \Big/ (\theta ch(r_1 b) - \theta\cos(r_2 b)) \tag{20c}$$

Finally, the mode shapes of the multi-span bridge deck are represented by

$$\phi_{ij}(x,y) = \varphi_i(x)\{\sin(r_2 y) + D_{ij}\cos(r_2 y) + E_{ij} sh(r_1 y) + F_{ij} ch(r_1 y)\} \tag{21}$$

3. Numerical example

In order to verify the suggested approach with other approaches, a numerical example was prepared. The bridge deck was modeled as an orthotropic, three-span plate. The following features of bridge deck were as reported elsewhere

Table 1
Mesh density convergence.

Mode shapes	Order of frequencies	Natural frequencies (Hz)		
		156 × 20	312 × 40	468 × 60
1	1.1	4.13	4.13	4.13
2	1.2	5.45	5.45	5.45
3	2.1	6.30	6.30	6.30
4	2.2	7.59	7.59	7.59
5	3.1	7.76	7.76	7.76
6	3.2	8.79	8.79	8.79
7	1.3	9.05	9.01	9.00
8	2.3	11.29	11.24	11.23
9	3.3	12.06	12.02	12.01
10	1.4	15.09	14.91	14.88
11	4.1	15.80	15.80	15.80
12	4.2	17.18	17.17	17.17
13	2.4	17.52	17.30	17.26
14	3.4	17.98	17.77	17.73
15	4.3	21.28	21.19	21.17
16	5.1	22.28	22.28	22.28

[13,26]: $l = 78\,\text{m}$, $l_1 = l_3 = 24\,\text{m}$, $l_2 = 30\,\text{m}$, $b = 13.715\,\text{m}$, $h = 0.21157\,\text{m}$, $\rho = 3265.295\,\text{kg}\,\text{m}^{-3}$, $D_x = 2.415 \times 10^9\,\text{N}\,\text{m}$, $D_y = 2.1807 \times 10^7\,\text{N}\,\text{m}$, $D_{xy} = 1.1424 \times 10^8\,\text{N}\,\text{m}$, $v_{xy} = 0.3$, $E_x = 3.0576 \times 10^{12}\,\text{N}\,\text{m}^{-2}$, $E_y = 2.7607 \times 10^{10}\,\text{N}\,\text{m}^{-2}$, $G_{xy} = 1.4475 \times 10^{11}\,\text{N}\,\text{m}^{-2}$.

To calculate the natural frequencies of the orthotropic bridge deck, first k_i values were calculated (see appendix) and the k_{1i} values using expression (8). Subsequently, Mathematica software was used to determine the roots r_{1ij} or r_{2ij} of the frequency Eq. (18). Finally, natural frequencies of the bridge ω_{ij} were calculated by expressions (11a) and (11b). By comparing the natural frequencies obtained by the newly devised approach with those previously published by Zhu and Law [13], and those calculated with a finite element method using ANSYS software v.10, the method was verified.

To obtain the ANSYS results, firstly all material properties of orthotropic three span bridge deck herein reported were numerically modeled. The bridge deck consisted of a fine mesh 468×60 resulting in 28 080 elements of type shell63 (6 DOF per node). The decision was made to calculate 16 modes. Convergence according to the mesh density is presented in Table 1.

Table 2 summarizes the differences between the values for the first 16 natural frequencies of the bridge. Excellent agreement is observed for all the frequencies with the ANSYS results (errors not exceeding 2 percent). This is mainly due to

Table 2
Comparison of natural frequencies of the bridge deck.

Mode shapes	Order of frequencies	Natural frequencies (Hz)			Error (%)	
		Proposed approach	ANSYS	Zhu and Law [13]	Proposed approach/ ANSYS	Proposed approach/Zhu and Law [13]
1	1.1	4.13	4.13	4.13	0.00	0.00
2	1.2	5.45	5.45	4.70	0.00	13.76
3	2.1	6.30	6.30	6.31	0.00	−0.16
4	2.2	7.59	7.59	6.86	0.00	9.62
5	3.1	7.75	7.76	7.76	−0.13	−0.13
6	3.2	8.77	8.79	8.20	−0.23	6.50
7	1.3	9.08	9.00	–	0.88	–
8	2.3	11.26	11.23	–	0.27	–
9	3.3	11.97	12.01	–	−0.33	–
10	1.4	15.07	14.88	–	1.26	–
11	4.1	15.79	15.80	15.81	−0.06	−0.13
12	4.2	17.16	17.17	16.39	−0.06	4.49
13	2.4	17.33	17.26	–	0.40	–
14	3.4	17.65	17.73	–	−0.45	–
15	4.3	21.19	21.17	–	0.09	–
16	5.1	22.27	22.28	22.29	−0.04	−0.09

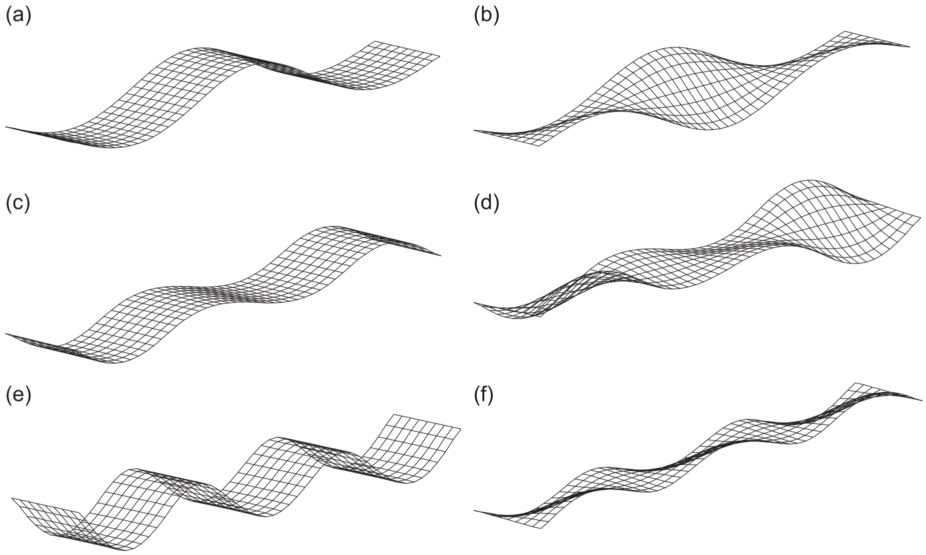

Fig. 2. The first six mode shapes of the three span, bridge deck obtained through the proposed approach. Modes: (a) 1, $f_1 = 4.13$ Hz; (b) 2, $f_2 = 5.45$ Hz; (c) 3, $f_3 = 6.30$ Hz; (d) 4, $f_4 = 7.59$ Hz; (e) 5, $f_5 = 7.75$ Hz; and (f) 6, $f_6 = 8.77$ Hz.

the weak influence of the side effects of shear deformation and rotary inertia, since the two ratios of span width and length of the bridge deck with respect to its height are very significant (65 and 114, respectively). According to Table 2, one sees that the natural frequencies of flexural mode shapes from Zhu and Law [13] are very close to the semi-analytically derived frequencies (error not exceeding 0.2 percent). This error becomes significant for the natural frequencies of torsional mode shapes. Zhu and Law [13] decomposed $\phi_{ij}(x,y)$ as the product of two admissible functions $\varphi_i(x)$ and $\psi_j(y)$, which are eigenfunctions of the continuous multi-span, simply supported beam, and eigenfunctions of a single-span, free beam,

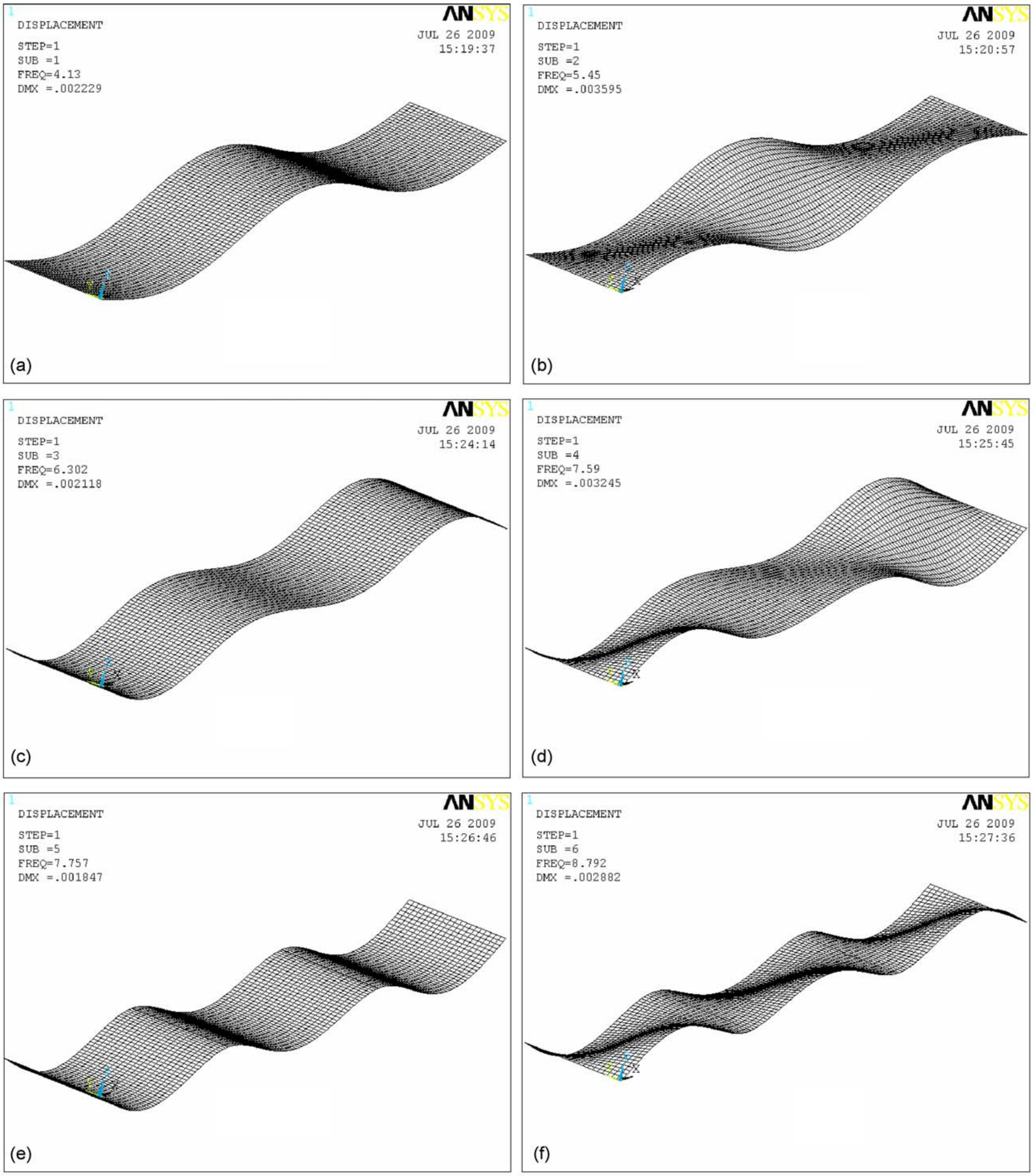

Fig. 3. The first six mode shapes of the three span, bridge deck obtained through ANSYS. Modes: (a) 1, $f_1 = 4.13$ Hz; (b) 2, $f_2 = 5.45$ Hz; (c) 3, $f_3 = 6.30$ Hz; (d) 4, $f_4 = 7.59$ Hz; (e) 5, $f_5 = 7.76$ Hz; and (f) 6, $f_6 = 8.79$ Hz.

respectively. This decomposition does not consider the effect of intermodal coupling. In contrast, the semi-analytic approach presented herein includes this effect by assuming that $\phi_{ij}(x,y) = \varphi_i(x)h_{ij}(y)$. Additionally, Zhu and Law [13] only considered two orders of torsional modes. In contrast, in Table 2 torsional modes of orders three and four appear before certain flexural and torsional modes of the first and second orders. Figs. 2 and 3 show the first six mode shapes of the bridge deck obtained by the proposed approach and ANSYS software, respectively. Excellent agreement between the mode shapes is seen.

4. Conclusion

In this paper, a new semi-analytical approach is described to determine natural frequencies and mode shapes of a multi-span, orthotropic, roadway bridge deck. This approach treats the function defining the mode shapes of the bridge deck as being the product of two admissible functions. One defines the longitudinal mode shapes of the bridge deck as being the mode shapes of a continuous simply supported beam. The other defines the mode shapes of a free beam with boundary conditions of a free plate, to incorporate the effect of intermodal coupling, which is usually neglected because of the further complexity. This decomposition leads to a differential equation with only space coordinates, which is highly complex. To solve this, an average meaning integration is introduced. The obtained results show agreement within 2 percent of previously published results, with the advantage of a vastly simplified implementation

Appendix A. Mode shapes of a three-span, simply supported beam

To determine the natural mode shapes of a three-span, continuous, simply supported beam (Fig. A1), it is necessary to determine the natural mode shapes for each span, while taking into account the boundary conditions and the continuity conditions at the intermediate supports. Assuming that the flexural rigidity of the beam is the same for all spans; the expression of ith mode shape for the transverse vibration in the rth span is [26] as reflected in

$$\varphi_{ri}(x_r) = A_{ri}\sin k_i x_r + B_{ri}\cos k_i x_r + C_{ri} shk_i x_r + D_{ri} chk_i x_r, \quad r = 1,2,3 \tag{A.1}$$

where A_{ri}, B_{ri}, C_{ri} and D_{ri}, are determined by the application of the boundary conditions and the continuity conditions at the intermediate supports 1 and 2, k_i is the eigenvalue of the ith mode shape of three-span beam vibration.

The boundary conditions are as follows: the vertical deflection is equal to zero at all supports, and the bending moments are equal to zero at the ends, i.e.

$$\varphi_r(x_r)|_{x_r=0} = \varphi_r(x_r)|_{x_r=l_r} = 0, \quad r = 1,2,3$$

$$\left.\frac{\partial^2\varphi_1}{\partial x_1^2}\right|_{x_1=0} = \left.\frac{\partial^2\varphi_3}{\partial x_3^2}\right|_{x_3=0} = 0 \tag{A.2}$$

The slope and bending moments at the intermediate supports are continuity conditions:

$$\left.\frac{\partial\varphi_1}{\partial x_1}\right|_{x_1=l_1} = \left.\frac{\partial\varphi_2}{\partial x_2}\right|_{x_2=0}, \quad \left.\frac{\partial^2\varphi_1}{\partial x_1^2}\right|_{x_1=l_1} = \left.\frac{\partial^2\varphi_2}{\partial x_2^2}\right|_{x_2=0}$$

$$\left.\frac{\partial\varphi_2}{\partial x_2}\right|_{x_2=l_2} = -\left.\frac{\partial\varphi_3}{\partial x_3}\right|_{x_3=l_3}, \quad \left.\frac{\partial^2\varphi_2}{\partial x_2^2}\right|_{x_2=l_2} = \left.\frac{\partial^2\varphi_3}{\partial x_3^2}\right|_{x_3=l_3} \tag{A.3}$$

Thus, there are 12 boundary conditions for a three-span beam. Substituting the boundary and continuity conditions (A.2) and (A.3) into expression (A.1), after simplifications, one obtains expressions (A.4) for mode shapes of a continuous,

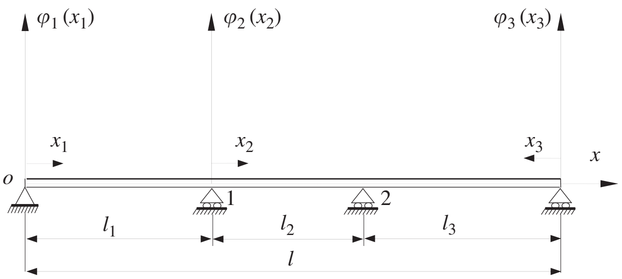

Fig. A1. Continuous three-span simply supported beam.

three-span simply supported beam:

$$\varphi_i(x) = \begin{cases} A_{1i}\left(\sin(k_i x) - \dfrac{\sin(k_i l_1)}{sh(k_i l_1)} sh(k_i x)\right) & \text{for } 0 \le x \le l_1 \\ A_{2i}\left(\sin(k_i(x-l_1)) - \dfrac{\sin(k_i l_2)}{sh(k_i l_2)} sh(k_i(x-l_1))\right) + B_{2i}\big(\cos(k_i(x-l_1)) & \\ \quad -ch(k_i(x-l_1)) + \dfrac{ch(k_i l_2) - \cos(k_i l_2)}{sh(k_i l_2)} sh(k_i(x-l_1))\big) & \text{for } l_1 \le x \le l_1 + l_2 \\ A_{3i}\left(\sin(k_i(l-x)) - \dfrac{\sin(k_i l_3)}{sh(k_i l_3)} sh(k_i(l-x))\right) & \text{for } l_1 + l_2 \le x \le l \end{cases} \tag{A.4}$$

with
A_{1i}: normalized values

$$A_{2i} = A_{1i}\left(\frac{\cos(k_i l_1) - \theta_1 ch(k_i l_1) - \Phi_2 \sin(k_i l_1)}{1-\theta_2}\right)$$

$$B_{2i} = A_{1i} \sin(k_i l_1)$$

$$A_{3i} = A_{2i}\left(\frac{\cos(k_i l_2) - \theta_2 ch(k_i l_2)}{\theta_3 ch(k_i l_3) - \cos(k_i l_3)}\right) - A_{1i}\left(\frac{\sin(k_i l_1)\sin(k l_2) + \sin(k_i l_1) sh(k_i l_2) - \Phi_2 \sin(k_i l_1) ch(k_i l_2)}{\theta_3 ch(k_i l_3) - \cos(k_i l_3)}\right)$$

$$\theta_r = \frac{\sin(k_i l_r)}{sh(k_i l_r)}, \quad r = 1,2,3 \quad \Phi_2 = \frac{ch(k_i l_2) - \cos(k_i l_2)}{sh(k_i l_2)} \tag{A.5}$$

The frequency equation is given by

$$\begin{aligned} & ch(k_i l_3)\sin(k_i l_3)(ch(k_i l_2)\sin(k_i l_1)\sin(k_i l_2) sh(k_i l_1) \\ & \quad + (ch(k_i l_1)\sin(k_i l_1)\sin(k_i l_2) - \sin(k_i l_1 + k_i l_2) sh(k_i l_1)) sh(k_i l_2)) \\ & \quad + (sh(k_i l_1)(2\sin(k_i l_1)\sin(k_i l_3) - ch(k_i l_2)(2\cos(k_i l_2)\sin(k_i l_1)\sin(k_i l_3) \\ & \quad + \sin(k_i l_2)\sin(k_i l_1 + k_i l_3)) + \cos(k_i l_1)\cos(k_i l_3)\sin(k_i l_2) sh(k_i l_2) \\ & \quad + \cos(k_i l_2)\sin(k_i l_1 + k_i l_3) sh(k_i l_2)) \\ & \quad + ch(k_i l_1)\sin(k_i l_1)(ch(k_i l_2)\sin(k_i l_2)\sin(k_i l_3) - \sin(k_i l_2 + k_i l_3) sh(k_i l_2)))sh(k_i l_3) = 0 \end{aligned} \tag{A.6}$$

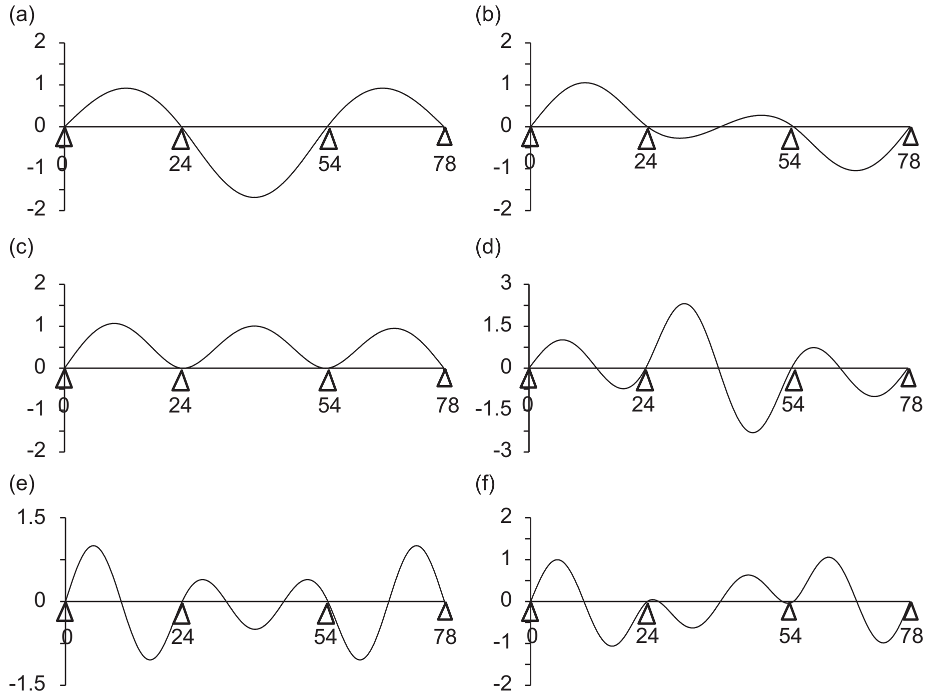

Fig. A2. Mode shape 1–6 for a continuous, three-span, simply supported beam. Modes: (a) 1, $k_1 = 0.1178$; (b) 2, $k_2 = 0.1455$; (c) 3, $k_3 = 0.1614$; (d) 4, $k_4 = 0.2304$; (e) 5, $k_5 = 0.2736$; and (f) 6, $k_6 = 0.2857$.

Mathematica enables determination of the roots k_i of the frequency Eq. (A.6) with respect to a beam 78 m long, with three spans of unequal (lengths $l_1 = l_3 = 24\,\text{m}$, $l_2 = 30\,\text{m}$) corresponding to the numerical example of the orthotropic, bridge deck presented in this article. The first six mode shapes of a continuous, three-span simply supported beam are shown in Fig. A2.

References

[1] C. Broquet, Dynamic behaviour of concrete bridge deck slabs solicited by the road trafic, Doctorate Thesis, EPFL, 1999.
[2] M.F. Green, D. Cebon, Dynamic response of highway bridges to heavy vehicle loads: theory and experimental validation, *Journal of Sound and Vibration* 170 (1994) 51–78.
[3] A.S. Veletsos, N.M. Newmark, Determination of natural frequencies of continuous plates hinged along two opposite edges, *Journal of Applied Mechanics* 23 (1956) 97–102.
[4] S.M. Dickinson, G.B. Warburton, Natural frequencies of plate systems using the edge-effect method, *Journal of Mechanical Engineering Science* 9 (1967) 318–332.
[5] V.V. Bolotin, A generalization of the asymptotic method of the eigenvalue problems for rectangular regions, *Inzhenernyi Zhurnal* 3 (1961) 86–92.
[6] V.V. Bolotin, An asymptotic method for the study of the problem of eigenvalues for rectangular regions, *Problems of Continuum Mechanics*, Philadelphia, 1961, pp. 56–68 (Dedicated to N.I. Muskhelishvili).
[7] K. Vijayakumar, A new method for analysis of flexural vibration of rectangular orthotropic plates, *Journal of the Aeronautical Society of India* 23 (1971) 197–204.
[8] I.B. Elishakoff, Vibration analysis of clamped square orthotropic plate, *AIAA Journal* 12 (1974) 92l–924.
[9] I.B. Elishakoff, A. Sternberg, Eigenfrequencies of continuous plates with arbitrary number of equal spans, *Journal of Applied Mechanics* 46 (1979) 656–662.
[10] S. Azimit, J.F. Hamilton, W. Soedel, The receptance method applied to the free vibration of continuous rectangular plates, *Journal of Sound and Vibration* 93 (1984) 9–29.
[11] D.J. Gorman, L. Garibaldi, Accurate analytical type solutions for free vibration frequencies and mode shapes of multi-span bridge decks: the span-by-span approach, *Journal of Sound and Vibration* 290 (2006) 321–336.
[12] D. Zhou, Eigenfrequencies of line supported rectangular plates, *International Journal of Solids and Structures* 31 (1994) 347–358.
[13] X.Q. Zhu, S.S. Law, Dynamic loads on continuous multi lane bridge deck from moving vehicles, *Journal of Sound and Vibration* 251 (2002) 697–716.
[14] S. Marcheseillo, A. Fasana, L. Garibaldi, B.A.D. Piombo, Dynamic of multi-span continuous straight bridges subject to multi-degrees of freedom moving vehicle excitation, *Journal of Sound and Vibration* 224 (1999) 541–561.
[15] D. Zhou, Y.K. Cheung, Free vibration of line supported rectangular plates using a set of static beam functions, *Journal of Sound and Vibration* 223 (1999) 231–245.
[16] M.M. Hrabok, T.M. Hrudley, A review and catalogue of plate bending elements, *Computers and Structures* 19 (1984) 479–495.
[17] I.M. Smith, D. William, The effectiveness of excessive nodal continuities in the finite element analysis of thin rectangular and skew plates in bending, *International Journal for Numerical Methods in Engineering* 2 (1970) 253–257.
[18] J.S. Wu, M.L. Lee, T.S. Lai, The dynamic analysis of a flat plate under a moving load by the finite element method, *International Journal for Numerical Methods in Engineering* 24 (1987) 743–762.
[19] M.S. Cheung, Y.K. Cheung, D.V. Reddy, Frequency analysis of certain single and continuous span bridges, *Developments in Bridges Design and Construction*, 1971, pp. 188–199.
[20] C.I. Wu, Y.K. Cheung, Frequency analysis of rectangular plates continuous in one or two directions, *Journal of Earthquake Engineering and Structural Dynamics* 3 (1974) 3–14.
[21] C.A. Mercer, C. Seavey, Prediction of natural frequencies and normal modes of skin-stringer panel rows, *Journal of Sound and Vibration* 6 (1967) 149–162.
[22] L.M. Keer, B. Stahl, Eigenvalue problems of rectangular plates with mixed edge conditions, *Journal of Applied Mechanics* 39 (1972) 513–520.
[23] R.E. Bellman, J. Casti, Differential quadrature and long-term integration, *Journal of Mathematical Analysis and Applications* 34 (1971) 235–238.
[24] P.A.A. Laura, R.H. Gutierrez, Analysis of vibrating rectangular plates with non-uniform boundary conditions by using the differential quadrature method, *Journal of Sound and Vibration* 173 (1994) 702–706.
[25] C.F. Lu, Z.C. Zhang, W.Q. Chen, Free vibration of generally supported rectangular Kirchhoff plates: state-space-based differential quadrature method, *International Journal for Numerical Methods in Engineering* 70 (2007) 1430–1450.
[26] A. Rezaiguia, Vibroacoustic Modelling of Highway Bridges Crossing by Moving Vehicles, Doctorate Thesis, Annaba University, Algeria, 2008.