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Analytic upper and lower bounds for the period of nonlinear oscillators

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ABSTRACT

We show that analytic expressions for the frequency of second-order nonlinear oscillations can be obtained from dual variational principles. At each amplitude two analytic expressions can be constructed which constitute upper and lower bounds to the exact value of the frequency. The results are accurate at small and large amplitude and compare well with the perturbative approach.

1. Introduction

The solution of second-order nonlinear oscillators

$$\ddot{x} + f(x) = 0,$$

where $-f(x)$ is a conservative force, while simple to understand qualitatively, cannot always be found explicitly. In particular the frequency of oscillations around an equilibrium point can be calculated analytically in few exactly solvable cases. For this reason different perturbative methods have been developed [1–3]. The simplest perturbation approach applies to weak nonlinearities only, while modified perturbation approaches have been developed which are valid at large amplitudes. In spite of the fact that this is an old and much studied problem, new and improved methods of approximation are being constantly developed. Some of the more recent developments can be found in [4–20] and references therein. The methods developed in these references rely on perturbation theory to obtain improved analytical approximations at each stage of the calculations.

The purpose of this work is to show that analytic approximations to the frequency can be obtained in a simple nonperturbative manner from variational principles. It was shown in [21,22] that the frequency of nonlinear oscillators can be derived from a variational principle which provides lower bounds on the frequency of oscillations. A dual variational principle was found in [23] from which upper bounds on the frequency can be obtained. In these previous works no emphasis was made in developing accurate analytical expressions. Here we show that the variational method gives closed form analytic expressions accurate for large amplitudes. A new straightforward proof of the second variational principle [23] is presented in Section 2.

In the perturbation approach the accuracy of the analytical approximation can be ascertained only by comparison to the exact solution be it analytical or numerical. In the present work we shall see that since there exist dual integral variational principles for the frequency of oscillations the degree of accuracy of the analytic approximation can be estimated without

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knowledge of the exact solution. While the method is of general validity, we will apply it to two simple examples: a piecewise linear potential and the Duffing nonlinear oscillator which has been extensively studied and therefore allows to compare the results obtained by different methods. In Section 2 we state the variational principles in general form, in Section 3 we apply them to the Duffing nonlinear oscillator and in Section 4 we apply them to a piecewise linear oscillator.

2. Variational principles

We consider systems with an equilibrium point $x=0$, around which the system oscillates. For the sake of simplicity, we assume that the force is an odd function of x, the results can be generalized in a simple way to a general force term.

If the force is odd, the period of oscillation can be evaluated by considering the motion through a quarter of a period. Choosing the quarter period in the quadrant ($\dot{x}<0, x>0$) of phase space, the period may be calculated solving $\ddot{x}=-f(x)$, with $x(0)=x_m$, $\dot{x}(0)=0$ and $x(T/4)=0$.

Introducing the new variable $\tau = 4t/T$, the problem reduces to finding the eigenvalue λ of

$$-x'' = \lambda f(x), \quad x(0)=x_m, \quad x'(0)=0, \quad x(1)=0 \tag{1}$$

with $\lambda = T^2/16$. Here, primes denote derivatives with respect to τ. Notice that the eigenvalue λ depends on the amplitude x_m. In terms of λ the frequency is then

$$\omega = \frac{\pi}{2\sqrt{\lambda}}.$$

2.1. Upper bounds for the frequency

In previous work we proved that the eigenvalue λ of problem (1) is characterized by the variational principle [21,22]

$$\lambda[x_m] = \max_g \frac{1}{2}\frac{(\int_0^{x_m} g'^{1/3}(x)\,\mathrm{d}x)^3}{\int_0^{x_m} f(x)g(x)\,\mathrm{d}x}, \tag{2}$$

where the maximum is taken over all positive functions $g(x)$ such that $g(0)=0$, $g'(x)>0$. The maximum is achieved for $g=\hat{g}$ which satisfies

$$\hat{g}'(x) = \frac{1}{(E-V(x))^{3/2}},$$

where $V(x)$ is the potential associated to the force $-f(x)$ and $E=V(x_m)$ is the total energy of the system.

A more convenient expression for (2) is obtained defining a new function

$$q(x) = (g'(x))^{1/3}.$$

After integrating by parts the denominator in (2), we find

$$\lambda[x_m] = \max_q \frac{1}{2}\frac{(\int_0^{x_m} q(x)\,\mathrm{d}x)^3}{\int_0^{x_m} q^3(V(x_m)-V(x))\,\mathrm{d}x}, \tag{3}$$

where

$$V(x) = \int_0^x f(s)\,\mathrm{d}s$$

is the potential. From (3) it is direct to verify that for the optimal trial function $\hat{g}(x)$, or equivalently for q,

$$\hat{q} = \frac{1}{\sqrt{E-V(x)}},$$

the exact formula for the period is recovered. If we evaluate the right side of (3) with an arbitrary $q(x)>0$ then we obtain a lower bound for λ and an upper bound on ω.

2.2. Lower bounds for the frequency

A second variational expression for λ is given by [23]

$$\frac{1}{\lambda} = \max_{\sigma\in H}\frac{2\int_0^{x_m}(1-2\sigma(s))\sqrt{(\sigma(s)+1)/h(s)}\,\mathrm{d}s}{(\int_0^{x_m}\sqrt{(\sigma(s)+1)/h(s)}\,\mathrm{d}s)^3}, \tag{4}$$

where $H=\{\sigma|\sigma\in C(0,y_m), 0\leqslant\sigma<\frac{1}{2}\}$. To simplify the notation it is convenient to define the function

$$h(x) = E - V(x) = V(x_m) - V(x).$$

From this variational expression upper bounds on λ, hence lower bounds for the frequency, can be obtained. The maximum is attained at $\sigma = 0$ and it is given precisely by

$$\frac{1}{\lambda} = 2\left(\int_0^{y_m} \frac{1}{\sqrt{h(s)}}\mathrm{d}s\right)^{-2}. \tag{5}$$

Before any application we will give an elementary proof of this variational expression. We need to show that for any $0 \leqslant \sigma < \frac{1}{2}$,

$$\frac{(\int_0^{x_m}\sqrt{(\sigma(s)+1)/h(s)}\,\mathrm{d}s)^3}{2\int_0^{x_m}(1-2\sigma(s))\sqrt{(\sigma(s)+1)/h(s)}\,\mathrm{d}s} \geq \frac{1}{2}\left(\int_0^{x_m}\sqrt{\frac{1}{h(s)}}\mathrm{d}s\right)^2. \tag{6}$$

First, it is evident that when $\sigma = 0$ equality holds and the exact value for λ is recovered. If $\sigma \geq 0$, the numerator on the left side of (6) satisfies

$$\int_0^{x_m}\sqrt{\frac{\sigma(s)+1}{h(s)}}\mathrm{d}s \geq \int_0^{x_m}\sqrt{\frac{1}{h(s)}}\mathrm{d}s. \tag{7}$$

In order to estimate the denominator consider the integrand as a function of σ and define

$$m(\sigma) = (1-2\sigma)\sqrt{\frac{\sigma+1}{h}}.$$

This is a decreasing function of σ, effectively

$$\frac{\mathrm{d}m}{\mathrm{d}\sigma} = -\frac{3}{2}\frac{2\sigma+1}{\sqrt{(1+\sigma)h}}$$

is always negative for $\sigma \geq 0$. Therefore $m(\sigma) \leqslant m(0)$. It follows then that the denominator on the left side of (6) satisfies

$$\int_0^{x_m}(1-2\,\sigma(s))\sqrt{(\sigma(s)+1)/h(s)}\,\mathrm{d}s \leqslant \int_0^{x_m}\sqrt{\frac{1}{h(s)}}\mathrm{d}s. \tag{8}$$

From (7) and (8) the desired result, Eq. (6) follows.

3. Application to the Duffing oscillator

We will apply the variational principles to obtain accurate analytic expressions for the frequency of the Duffing oscillator

$$\ddot{x} = -\lambda(x + \delta x^3) \tag{9}$$

with $\dot{x}(0) = 0$ and $x(1) = 0$. The amplitude of the oscillations $x_m = x(0)$.

In order to obtain analytical expressions for the frequency of oscillations we must construct suitable trial functions to be used in the variational principles. Upper bounds for the frequency (lower bounds for λ) follow from the variational expression equation (3). In previous work we showed that a simple closed analytic approximation can be obtained by taking as a trial function $q(x)$ the exact trial function for the linear problem, q_L, which is given by

$$q_L(x) = \frac{\sqrt{2}}{\sqrt{x_m^2 - x^2}}. \tag{10}$$

Performing the integrals in Eq. (3) we obtain

$$\omega \leqslant \omega_{U0} = \sqrt{1 + \tfrac{3}{4}\delta x_m^2}. \tag{11}$$

This approximation for the frequency gives a close estimate for all amplitudes, and has been obtained by other methods as well. Here, however, we establish that it constitutes an upper bound to the exact frequency. This first expression ω_{U0} was obtained by using the simplest possible trial function. In order to construct a closer approximation we make use of the fact that the optimal trial function $\hat{q}$ is known. For our example it is given by

$$\hat{q}_{\text{exact}} = \frac{\sqrt{2}}{\sqrt{x_m^2 - x^2}}\frac{1}{\sqrt{1 + \frac{\delta}{2}(x_m^2 + x^2)}}.$$

If we use the above function we will obtain the exact value of the frequency since this is an exactly solvable example. A simple way to obtain analytic upper bounds is to approximate the second factor in the right side by a series expansion. We find it convenient to define

$$p(x) = \frac{1}{\sqrt{1 + \frac{\delta}{2}(x_m^2 + x^2)}}.$$

As a first trial we may approximate $p(x)$ by a series expansion of degree 2, centered at $x_m/2$, that is

$$p(x) \approx \frac{1}{\sqrt{\frac{5\delta x_m^2}{8} + 1}} - \frac{\delta x_m \left(x - \frac{x_m}{2}\right)}{4\left(\frac{5\delta x_m^2}{8} + 1\right)^{3/2}} - \frac{8(\sqrt{2}\delta^2 x_m^2 + 4\sqrt{2}\delta)\left(x - \frac{x_m}{2}\right)^2}{(5\delta x_m^2 + 8)^{5/2}}.$$

Then, with the trial function $q_1(x) = q_L(x)p(x)$ we perform the integrals in (3) to obtain

$$\omega \leqslant \omega_{U1} = \frac{A1}{B1},$$

where

$$\begin{aligned} A_1 = 2\pi\sqrt{\frac{2}{35}}(&9175040\pi + 39567360\delta\pi x_m^2 + 107520\delta^2(-48 + 695\pi)x_m^4 \\ &+11200\delta^3(-1440 + 7157\pi)x_m^6 + 4032\delta^4(-5157 + 13115\pi)x_m^8 \\ &+120\delta^5(-116064 + 179795\pi)x_m^{10} + 9\delta^6(-541276 + 565145\pi)x_m^{12} \\ &+6\delta^7(-121119 + 90755\pi)x_m^{14})^{1/2}, \end{aligned}$$

$$B_1 = [128\pi + 152\delta * \pi x_m^2 + 6\delta^2(-4 + 9\pi)x_m^4]^{3/2}.$$

A simpler expression is obtained approximating $p(x)$ by a Padé polynomial of first degree in the numerator, that is,

$$p(x) \approx \frac{1}{\sqrt{\delta x_m^2 + 1}} - \frac{\delta x_m(x - x_m)}{2(\delta x_m^2 + 1)^{3/2}}$$

with which the trial function becomes

$$q_2(x) = \frac{\sqrt{2}}{\sqrt{x_m^2 - x^2}}\left(\frac{1}{\sqrt{\delta x_m^2 + 1}} - \frac{\delta x_m(x - x_m)}{2(\delta x_m^2 + 1)^{3/2}}\right).$$

Performing the integrals in Eq. (3) we obtain a second analytical upper bound,

$$\omega \leqslant \omega_{U2} = \frac{A_2}{B_2}, \tag{12}$$

where

$$A_2 = \pi(8\pi + 6(7\pi - 4)\delta x_m^2 + 4(21\pi - 23)\delta^2 x_m^4 + \tfrac{1}{24}(1791\pi - 2768)\delta^3 x_m^6 + \tfrac{3}{80}(645\pi - 1232)\delta^4 x_m^8)^{1/2},$$

$$B_2 = 2\sqrt{2}\left(\left(\frac{3\pi}{2} - 1\right)\delta x_m^2 + \pi\right)^{3/2}.$$

Better accuracy is obtained approximating $p(x)$ by a series expansion of degree 3, centered at $x_m/2$, we obtain

$$\omega \leqslant \omega_{U3} = \frac{\pi}{2\sqrt{2}}\frac{A_3}{B_3}$$

with

$$\begin{aligned} A_3 = \Bigg(&\frac{25(2688256 + 2683065\pi)\delta^{10}x_m^{20}}{4928} + \frac{5(237126448 + 243489015\pi)\delta^9 x_m^{18}}{5544} \\ &+\frac{5}{616}(180808736 + 198264297\pi)\delta^8 x_m^{16} + \frac{1}{231}(1328298320 + 1631028861\pi)\delta^7 x_m^{14} \\ &+\frac{8}{105}(183586828 + 269824695\pi)\delta^6 x_m^{12} + \frac{16}{35}(46827208 + 90738165\pi)\delta^5 x_m^{10} \\ &+\frac{128}{35}(5429638 + 15998115\pi)\delta^4 x_m^8 + 2560(3952 + 22075\pi)\delta^3 x_m^6 \\ &+24576(88 + 1443\pi)\delta^2 x_m^4 + 12976128\pi\delta x_m^2 + 2097152\pi\Bigg)^{1/2}, \end{aligned}$$

$$B_3 = (\tfrac{5}{6}(8+15\pi)\delta^3 x_m^6 + 2(11+32\pi)\delta^2 x_m^4 + 116\pi\delta x_m^2 + 64\pi)^{3/2}.$$

We shall now use the dual variational principle (4) to obtain lower bounds on ω. The exact solution is obtained when $\sigma = 0$. Our purpose is to illustrate the method, therefore we need to find a good trial function $\sigma(x)$. It is convenient to define

$$\not{p}(x) = \sqrt{\frac{h_0(\sigma+1)}{h}},$$

where $h_0 = (x_m^2 - x^2)/2$ in terms of which

$$\sigma = (\not{p})^2 \frac{h}{h_0} - 1.$$

The exact value from the dual principle is obtained when $\sigma = 0$, that is, when $\not{p} = p$. We construct a trial function σ replacing $\not{p}$ with a suitable approximation for p as was done to construct the upper bounds. A simple way to obtain an analytic approximate value for the frequency is to use a Padé approximant of degree 2 in the numerator and denominator for $p(x)$. The trial function constructed in this way is

$$\sigma(x) = \frac{\delta^3 x^6}{(\delta x_m^2 + 2)(3\delta x^2 + 4\delta x_m^2 + 8)^2},$$

which satisfies the condition $\sigma \geq 0$. The lower bound calculated performing the integrations in (4) is given by

$$\omega_{L1} = \frac{\sqrt{N}}{D},$$

where

$$N = \frac{928}{\sqrt{7\delta x_m^2 + 8}} - \frac{512}{(7\delta x_m^2 + 8)^{3/2}} + \frac{1152}{(7\delta x_m^2 + 8)^{5/2}} + \frac{196}{\sqrt{\delta x_m^2 + 2}} + \frac{49}{(\delta x_m^2 + 2)^{3/2}},$$

$$D = 7\left(\frac{4}{\sqrt{7\delta x_m^2 + 8}} + \frac{1}{\sqrt{\delta x_m^2 + 2}}\right)^{3/2}.$$

This lower bound gives quite accurate values for the frequency as we show in the table below. If higher precision is desired we may choose a high order polynomial trial function, such as

$$\begin{aligned}\sigma(x) = -1 + \frac{8}{(8+5\delta x_m^2)^1 5}\left[1 + \frac{1}{2}\delta(x^2 + x_m^2)\right] \\ \times(\delta(-2x + x_m)^2(4+\delta x_m^2)(8+5\delta x_m^2)^5 + \delta x_m(2x - x_m)(8+5\delta x_m^2)^6 \\ -(8+5\delta x_m^2)^7 + \delta^2 x_m(2x - x^m)^3(8+5\delta x_m^2)^4(12+5\delta x_m^2) \\ +\delta^2(2x - x_m)^4(8+5\delta x_m^2)^3(-24+5\delta^2 x_m^4) \\ +\delta^3 x_m(2x - x_m)^5(8+5\delta x_m^2)^2(120+80\delta x_m^2 + 11\delta^2 x_m^4) \\ +\delta^4 x_m(2x - x_m)^7(-1120 - 840\delta x_m^2 - 84\delta^2 x_m^4 + 29\delta^3 x_m^6) \\ -\delta^3(2x - x_m)^6(8+5\delta x_m^2)(-160 + 120\delta x_m^2 + 180\delta^2 x_m^4 + 41\delta^3 x_m^6))^2.\end{aligned}$$

The integrals can be performed and a high order rational polynomial approximation for the frequency, which we call ω_{L2} is found. We do not give the analytical expression, only its numerical value at different values of the amplitude.

Numerical values for the upper and lower bounds for the frequency are given in the following table:

x_m	ω_{L1}	ω_{L2}	ω_{true}	ω_{U0}	ω_{U1}	ω_{U2}	ω_{U3}
1	1.31686	1.31777	1.31778	1.32288	1.3178	1.31873	1.31778
10	8.4551	8.53225	8.53359	8.7178	8.53759	8.53661	8.53376
100	83.9146	84.714	84.7275	86.6083	84.769	84.5672	84.7294

From the table we see that with adequate trial functions we obtain analytic bounds which can be as accurate as desired. In the example, at the amplitude $x_m = 100$ we have $84.714 \leqslant \omega \leqslant 84.7249$, a degree of certainty which cannot be achieved by other methods.

4. Application to a piecewise linear oscillator

As a second example consider the piecewise linear oscillator

$$\ddot{y}+f(y)=0$$

with

$$f(y)=\begin{cases} y & \text{for } y\leqslant 1, \\ y+\mu(y-1) & \text{for } y>1 \end{cases}$$

with initial conditions $y(0)=1$, $\dot{y}(0)=\dot{y}_0>0$. The potential associated to the force $-f$ is given by

$$V(y)=\begin{cases} y^2/2 & \text{for } y<1, \\ y^2/2+\mu(y-1)^2/2 & \text{for } y>1. \end{cases}$$

This is a linear problem, easily solvable, the solution of which is given by

$$T=2\pi+\frac{4}{\sqrt{1+\mu}}\arctan(\dot{y}_0\sqrt{1+\mu})-4\arctan(\dot{y}_0). \tag{13}$$

The amplitude of the motion $A>1$ can be expressed in terms of the initial conditions solving

$$E=\frac{1}{2}+\frac{\dot{y}_0^2}{2}=\frac{A^2}{2}+\frac{\mu(A-1)^2}{2}.$$

Replacing $\dot{y}_0$ in terms of the amplitude A in Eq. (13) we obtain the exact expression for the period in terms of the amplitude:

$$T_E=2\pi+\frac{4}{\sqrt{1+\mu}}\arctan(\sqrt{1+\mu}\sqrt{A^2+\mu(A-1)^2-1}) \tag{14}$$

$$-4\ \arctan(\sqrt{A^2+\mu(A-1)^2-1}). \tag{15}$$

We will show here that an accurate approximation for the period can be obtained from the variational principles. We shall use the first variational principle to construct a simple analytic approximation to the exact solution, and the second variational principle to establish that the exact solution constitutes an upper bound.

To obtain a lower bound for the period choose the trial function

$$q(y)=\frac{1}{\sqrt{A^2-y^2}}$$

and evaluate (3). The integrals are not difficult to perform and we obtain

$$T\geq T_L\equiv\frac{\pi\sqrt{2\pi}}{\left[\arcsin A^{-1}+(1+\mu)\arccos A^{-1}-\frac{\mu}{A^2}\sqrt{A^2-1}\right]^{1/2}}. \tag{16}$$

With the trial function $\sigma=0$ in the variational principle (4) we obtain $T<T_E$, the exact value.

Let us now compare the numerical accuracy of the lower bound (16) with the exact value, which we have established as an upper bound.

A	$\mu=0.5$		$\mu=2$		$\mu=10$	
	T_L	T_E	T_L	T_E	T_L	T_E
2	5.74652	5.74972	4.7068	4.7262	2.83556	2.87603
10	5.24246	5.24264	3.79182	3.79238	2.01441	2.01501
100	5.14112	5.14112	3.64309	3.6431	1.90551	1.90552

From the table we see that the lower bound T_L gives a very accurate analytical approximation to the exact value of the period particularly at large amplitudes. Since this is a linear problem the exact solution is already simple, the lower bound is also simple and clearly it cannot be obtained by a series expansion of the exact solution.

In summary, we have shown that from variational principles the frequency of second-order nonlinear oscillators can be determined accurately at low and large amplitudes. We illustrated the method through an application to the Duffing equation to allow comparison with the results obtained by other methods, and to a piecewise linear oscillator. For any oscillator we may find the frequency with any degree of accuracy as desired by proper choice of trial functions using the standard variational methods.

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