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A boundary perturbation method for circularly periodic plates with a core

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ABSTRACT

Fundamental frequencies of vibrating circularly periodic plates with a circular core have been determined analytically. A boundary perturbation method is developed to extract the fundamental eigenvalue of the governing biharmonic boundary value problem. The method is then applied to wavy and polygonal plates with clamped and simply supported outer boundary conditions. Clamped, simply supported, and free circular cores are considered. Approximate analytical formulations of the fundamental frequency for such plates with core are obtained.

1. Introduction

Fundamental frequencies of vibrating plates have been determined analytically only for limited classes of plate geometries. Leissa, in his monograph, gave an extensive survey of plates [1]. The problem has a general analytical solution in a circular domain in terms of a linear combination of the Bessel functions [1,2]. However, difficulty in finding analytic solutions arises when the domain is no longer circular. If the domain is a rectangle, Navier's double series solution and Levy's single series solution are possible for certain boundary conditions [1]. But for other domains, numerical methods are necessary. Asymptotic approximations of annular plates have been investigated by [3,4]. In the case of polygonal plates (without core support), various numerical solutions are available [5–9]. However, in some cases, especially when numerical methods become inadequate, such as plates with small internal supports, numerical methods often encounter the problem of singularity, scaling, and sensitivity to the boundary conditions. This leads us to a special formulation of perturbation theory to improve accuracy and reliability of the fundamental frequency. Recently, a boundary perturbation method (BPM) is developed by Yüce and Wang [10] for the circularly periodic plates with clamped boundary conditions. A recent important work dealing with a different BPM to approximate the frequencies and modes of circular and rectangular plates with discontinuous boundary conditions (i.e. the situation where a portion of an edge is simply supported while the remainder is clamped) is given by Febbo et al. [11]. The method presented in [10] uses not only the perturbed boundary to obtain the desirable geometry, but also uses perturbed modes and frequencies. In this present work, we extend the method developed in [10] to doubly connected domains by placing a concentric circular core into wavy and polygonal plates. Also the extension includes perturbation of simply supported boundary conditions. The inner boundary conditions (boundary conditions of the core) for the plates include clamped, simply supported, and free. The purpose of this study is to examine the fundamental frequencies of vibrating polygonal plates with a concentric circular core. Perturbation solutions yield analytic approximate formulations of the fundamental frequencies for such plates. We consider circularly periodic plates

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with an internal core support and we develop analytic approximate formulations of the fundamental frequencies of such plates, using the boundary perturbation method in the case of clamped and simply supported outer boundary conditions.

The governing equation for the transverse vibration of a thin plate is given by

$$\nabla^4 W - k^4 W = 0, \tag{1}$$

where $k^2 = \omega L^2 \sqrt{\rho/D}$, ω is the natural frequency, ρ is the density, D is the flexural rigidity, L is a length scale. The fundamental frequency is the smallest eigenvalue of the Eq. (1). However, the fundamental frequency may switch from no nodal diameter to one nodal diameter in the case of annular plates as the radius of inner circle is decreased (with clamped, simply supported, and free boundary conditions). We call this point of switch the *transition point*. The fundamental frequency is essential in applied sciences before finalizing the design. It is desirable to increase the fundamental frequency, below which no vibration would occur, by placing internal core supports.

2. Formulation of the perturbation method

Let $L = 1$ be the normalized average radius. The boundary is given by $r = 1 + \varepsilon f(\theta)$ where $f(\theta)$ is the boundary function of zero mean and ε is the small amplitude of the boundary as shown in Fig. 1. Perturb the solution $W(r,\theta)$ and the fundamental frequency k about the circular state as in [10]

$$W(r,\theta) = W_0(r) + \varepsilon W_1(r,\theta) + \varepsilon^2 W_2(r,\theta) + O(\varepsilon^3), \tag{2}$$

$$k^4 = k_0^4[1 + \varepsilon^2 b + O(\varepsilon^4)], \tag{3}$$

where $\varepsilon^2 b$ is the correction to the fundamental frequency and $b = (k_1/k_0)^4$. Expansion of the clamped boundary conditions is given by [10]

$$0 = W(1+\varepsilon f(\theta),\theta) = W_0(1) + \varepsilon[W_1(1,\theta) - \Phi_1(1,\theta)] + \varepsilon^2[W_2(1,\theta) - \Phi_2(1,\theta)] + \cdots, \tag{4}$$

$$0 = \left.\frac{\partial W}{\partial \mathbf{n}}\right|_{r=1+\varepsilon f(\theta)} = W_{0_r}(1) + \varepsilon[W_{1_r}(1,\theta) - \Psi_1(1,\theta)] + \varepsilon^2[W_{2_r}(1,\theta) - \Psi_2(1,\theta)] + \cdots, \tag{5}$$

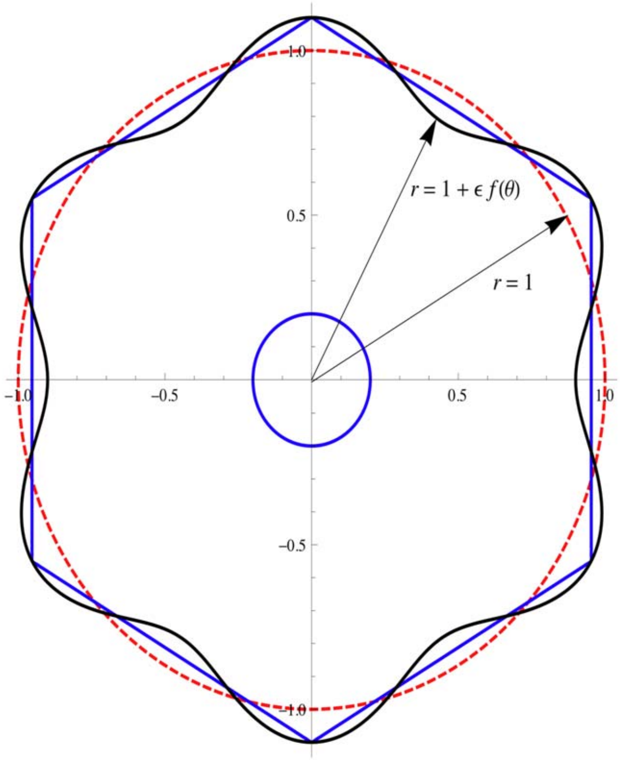

Fig. 1. The boundary perturbation about the unit circle.

where $\mathbf{n}$ the unit normal vector and $\Phi_1(1,\theta)$, $\Phi_2(1,\theta)$, $\Psi_1(1,\theta)$, and $\Psi_2(1,\theta)$ are given in Appendix A. Then both Eqs. (4) and (5) give the perturbed clamped boundary conditions.

At $r = 1 + \varepsilon f(\theta)$, simply supported boundary conditions are given by

$$W(1 + \varepsilon f(\theta), \theta) = 0, \tag{6}$$

$$\mathfrak{M}[W(1 + \varepsilon f(\theta), \theta)] = 0, \tag{7}$$

where

$$\mathfrak{M}(W(r,\theta)) = \frac{\partial^2 W(r,\theta)}{\partial \mathbf{n}^2} + \frac{v}{R}\frac{\partial W(r,\theta)}{\partial \mathbf{n}} + v\frac{\partial^2 W(r,\theta)}{\partial \mathbf{s}^2}$$

is the moment operator with $\mathbf{n}$ the unit normal vector, $\mathbf{s}$ the unit tangential vector, v the Poisson ratio, and R the normalized radius of curvature. Since there is no tangential deflection on the boundary of the plate, the tangential derivatives in the moment vanish in the boundary condition (7). The first boundary condition (6) has the same expansion as in Eq. (4). The second boundary condition (7) can be written as

$$\frac{\nabla F}{|\nabla F|} \cdot \nabla\left(\frac{\nabla F}{|\nabla F|} \cdot \nabla W\right) + \frac{v}{R}\left(\frac{\nabla F}{|\nabla F|} \cdot \nabla W\right) = 0, \tag{8}$$

where $F(r,\theta) = r - 1 - \varepsilon f(\theta) = 0$. Then we have

$$\frac{1}{|\nabla F|}\left(F_r G_r + \frac{1}{r^2} F_\theta G_\theta\right) + \frac{v}{R} G = 0, \tag{9}$$

where

$$G = \frac{1}{|\nabla F|}\left(F_r W_r + \frac{1}{r^2} F_\theta W_\theta\right).$$

Using the radius of curvature in polar coordinates,

$$R(\theta) = \frac{\left[\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 + r^2\right]^{3/2}}{r^2 + 2\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 - r\frac{\mathrm{d}^2 r}{\mathrm{d}\theta^2}}. \tag{10}$$

We obtain the asymptotic expansion of the second boundary condition as

$$\begin{aligned} 0 = {} & [W_{0_{rr}}(1,\theta) + vW_{0_r}(1,\theta)] + \varepsilon[W_{1_{rr}}(1,\theta) + vW_{1_r}(1,\theta) - \Omega_1(1,\theta)] \\ & + \varepsilon^2[W_{2_{rr}}(1,\theta) + vW_{2_r}(1,\theta) - \Omega_2(1,\theta)] + \cdots, \end{aligned} \tag{11}$$

where Ω_1 and Ω_2 are given in Appendix A. Thus both Eqs. (4) and (11) give the perturbed simply supported boundary conditions for a boundary function $f(\theta)$.

Substituting Eqs. (2) and (3) into Eq. (1), we obtain the following sequential boundary value problems as in [10]

$$\begin{cases} \nabla^4 W_0(r) - k_0^4 W_0(r) = 0, \\ \nabla^4 W_1(r,\theta) - k_0^4 W_1(r,\theta) = 0, \\ \nabla^4 W_2(r,\theta) - k_0^4 W_2(r,\theta) = bk_0^4 W_0(r), \\ \vdots \end{cases} \tag{12}$$

Solutions of sequential Eqs. (12) with clamped and simply supported outer boundary conditions of specific shapes are considered below.

3. Wavy boundary plates

In this section, we consider a wavy circular boundary by taking $f(\theta) = \cos(M\theta)$, where $M \geq 2$ is the number of circumferential waves. Then the radius is defined by $r = 1 + \varepsilon\cos(M\theta)$, where $\varepsilon \ll 1$ is a small amplitude.

3.1. Clamped–clamped boundary conditions (CC)

We proceed with a perturbation scheme adopting the boundary perturbation method presented in Section 2. Let $c < 1 - \varepsilon$ be the radius of the concentric circular core. We perturb the solution $W(r,\theta)$ and the fundamental frequency k

about the circular state as before in Eqs. (2) and (3). We take the boundary function to be $f(\theta) = \cos(M\theta)$. Then the clamped boundary conditions (both inner and outer boundary) are

$$W(1+\varepsilon f(\theta),\theta)=0, \quad \left.\frac{\partial W}{\partial \mathbf{n}}\right|_{r=1+\varepsilon f(\theta)}=0, \tag{13}$$

$$W(c,\theta)=0, \quad \left.\frac{\partial W}{\partial \mathbf{n}}\right|_{r=c}=0, \tag{14}$$

where **n** is the unit normal vector. The 0th order equation, with corresponding 0th order homogenous clamped boundary conditions, is given by

$$\begin{cases} \nabla^4 W_0(r,\theta)-k_0^4 W_0(r,\theta)=0, \\ W_0(1,\theta)=0, \quad W_0(c,\theta)=0, \\ W_{0_r}(1,\theta)=0, \quad W_{0_r}(c,\theta)=0. \end{cases} \tag{15}$$

Eq. (15) corresponds to the CC annular plate (unperturbed case). The solution of Eq. (15) with the homogenous boundary conditions is then given by

$$W_0(r,\theta)=J_0(rk_0)+\alpha_{01}^{cc}Y_0(rk_0)+\alpha_{02}^{cc}I_0(rk_0)+\alpha_{03}^{cc}K_0(rk_0), \tag{16}$$

where α_{01}^{cc}, α_{02}^{cc}, and α_{03}^{cc} are given in Appendix B.1. The solution given by Eq. (16) is finite, since r cannot be zero due to the concentric circular core. Imposing the clamped edge boundary conditions and using the recursion formulas for the Bessel functions, we obtain the following frequency equation when $c=0$ and $n=1$:

$$J_1(k_0)[I_0(k_0)+I_2(k_0)]-I_1(k_0)[J_0(k_0)-J_2(k_0)]=0. \tag{17}$$

The fundamental frequency rises from $k_0 = 4.611$ (for mode $n = 1$) which is the first root of Eq. (17). The transition location (from mode $n = 1$ to mode $n = 0$) is at $c = 0.00132$ and the frequency at the transition location is $k_0 = 4.769$ [3]. Here, we shall note that, unlike membranes, the limiting case as $c \to 0$ does not give the fundamental frequency of the circular plate without a core ($k_0 = 3.19$). Fig. 2 shows the fundamental frequency of the annular plate with clamped inner and outer boundary conditions for $n = 0$ and 1 modes with respect to the radius of the core c. For the transition points of other boundary conditions see Wang et al. [3]. Fig. 3 shows fundamental frequencies of CC wavy plates with different number of sides for $c \geq 0.00132$.

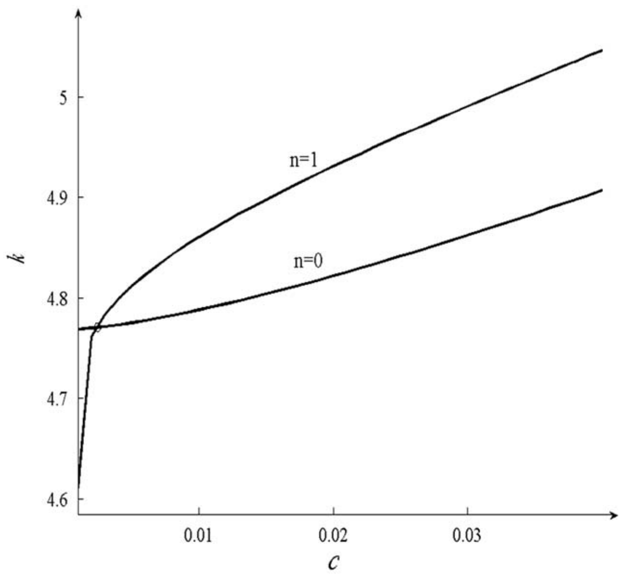

Fig. 2. Fundamental frequency vs. radius of the core of CC annular plate.

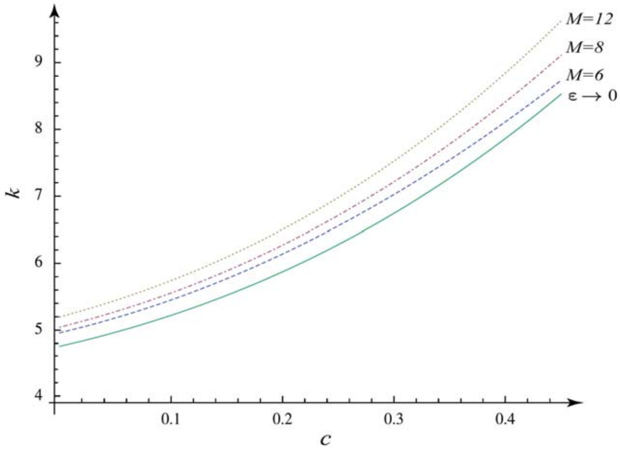

Fig. 3. Fundamental frequency vs. radius of the core of CC wavy plate.

The 1st order $O(\varepsilon)$ equation with corresponding 1st order clamped boundary conditions

$$\begin{cases} \nabla^4 W_1(r,\theta) - k_0^4 W_1(r,\theta) = 0, \\ W_1(1,\theta) = 0, \quad W_1(c,\theta) = 0, \\ W_{1_r}(1,\theta) = -f(\theta) W_{0_{rr}}(1,\theta), \quad W_{1_r}(c,\theta) = 0 \end{cases} \tag{18}$$

has the following solution:

$$W_1(r,\theta) = \sum_{n=1}^{\infty} W_1^{nM}(r)\cos(nM\theta), \tag{19}$$

where

$$W_1^{nM}(r) = \delta_{n1}^{cc} J_{nM}(rk_0) + \delta_{n2}^{cc} Y_{nM}(rk_0) + \delta_{n3}^{cc} I_{nM}(rk_0) + \delta_{n4}^{cc} K_{nM}(rk_0), \tag{20}$$

and δ_{n1}^{cc}, δ_{n2}^{cc}, δ_{n3}^{cc}, δ_{n4}^{cc} for $n = 1, 2, \ldots$ are the coefficients given in Appendix B.1. The boundary conditions in Eq. (18) suggest that the solution of the 1st order equation has the following form:

$$W_1^M(r)\cos(M\theta) = [\delta_{11}^{cc} J_M(rk_0) + \delta_{12}^{cc} Y_M(rk_0) + \delta_{13}^{cc} I_M(rk_0) + \delta_{14}^{cc} K_M(rk_0)]\cos(M\theta). \tag{21}$$

The 2nd order $O(\varepsilon^2)$ equation with corresponding 2nd order clamped boundary conditions is

$$\begin{cases} \nabla^4 W_2(r,\theta) - k_0^4 W_2(r,\theta) = b k_0^4 W_0(r), \\ W_2(1,\theta) = \Phi_2(1,\theta), \quad W_2(c,\theta) = 0, \\ W_{2_r}(1,\theta) = \Psi_2(1,\theta), \quad W_{2_r}(c,\theta) = 0. \end{cases} \tag{22}$$

The outer boundary conditions of Eq. (22) consist of two parts:

$$\Phi_2(r,\theta) = \varphi_0(r) + \varphi_2(r)\cos(2M\theta),$$

$$\Psi_2(r,\theta) = \xi_0(r) + \xi_2(r)\cos(2M\theta), \tag{23}$$

where $\varphi_0(r) = -W_{0_{rr}}(r)/4$ and $\xi_0(r) = M^2 W_1^M(r)/2 - W_{1_{rr}}^M(r)/2 - W_{0_{rrr}}(r)/4$. Since the contribution to the fundamental frequency comes from the θ-independent terms, we can ignore the functions φ_2 and ξ_2. However, in the case of clamped outer boundary condition, we have $\varphi_0(r) = \varphi_2(r)$ and $\xi_0(r) = \xi_2(r)$. The 2nd order boundary conditions suggest that we have a solution of type $W_2(r,\theta) = U(r) + V(r)\cos(2M\theta)$, which gives us the following θ-independent equation:

$$\nabla^4 U(r) - k_0^4 U(r) = b k_0^4 W_0(r), \tag{24}$$

which has the following general solution:

$$U(r) = B_1 J_0(k_0 r) + B_2 Y_0(k_0 r) + B_3 I_0(k_0 r) + B_4 K_0(k_0 r) - \frac{b k_0 r}{4}(J_1(k_0 r) + \alpha_{01}^{cc} Y_1(k_0 r) - \alpha_{02}^{cc} I_1(k_0 r) + \alpha_{03}^{cc} K_1(k_0 r)), \tag{25}$$

Table 1
Fundamental frequency k of M-sided CC wavy boundary with core radius c for $\varepsilon = 0.1$.

$M\backslash c$	0.4	0.3	0.2	0.1	0.05	0.00132
5	7.89626	6.89962	6.06790	5.39755	5.12037	4.92724
6	8.11501	7.01781	6.14575	5.45469	5.17060	4.97287
7	8.27750	7.11699	6.21573	5.50789	5.21786	5.01608
8	8.41292	7.20617	6.28104	5.55851	5.26309	5.05760
12	8.84567	7.51468	6.51594	5.74450	5.43054	5.21205

where B_1, B_2, B_3, B_4 are constants. Imposing the boundary conditions

$$\begin{cases} U(1) = \varphi_0(1), & U(c) = 0, \\ U_r(1) = \xi_0(1), & U_r(c) = 0 \end{cases} \tag{26}$$

on Eq. (24), we obtain

$$\begin{cases} B_1 J_0(k_0) + B_2 Y_0(k_0) + B_3 I_0(k_0) + B_4 K_0(k_0) = bF(1) + \varphi_0(1), \\ B_1 J_1(k_0) + B_2 Y_1(k_0) - B_3 I_1(k_0) + B_4 K_1(k_0) = -\dfrac{b}{k_0} F'(1) + \xi_0(1), \\ B_1 J_0(ck_0) + B_2 Y_0(ck_0) + B_3 I_0(ck_0) + B_4 K_0(ck_0) = bF(c), \\ B_1 J_1(ck_0) + B_2 Y_1(ck_0) - B_3 I_1(ck_0) + B_4 K_1(ck_0) = -\dfrac{b}{k_0} F'(c), \end{cases} \tag{27}$$

where $F(r) = k_0 r/4[J_1(k_0 r) + \alpha_{01}^{cc} Y_1(k_0 r) - \alpha_{02}^{cc} I_1(k_0 r) + \alpha_{03}^{cc} K_1(k_0 r)]$. Since the rank of the coefficient matrix on the left-hand side of Eq. (27) and the augmented matrix of Eq. (27) are the same and <4, every possible 4×4 matrix coming from the system Eq. (27) has determinant zero (solvability condition). Thus, we obtain a unique solution of b,

$$\begin{vmatrix} Y_0(k_0) & I_0(k_0) & K_0(k_0) & bF(1) + \varphi_0(1) \\ Y_1(k_0) & -I_1(k_0) & K_1(k_0) & -\dfrac{b}{k_0} F'(1) + \xi_0(1) \\ Y_0(ck_0) & I_0(ck_0) & K_0(ck_0) & bF(c) \\ Y_1(ck_0) & -I_1(ck_0) & K_1(ck_0) & -\dfrac{b}{k_0} F'(c) \end{vmatrix} = 0. \tag{28}$$

The product $\varepsilon^2 b$ is the first correction to the square of the fundamental frequency of the CC wavy boundary plate. Table 1 lists the values of the fundamental frequencies k with various core radii c for the CC wavy plates of amplitude $\varepsilon = 0.1$. Notice that for $\varepsilon = 0$, the fundamental frequency of the CC annular plate is recovered. Frequency values in Table 1 are obtained down to the transition point $c = 0.00132$. Since the area of each wavy boundary plate is the same, $\pi(1 + (\varepsilon^2/2) - c^2)$, and independent of the number of sides, the fundamental frequency increases as M increases for the same c and fixed ε. Note that the area of the annular state ($\varepsilon \to 0$) is slightly smaller than the area of the M-sided wavy plates. The frequency comparisons between annular and wavy plates can be made only if the outer radius of the annular plate is $r = \sqrt{1 + (\varepsilon^2/2)}$, resulting the same area as the wavy plates. In this case, the CC annular plate has smaller frequency than CC wavy plates, which verifies Pólya and Szegö [12] (the smallest eigenvalue is attained in a circular domain). Note also that larger areas result smaller frequencies of the same shape plates.

3.2. Simply supported–clamped boundary conditions (SC)

The clamped outer and the simply supported inner boundary conditions are given by

$$W(1 + \varepsilon f(\theta), \theta) = 0, \quad \left.\frac{\partial W}{\partial \mathbf{n}}\right|_{r=1+\varepsilon f(\theta)} = 0, \tag{29}$$

$$W(c, \theta) = 0, \quad \mathfrak{M}[W(c, \theta)] = 0. \tag{30}$$

The 0th order equation with simply supported inner, clamped outer boundary conditions is given by

$$\begin{cases} \nabla^4 W_0(r,\theta) - k_0^4 W_0(r,\theta) = 0, \\ W_0(1,\theta) = 0, \quad W_0(c,\theta) = 0, \\ W_{0_r}(1,\theta) = 0, \quad \mathfrak{M}[W_0(c,\theta)] = 0, \end{cases} \tag{31}$$

where the moment in polar coordinates for W_0 is $\mathfrak{M}[W_0(r,\theta)] = W_{0_{rr}}(r,\theta) + (\nu/r) W_{0_r}(r,\theta)$. Eq. (30) corresponds to the annular plate and its solution is given by Eq. (16) with new coefficients α_{01}^{sc}, α_{02}^{sc}, α_{03}^{sc} given in Appendix B.2. In this case we have

$\alpha_{01}^{sc} = \alpha_{01}^{cc}$, $\alpha_{02}^{sc} = \alpha_{02}^{cc}$, and $\alpha_{03}^{sc} = \alpha_{03}^{cc}$, because the last boundary condition (the only difference from CC case) involving moment includes the first and the second derivatives of Bessel functions. Due to recurrence relations of Bessel functions, the moment equation is linearly dependent with other boundary conditions. For a small c, the fundamental frequency of SC annular plate is governed by the $n = 1$ mode and its value is $k_0 = 4.611$, which is the first root of Eq. (17), when $c = 0$. As c increases, the frequency k rises singularly as $|\ln c|^{-1}$ and crosses the $n = 0$ mode at $c = 0.0042$ for the Poisson ratio $\nu = 0.3$ [3].

The 1st order $O(\varepsilon)$ equation with simply supported inner and clamped outer boundary conditions is given by

$$\begin{cases} \nabla^4 W_1(r,\theta) - k_0^4 W_1(r,\theta) = 0, \\ W_1(1,\theta) = \Phi_1(1,\theta), \quad W_1(c,\theta) = 0, \\ W_{1_r}(1,\theta) = \Psi_1(1,\theta), \quad \mathfrak{M}[W_1(c,\theta)] = 0. \end{cases} \tag{32}$$

The solution of Eq. (32) is given by Eq. (21) with new coefficients δ_{11}^{sc}, δ_{12}^{sc}, δ_{13}^{sc}, δ_{14}^{sc} given in Appendix B.2. The 2nd order $O(\varepsilon^2)$ equation with simply supported inner, clamped outer boundary conditions is given by

$$\begin{cases} \nabla^4 W_2(r,\theta) - k_0^4 W_2(r,\theta) = bk_0^4 W_0(r), \\ W_2(1,\theta) = \Phi_2(1,\theta), \quad W_2(c,\theta) = 0, \\ W_{2_r}(1,\theta) = \Psi_2(1,\theta), \quad \mathfrak{M}[W_2(c,\theta)] = 0. \end{cases} \tag{33}$$

The outer boundary conditions of Eq. (33) consist of two parts given by Eq. (23). Then we have a solution of type $W_2(r,\theta) = U(r) + V(r)\cos(2M\theta)$. The contribution to the first correction term $\varepsilon^2 b$ of the square of the fundamental frequency comes from the non-homogenous Eq. (24), the general solution of which is given by Eq. (25) with coefficients α_{01}^{sc}, α_{02}^{sc}, and α_{03}^{sc} in the part of the particular solution. Imposing SC boundary conditions

$$\begin{cases} U(1) = \varphi_0(1), \quad U(c) = 0, \\ U_r(1) = \xi_0(1), \quad U_{rr}(c) + (\nu/c)U_r(c) = 0 \end{cases} \tag{34}$$

into the solution gives us a linear system of equations which leads to a unique solution of b,

$$\begin{vmatrix} Y_0(k_0) & I_0(k_0) & K_0(k_0) & bF(1) + \varphi_0(1) \\ Y_1(k_0) & -I_1(k_0) & K_1(k_0) & -\dfrac{b}{k_0}F'(1) + \xi_0(1) \\ Y_0(ck_0) & I_0(ck_0) & K_0(ck_0) & bF(c) \\ \mathfrak{M}(Y_0(ck_0)) & \mathfrak{M}(I_0(ck_0)) & \mathfrak{M}(K_0(ck_0)) & b\mathfrak{M}(F(c)) \end{vmatrix} = 0, \tag{35}$$

where $F(r) = k_0 r/4[J_1(k_0 r) + \alpha_{01}^{sc} Y_1(k_0 r) - \alpha_{02}^{sc} I_1(k_0 r) + \alpha_{03}^{sc} K_1(k_0 r)]$.

Fig. 4 shows fundamental frequencies of wavy plates with simply supported inner, clamped outer boundary conditions for the $n = 0$ mode with respect to the core radius $c \geq 0.0042$. Table 2 lists the values of the fundamental frequencies k with

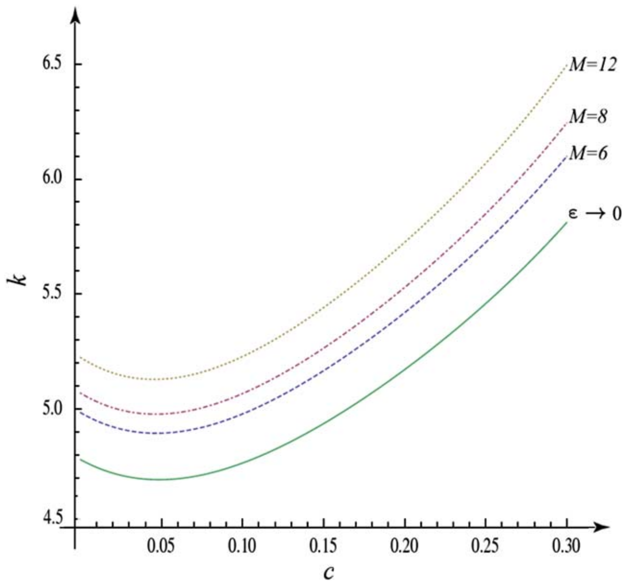

Fig. 4. Fundamental frequency vs. radius of the core of SC wavy plate.

Table 2
Fundamental frequency k of M-sided SC wavy boundary plate with core radius c for $\varepsilon = 0.1$ and $\nu = 0.3$.

$M \backslash c$	0.3	0.2	0.1	0.05	0.01	0.0042
5	6.01485	5.35819	4.93049	4.84897	4.90484	4.91991
6	6.09833	5.41869	4.97828	4.89362	4.94996	4.96536
7	6.17351	5.47500	5.02350	4.93598	4.99271	5.00840
8	6.24364	5.52853	5.06689	4.97673	5.03381	5.04977
12	6.49491	5.72464	5.22795	5.12848	5.18677	5.20369

various core radii c for the SC wavy plates of amplitude $\varepsilon = 0.1$. Notice that $\varepsilon = 0$ gives the fundamental frequency of the SC annular plate. Frequency values in Table 2 are obtained down to the transition point $c = 0.0042$.

3.3. Free–clamped boundary conditions (FC)

The boundary conditions of the FC plate are given by

$$W(1+\varepsilon f(\theta),\theta) = 0, \quad \left.\frac{\partial W}{\partial \mathbf{n}}\right|_{r=1+\varepsilon f(\theta)} = 0, \tag{36}$$

$$\mathfrak{M}(W(c,\theta)) = 0, \quad \mathfrak{B}(W(c,\theta)) = 0, \tag{37}$$

where $\mathfrak{M}$ is the moment and $\mathfrak{B}$ is the shear operator:

$$\mathfrak{M}(W(r,\theta)) = \frac{\partial^2 W}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2}\right), \tag{38}$$

$$\mathfrak{B}(W(r,\theta)) = \frac{\partial}{\partial r}(\nabla^2 W) + \frac{(1-\nu)}{r^2}\frac{\partial^2}{\partial \theta^2}\left(\frac{\partial W}{\partial r} - \frac{W}{r}\right). \tag{39}$$

The 0th order equation with free inner, clamped outer boundary conditions is

$$\begin{cases} \nabla^4 W_0(r,\theta) - k_0^4 W_0(r,\theta) = 0, \\ W_0(1,\theta) = 0, \quad \mathfrak{M}(W_0(c,\theta)) = 0, \\ W_{0_r}(1,\theta) = 0, \quad \mathfrak{B}(W_0(c,\theta)) = 0, \end{cases} \tag{40}$$

which corresponds to the annular plate. The solution of Eq. (40) is given by Eq. (16) with the corresponding coefficients $\alpha_{01}^{\mathrm{fc}}$, $\alpha_{02}^{\mathrm{fc}}$, and $\alpha_{03}^{\mathrm{fc}}$ given in Appendix B.3.

In the case of the FC boundary conditions, the $n = 0$ mode gives the fundamental frequency with no singular rise. Fig. 5 shows fundamental frequencies of wavy plates with free inner, clamped outer boundary conditions for the $n = 0$ mode with respect to the core radius $c \geq 0$. For a small c, the fundamental frequency of the annular plate is governed by the $n = 0$ mode and when $c = 0$, its value is $k_0 = 3.19622$. The 1st order $O(\varepsilon)$ equation with free inner, clamped outer boundary conditions is given by

$$\begin{cases} \nabla^4 W_1(r,\theta) - k_0^4 W_1(r,\theta) = 0, \\ W_1(1,\theta) = \Phi_1(1,\theta), \quad \mathfrak{M}(W_1(c,\theta)) = 0, \\ W_{1_r}(1,\theta) = \Psi_1(1,\theta), \quad \mathfrak{B}(W_1(c,\theta)) = 0. \end{cases} \tag{41}$$

The boundary conditions in Eq. (41) suggest that the 1st order equation has the solution Eq. (21) with corresponding coefficients $\delta_{11}^{\mathrm{fc}}$, $\delta_{12}^{\mathrm{fc}}$, $\delta_{13}^{\mathrm{fc}}$, $\delta_{14}^{\mathrm{fc}}$ given in Appendix B.3 for $n = 1$. The 2nd order $O(\varepsilon^2)$ equation with free inner, clamped outer boundary conditions is given by

$$\begin{cases} \nabla^4 W_2(r,\theta) - k_0^4 W_2(r,\theta) = bk_0^4 W_0(r), \\ W_2(1,\theta) = \Phi_2(1,\theta), \quad \mathfrak{M}(W_2(c,\theta)) = 0, \\ W_{2_r}(1,\theta) = \Psi_2(1,\theta), \quad \mathfrak{B}(W_2(c,\theta)) = 0. \end{cases} \tag{42}$$

The first correction term $\varepsilon^2 b$ of the square of the fundamental frequency can be obtained from the non-homogenous Eq. (24), the solution of which is given by Eq. (25) with the corresponding coefficients $\alpha_{01}^{\mathrm{fc}}$, $\alpha_{02}^{\mathrm{fc}}$, and $\alpha_{03}^{\mathrm{fc}}$ (for the particular solution) given in Appendix B.3. Imposing FC boundary conditions

$$\begin{cases} U(1) = \varphi_0(1), \quad \mathfrak{M}(U(c)) = 0, \\ U_r(1) = \xi_0(1), \quad \mathfrak{B}(U(c)) = 0 \end{cases} \tag{43}$$

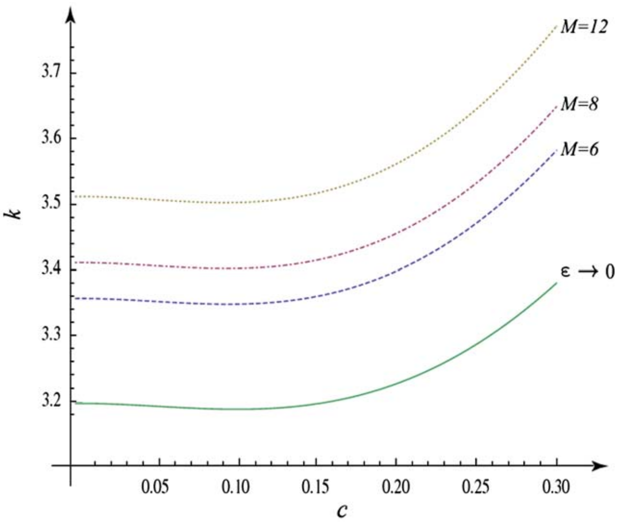

Fig. 5. Fundamental frequency vs. radius of the core of FC wavy plate.

Table 3
Fundamental frequency k of M-sided FC wavy boundary plate with core radius c for $\varepsilon = 0.1$ and $\nu = 0.3$.

$M\backslash c$	0.3	0.2	0.1	0.05	0.01	0.001
5	3.54707	3.36795	3.31970	3.32384	3.32805	3.32827
6	3.58191	3.39787	3.34791	3.35206	3.35636	3.35659
7	3.61577	3.42687	3.37528	3.37944	3.38383	3.38406
8	3.64863	3.45507	3.40193	3.40610	3.41057	3.41081
12	3.77149	3.56117	3.50245	3.50665	3.51141	3.51167

into the solution gives us a linear system of equations which leads to a unique solution of b,

$$\begin{vmatrix} Y_0(k_0) & I_0(k_0) & K_0(k_0) & bF(1)+\varphi_0(1) \\ Y_1(k_0) & -I_1(k_0) & K_1(k_0) & -\dfrac{b}{k_0}F'(1)+\xi_0(1) \\ \mathfrak{M}(Y_0(ck_0)) & \mathfrak{M}(I_0(ck_0)) & \mathfrak{M}(K_0(ck_0)) & b\mathfrak{M}(F(c)) \\ \mathfrak{V}(Y_0(ck_0)) & \mathfrak{V}(I_0(ck_0)) & \mathfrak{V}(K_0(ck_0)) & b\mathfrak{V}(F(c)) \end{vmatrix} = 0, \tag{44}$$

where $F(r) = (k_0 r/4)[J_1(k_0 r) + \alpha_{01}^{\mathrm{fc}} Y_1(k_0 r) - \alpha_{02}^{\mathrm{fc}} I_1(k_0 r) + \alpha_{03}^{\mathrm{fc}} K_1(k_0 r)]$. Table 3 lists the values of the fundamental frequencies k with various core radii c for the FC wavy plates of amplitude $\varepsilon = 0.1$. Note that $\varepsilon = 0$ gives the fundamental frequency of the FC annular plate.

3.4. Clamped–simply supported boundary conditions (CS)

The boundary conditions for the CS plate are given by

$$W(1+\varepsilon f(\theta),\theta) = 0, \quad \mathfrak{M}[W(1+\varepsilon f(\theta),\theta)] = 0, \tag{45}$$

$$W(c,\theta) = 0, \quad \left.\frac{\partial W}{\partial \mathbf{n}}\right|_{r=c} = 0. \tag{46}$$

The 0th order equation with the CS boundary conditions is

$$\begin{cases} \nabla^4 W_0(r,\theta) - k_0^4 W_0(r,\theta) = 0, \\ W_0(c,\theta) = 0, \quad W_0(1,\theta) = 0, \\ W_{0_r}(c,\theta) = 0, \quad \mathfrak{M}[W_0(1,\theta)] = 0, \end{cases} \tag{47}$$

which corresponds to the CS annular plate. Then the solution of Eq. (47) is given by Eq. (16) with corresponding coefficients $\alpha_{01}^{\mathrm{cs}}$, $\alpha_{02}^{\mathrm{cs}}$, and $\alpha_{03}^{\mathrm{cs}}$ given in Appendix B.4. For small c, the fundamental frequency is governed by the $n = 1$ mode. When $c = 0$, the $n=1$ mode rises singularly as $|\ln c|^{-1}$ from the first root ($k_0 = 3.728$ for $\nu = 0.3$) of the equation

$$\{2\nu[I_0(k_0)+I_2(k_0)]+k_0I_3(k_0)\}J_1(k_0)-\{2\nu[J_0(k_0)-J_2(k_0)]-k_0[6J_1(k_0)-J_3(k_0)]\}I_1(k_0)=0, \tag{48}$$

given in [3]. As c increases, the frequency k rises singularly as $|\ln c|^{-1}$ and crosses the $n=0$ mode at $c = 0.00034$ with the value $k_0 = 3.849$ for the Poisson ratio $\nu = 0.3$ [3]. The 1st order, $O(\varepsilon)$, equation with the CS boundary conditions is given by

$$\begin{cases} \nabla^4 W_1(r,\theta) - k_0^4 W_1(r,\theta) = 0, \\ W_1(1,\theta) = \Phi_1(1,\theta), \quad W_1(c,\theta)=0, \\ \mathfrak{M}[W_1(1,\theta)] = \Omega_1(1,\theta), \quad W_{1_r}(c,\theta) = 0. \end{cases} \tag{49}$$

The solution of which is given by Eq. (21) with corresponding coefficients δ_{11}^{cs}, δ_{12}^{cs}, δ_{13}^{cs}, δ_{14}^{cs} given in Appendix B.4. The 2nd order, $O(\varepsilon^2)$, equation with the CS boundary conditions is given by

$$\begin{cases} \nabla^4 W_2(r,\theta) - k_0^4 W_2(r,\theta) = bk_0^4 W_0(r), \\ W_2(1,\theta) = \Phi_2(1,\theta), \quad W_2(c,\theta)=0, \\ \mathfrak{M}[W_2(1,\theta)] = \Omega_2(1,\theta), \quad W_{2_r}(c,\theta) = 0. \end{cases} \tag{50}$$

Then the outer boundary conditions of Eq. (50) consist of two parts:

$$\Phi_2(r,\theta) = \varphi_0(r) + \varphi_2(r)\cos(2M\theta),$$

$$\Omega_2(r,\theta) = \omega_0(r) + \omega_2(r)\cos(2M\theta), \tag{51}$$

where

$$\varphi_0(r) = -\tfrac{1}{4}W_{0_{rr}}(r) - \tfrac{1}{2}W_{1_r}^M(r),$$

$$\omega_0(r) = \frac{1}{2}W_{0_{rr}}(r)[M^2+\nu(1-M^2)] - \frac{\nu}{4}W_{0_{rrr}}(r) - \frac{1}{4}W_{0_{rrrr}}(r) - \frac{1}{2}M^2(2-\nu)W_1^M(r) + \left[M^2 + \frac{\nu}{2}(1-M^2)\right]W_{1_r}^M(r) - \frac{\nu}{2}W_{1_{rr}}^M(r) - \frac{1}{2}W_{1_{rrr}}^M(r). \tag{52}$$

Since the contribution to the fundamental frequency comes from the θ-independent terms, we can ignore the functions φ_2 and ω_2. In the case of simply supported outer boundary condition, we have $\varphi_0(r) = \varphi_2(r)$, but $\omega_0(r) \neq \omega_2(r)$. The 2nd order boundary conditions suggest that we have a solution of type $W_2(r,\theta) = U(r) + V(r)\cos(2M\theta)$, which gives us the θ-independent Eq. (24), which has the general solution Eq. (25) with corresponding coefficients α_{01}^{cs}, α_{02}^{cs}, and α_{03}^{cs} given in Appendix B.4. Imposing the CS boundary conditions

$$\begin{cases} U(1) = \varphi_0(1), & U(c) = 0, \\ \mathfrak{M}[U(1)] = \omega_0(1), & U_r(c) = 0 \end{cases} \tag{53}$$

into the solution gives us a linear system of equations which leads to a unique solution of b,

$$\begin{vmatrix} Y_0(k_0) & I_0(k_0) & K_0(k_0) & bF(1)+\varphi_0(1) \\ \mathfrak{M}[Y_0(k_0)] & \mathfrak{M}[I_0(k_0)] & \mathfrak{M}[K_0(k_0)] & b\mathfrak{M}[F(1)]+\omega_0(1) \\ Y_0(ck_0) & I_0(ck_0) & K_0(ck_0) & bF(c) \\ Y_1(ck_0) & -I_1(ck_0) & K_1(ck_0) & -\dfrac{b}{k_0}F'(c) \end{vmatrix} = 0, \tag{54}$$

where $F(r) = (k_0 r/4)[J_1(k_0 r) + \alpha_{01}^{cs} Y_1(k_0 r) - \alpha_{02}^{cs} I_1(k_0 r) + \alpha_{03}^{cs} K_1(k_0 r)]$.

Fig. 6 shows fundamental frequencies of wavy plates with clamped inner, simply supported outer boundary conditions for the $n=0$ mode with respect to the core radius $c \geq 0.00034$. Table 4 lists the values of the fundamental frequencies k with various core radii c for the CS wavy plates of amplitude $\varepsilon = 0.1$. Frequency values in Table 4 are obtained down to the transition point $c = 0.00034$.

3.5. Simply supported–simply supported boundary conditions (SS)

The boundary conditions of the SS plate are given by

$$W(1+\varepsilon f(\theta),\theta) = 0, \quad \mathfrak{M}[W(1+\varepsilon f(\theta),\theta)] = 0, \tag{55}$$

$$W(c,\theta) = 0, \quad \mathfrak{M}[W(c,\theta)] = 0. \tag{56}$$

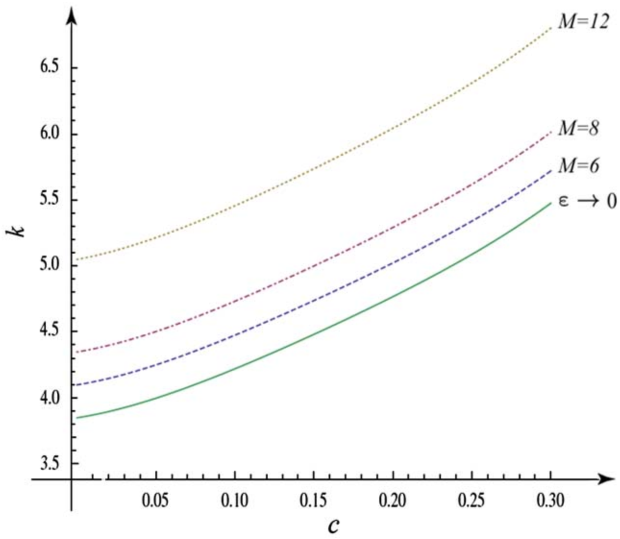

Fig. 6. Fundamental frequency vs. radius of the core of CS wavy plate.

Table 4
Fundamental frequency k of M-sided CS wavy boundary plate with core radius c for $\varepsilon = 0.1$ and $\nu = 0.3$.

$M \backslash c$	0.3	0.2	0.1	0.05	0.01	0.001	0.00034
5	5.59360	4.91488	4.37478	4.15478	4.02084	4.00602	4.00570
6	5.72013	5.02189	4.47292	4.25021	4.11459	4.09955	4.09922
7	5.85816	5.14675	4.59130	4.36642	4.22934	4.21408	4.21374
8	6.01294	5.29126	4.73019	4.50325	4.36467	4.34919	4.34885
12	6.80398	6.04377	5.45499	5.21607	5.06877	5.05207	5.05170

The 0th order equation with the SS boundary conditions is

$$\begin{cases} \nabla^4 W_0(r,\theta) - k_0^4 W_0(r,\theta) = 0, \\ W_0(c,\theta) = 0, \quad W_0(1,\theta) = 0, \\ \mathfrak{M}[W_0(c,\theta)] = 0, \quad \mathfrak{M}[W_0(1,\theta)] = 0, \end{cases} \tag{57}$$

which corresponds to the SS annular plate with solution Eq. (16), where the coefficients are replaced by α_{01}^{ss}, α_{02}^{ss}, and α_{03}^{ss} given in Appendix B.5. For a small c, the fundamental frequency is governed by the $n = 1$ mode. When $c = 0$, the $n = 1$ mode rises singularly as $|\ln c|^{-1}$ from the first root ($k_0 = 3.728$ for $\nu = 0.3$) of Eq. (48), which is the same as the CS case except the transition point is much bigger than that of the SS case. As c increases, the frequency k rises singularly as $|\ln c|^{-1}$ and crosses the $n = 0$ mode at $c = 0.0013$ with the value $k_0 = 3.848$ for the Poisson ratio $\nu = 0.3$ [3]. The 1st order, $O(\varepsilon)$, equation with simply supported inner and outer boundary conditions is given by

$$\begin{cases} \nabla^4 W_1(r,\theta) - k_0^4 W_1(r,\theta) = 0, \\ W_1(1,\theta) = \Phi_1(1,\theta), \quad W_1(c,\theta) = 0, \\ \mathfrak{M}[W_1(1,\theta)] = \Omega_1(1,\theta), \quad \mathfrak{M}[W_1(c,\theta)] = 0. \end{cases} \tag{58}$$

The solution of which is given by Eq. (21) with the corresponding coefficients δ_{11}^{ss}, δ_{12}^{ss}, δ_{13}^{ss}, δ_{14}^{ss} given in Appendix B.5. The 2nd order, $O(\varepsilon^2)$, equation with the SS boundary conditions is given by

$$\begin{cases} \nabla^4 W_2(r,\theta) - k_0^4 W_2(r,\theta) = b k_0^4 W_0(r), \\ W_2(1,\theta) = \Phi_2(1,\theta), \quad W_2(c,\theta) = 0, \\ \mathfrak{M}[W_2(1,\theta)] = \Omega_2(1,\theta), \quad \mathfrak{M}[W_2(c,\theta)] = 0. \end{cases} \tag{59}$$

The θ-independent non-homogenous part of Eq. (59) is given by Eq. (24), the solution of which is Eq. (25) with the corresponding coefficients α_{01}^{ss}, α_{02}^{ss}, and α_{03}^{ss} (for the particular solution) given in Appendix B.5. Imposing the SS

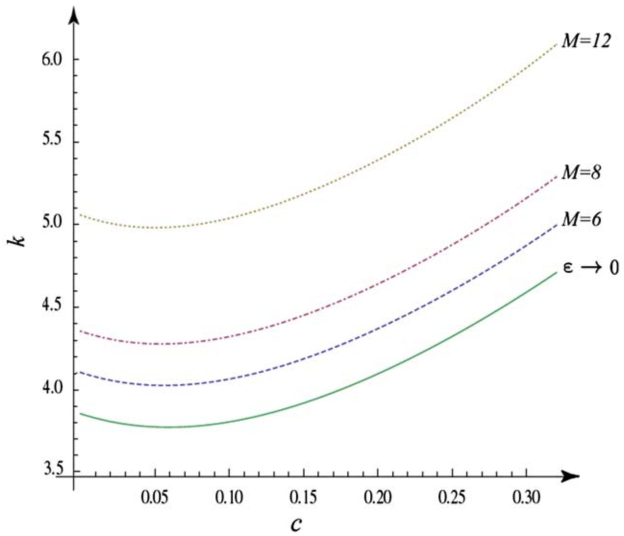

Fig. 7. Fundamental frequency vs. radius of the core of SS wavy plate.

Table 5
Fundamental frequency k of M-sided SS wavy boundary plate with core radius c for $\varepsilon = 0.1$ and $\nu = 0.3$.

$M\backslash c$	0.3	0.2	0.1	0.05	0.01	0.0013
5	4.76073	4.26665	3.96902	3.93136	3.98899	4.00483
6	4.87437	4.36993	4.06545	4.02517	4.08230	4.09833
7	5.00747	4.49443	4.18334	4.14022	4.19664	4.21284
8	5.16116	4.64002	4.32207	4.27591	4.33155	4.34794
12	5.95033	5.39027	5.03876	4.97952	5.03326	5.05071

boundary conditions

$$\begin{cases} U(1) = \varphi_0(1), & U(c) = 0, \\ \mathfrak{M}[U(1)] = \omega_0(1), & \mathfrak{M}[U(c)] = 0 \end{cases} \tag{60}$$

into the solution gives us a linear system of equations which leads to a unique solution of b,

$$\begin{vmatrix} Y_0(k_0) & I_0(k_0) & K_0(k_0) & bF(1) + \varphi_0(1) \\ \mathfrak{M}[Y_0(k_0)] & \mathfrak{M}[I_0(k_0)] & \mathfrak{M}[K_0(k_0)] & b\mathfrak{M}[F(1)] + \omega_0(1) \\ Y_0(ck_0) & I_0(ck_0) & K_0(ck_0) & bF(c) \\ \mathfrak{M}\left[Y_0(ck_0)\right] & \mathfrak{M}\left[I_0(ck_0)\right] & \mathfrak{M}\left[K_0(ck_0)\right] & b\,\mathfrak{M}[F(c)] \end{vmatrix} = 0, \tag{61}$$

where $F(r) = (k_0 r/4)[J_1(k_0 r) + \alpha_{01}^{SS} Y_1(k_0 r) - \alpha_{02}^{SS} I_1(k_0 r) + \alpha_{03}^{SS} K_1(k_0 r)]$.

Fig. 7 shows fundamental frequencies of wavy plates with simply supported inner and outer boundary conditions for the $n = 0$ mode with respect to the core radius $c \geq 0.0013$. Table 5 lists the values of the fundamental frequencies k with various core radii c for the SS wavy plates of amplitude $\varepsilon = 0.1$. Frequency values in Table 5 are obtained down to the transition point $c = 0.0013$.

3.6. Free–simply supported boundary conditions (FS)

The boundary conditions of the FS plate are given by

$$W(1 + \varepsilon f(\theta), \theta) = 0, \quad \mathfrak{M}[W(1 + \varepsilon f(\theta), \theta)] = 0, \tag{62}$$

$$\mathfrak{M}[W(c, \theta)] = 0, \quad \mathfrak{V}[W(c, \theta)] = 0. \tag{63}$$

The 0th order equation with the FS boundary conditions is

$$\begin{cases} \nabla^4 W_0(r,\theta) - k_0^4 W_0(r,\theta) = 0, \\ \mathfrak{M}[W_0(c,\theta)] = 0, \quad W_0(1,\theta) = 0, \\ \mathfrak{B}[W_0(c,\theta)] = 0, \quad \mathfrak{M}[W_0(1,\theta)] = 0. \end{cases} \tag{64}$$

Eq. (64) corresponds to annular plate (unperturbed case) and its solution is given by Eq. (16) with the corresponding coefficients α_{01}^{fs}, α_{02}^{fs}, and α_{03}^{fs} given in Appendix B.6. There is no singular rise in the fundamental frequency as $c \to 0$ for the FS case, which gives the following characteristic equation for the axisymmetric mode ($n = 0$):

$$[2\nu I_1(k_0) + k_0 I_2(k_0)]J_0(k_0) + [2k_0 J_0(k_0) + 2\nu J_1(k_0) - k_0 J_2(k_0)]I_0(k_0) = 0 \tag{65}$$

and the value for the fundamental frequency, when $c = 0$, is $k_0 = 2.222$ for the Poisson ratio $\nu = 0.3$ [3]. The 1st order, $O(\varepsilon)$, equation with the FS boundary conditions is

$$\begin{cases} \nabla^4 W_1(r,\theta) - k_0^4 W_1(r,\theta) = 0, \\ W_1(1,\theta) = \Phi_1(1,\theta), \quad \mathfrak{M}[W_1(c,\theta)] = 0, \\ \mathfrak{M}[W_1(1,\theta)] = \Omega_1(1,\theta), \quad \mathfrak{B}[W_1(c,\theta)] = 0. \end{cases} \tag{66}$$

The solution of which is given by Eq. (21) with the corresponding coefficients δ_{n1}^{fs}, δ_{n2}^{fs}, δ_{n3}^{fs}, δ_{n4}^{fs} for $n = 1$ given in Appendix B.6. The 2nd order, $O(\varepsilon^2)$, equation with the FS boundary conditions is given by

$$\begin{cases} \nabla^4 W_2(r,\theta) - k_0^4 W_2(r,\theta) = bk_0^4 W_0(r), \\ W_2(1,\theta) = \Phi_2(1,\theta), \quad \mathfrak{M}[W_2(c,\theta)] = 0, \\ \mathfrak{M}[W_2(1,\theta)] = \Omega_2(1,\theta), \quad \mathfrak{B}[W_2(c,\theta)] = 0. \end{cases} \tag{67}$$

The θ-independent non-homogenous part of Eq. (67) is given by Eq. (24), the solution of which is Eq. (25) with the corresponding coefficients α_{01}^{fs}, α_{02}^{fs}, and α_{03}^{fs} (for the particular solution) given in Appendix B.6. Imposing the FS boundary conditions

$$\begin{cases} U(1) = \varphi_0(1), \quad \mathfrak{M}[U(c)] = 0, \\ \mathfrak{M}[U(1)] = \omega_0(1), \quad \mathfrak{B}[U(c)] = 0 \end{cases} \tag{68}$$

into the solution gives us a linear system of equations which leads to a unique solution of b,

$$\begin{vmatrix} Y_0(k_0) & I_0(k_0) & K_0(k_0) & bF(1) + \varphi_0(1) \\ \mathfrak{M}[Y_0(k_0)] & \mathfrak{M}[I_0(k_0)] & \mathfrak{M}[K_0(k_0)] & b\mathfrak{M}[F(1)] + \omega_0(1) \\ \mathfrak{M}[Y_0(ck_0)] & \mathfrak{M}[I_0(ck_0)] & \mathfrak{M}[K_0(ck_0)] & b\mathfrak{M}[F(c)] \\ \mathfrak{B}[Y_0(ck_0)] & \mathfrak{B}[I_0(ck_0)] & \mathfrak{B}[K_0(ck_0)] & b\mathfrak{B}[F(c)] \end{vmatrix} = 0, \tag{69}$$

where $F(r) = (k_0 r/4)[J_1(k_0 r) + \alpha_{01}^{fs} Y_1(k_0 r) - \alpha_{02}^{fs} I_1(k_0 r) + \alpha_{03}^{fs} K_1(k_0 r)]$.

Fig. 8 shows fundamental frequencies of wavy plates with free inner and simply supported outer boundary conditions for the mode $n = 0$ with respect to the core radius $c \geq 0$. Table 6 lists the values of the fundamental frequencies k with various core radii c for the FS wavy plates of amplitude $\varepsilon = 0.1$.

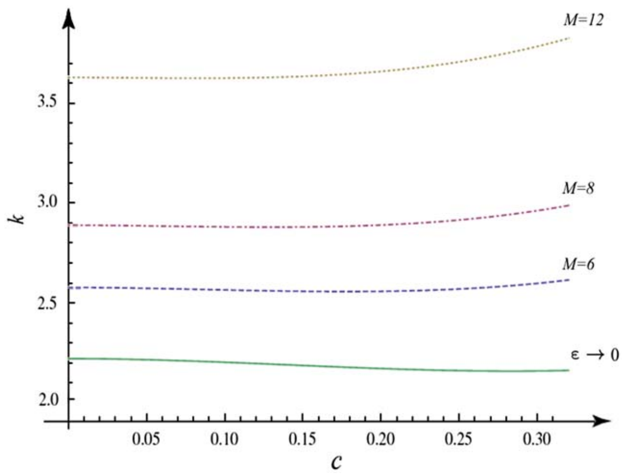

Fig. 8. Fundamental frequency vs. radius of the core of FS wavy plate.

Table 6
Fundamental frequency k of M-sided FS wavy boundary plate with core radius c for $\varepsilon = 0.1$ and $\nu = 0.3$.

$M\backslash c$	0.3	0.2	0.1	0.05	0.01	0.001	0.0001
5	2.44617	2.42291	2.43745	2.44772	2.42202	2.45221	2.45222
6	2.59777	2.55830	2.56472	2.57342	2.57730	2.57748	2.57748
7	2.77204	2.71600	2.71421	2.72135	2.72482	2.72498	2.72499
8	2.96166	2.88978	2.88023	2.88595	2.88905	2.88920	2.88921
12	3.78321	3.65848	3.62452	3.62623	3.62841	3.62853	3.62853

4. Polygonal plates

This section uses a boundary perturbation method (BPM) given in Section 2 to study free vibration of polygonal plates with a concentric circular core. The natural frequencies of these plates are determined by calculating the eigenvalues of the governing equations using the BPM. The boundary of the polygon can be written as

$$r = 1 + f(\theta) = 1 + \sum_{n=1}^{\infty} c_n \cos(nM\theta), \tag{70}$$

where

$$c_n = \frac{2a}{\beta}\int_0^{\beta} \frac{\cos(nM\theta)}{\cos\theta}\,d\theta, \quad \beta = \pi/M. \tag{71}$$

Here, we determine a such that the mean radius of the polygon is 1. See [13] for tabulated values of c_n. Thus the boundary perturbation in Section 2 applies to $r = 1 + f(\theta)$. Now f is $O(\varepsilon)$, instead of the previous order $O(1)$, due to known Fourier coefficients (71). The more terms in the Fourier series (70) will result in a better-approximated polygon. While very few results are available about circularly periodic plates with a core, Grossi et al. [14], and Huang and Sakiyama [15] gave numerical results for fundamental frequencies of clamped rectangular plates with a circular cutout, FC case. Gutierrez et al. [16] approximate the frequency by means of conformal mapping-Rayleigh–Ritz approach and finite element method for SC and SS plates. We compared our results with the available literature in the case of FC, SC, and SS square plates ($M = 4$) in the corresponding sections below.

4.1. Clamped–clamped boundary conditions (CC)

Let c be the radius of the concentric circular core. We perturbed the outer boundary of an annular plate to obtain an M-sided regular polygonal plate with a concentric circular core. The general solution of a circular plate with a concentric circular core is given by

$$W(r,\theta) = \sum_{n=0}^{\infty}[C_{n1}J_n(kr) + C_{n2}Y_n(kr) + C_{n3}I_n(kr) + C_{n4}K_n(kr)]\cos(n\theta), \tag{72}$$

where $C_{n1}, C_{n2}, C_{n3}, C_{n4}$ are arbitrary constants to be determined. We proceed with a perturbation scheme adopting the boundary perturbation method presented in Section 2. Let b be the correction parameter to the fundamental frequency. We perturb the solution $W(r,\theta)$ and the fundamental frequency k about the circular state as before in Eqs. (2) and (3). We take the boundary function $f(\theta)$ given by Eq. (70). Solutions of the 0th and the 1st order equations with the CC boundary conditions are given in Section 3. The 2nd order, $O(\varepsilon^2)$, equation with corresponding 2nd order clamped boundary conditions is given by Eq. (22). Boundary conditions of Eq. (22) suggest that we have a solution of type $W_2(r,\theta) = U(r) + \sum_{n=1}^{\infty} V_n(r)\cos(nM\theta)$. Then we obtain

$$\begin{cases} \sum_{n=1}^{\infty} \nabla^4 V_n(r,\theta) - k_0^4 V_n(r,\theta) = 0, \\ \nabla^4 U(r) - k_0^4 U(r) = bk_0^4 W_0(r). \end{cases} \tag{73}$$

The solution of the homogenous part of Eq. (73) has no effect on the correction parameter, b, of the fundamental frequency. Therefore, we solve the non-homogenous part of Eq. (73), the solution of which is given by Eq. (25). The outer boundary conditions for $\nabla^4 U(r) - k_0^4 U(r) = bk_0^4 W_0(r)$ are the θ-independent part of

$$\Phi_2(r,\theta) = \varphi_0(r) + \sum_{n=1}^{\infty} \varphi_n(r)\cos nM\theta,$$

$$\Psi_2(r,\theta) = \xi_0(r) + \sum_{n=1}^{\infty} \xi_n(r)\cos nM\theta. \tag{74}$$

Table 7
Fundamental frequency k of M-sided CC polygonal plate with core radius c and N iterations.

	$N\backslash c$	0.3	0.2	0.1	0.05	0.00132
$M=5$	3	6.91338	6.06538	5.38948	5.11118	4.94787
	6	6.92578	6.07494	5.39711	5.11807	4.95431
	12	6.92982	6.07802	5.39954	5.12026	4.95636
$M=6$	3	6.86461	6.00014	5.32464	5.04868	4.88731
	6	6.87090	6.00502	5.32854	5.05219	4.89060
	12	6.87291	6.00657	5.32978	5.05330	4.89164
$M=7$	3	6.82564	5.96114	5.28916	5.01525	4.85531
	6	6.82933	5.96401	5.29145	5.01731	4.85724
	12	6.83050	5.96491	5.29216	5.01795	4.85784
$M=8$	3	6.79942	5.93732	5.26835	4.99589	4.83690
	6	6.80179	5.93916	5.26982	4.99721	4.83813
	12	6.80253	5.93973	5.27027	4.99762	4.83851
$M=12$	3	6.75570	5.90030	5.23716	4.96721	4.80979
	6	6.75636	5.90081	5.23756	4.96757	4.81013
	12	6.75656	5.90096	5.23768	4.96768	4.81023
$\varepsilon\to 0$	N/A	6.73396	5.88296	5.22308	4.95444	4.76911

Then we have the following CC boundary conditions:

$$\begin{cases} U(1)=\varphi_0(1), & U(c)=0, \\ U_r(1)=\xi_0(1), & U_r(c)=0, \end{cases} \tag{75}$$

where the constant terms $\varphi_0(1)$ and $\xi_0(1)$ are

$$\varphi_0(1)=-\frac{1}{4}W_{0_{rr}}(1)\sum_{n=1}^{\infty}c_n^2-\frac{1}{2}\sum_{n=1}^{\infty}c_nW_{1_r}^{nM}(1),$$

$$\xi_0(1)=-\frac{1}{4}W_{0_{rrr}}(1)\sum_{n=1}^{\infty}c_n^2+\frac{1}{2}\sum_{n=1}^{\infty}c_n(nM)^2W_1^{nM}(1)-\frac{1}{2}\sum_{n=1}^{\infty}c_nW_{1_{rr}}^{nM}(1). \tag{76}$$

Imposing the boundary conditions (75) into Eq. (25), gives us a linear system of equations which leads to a unique solution of b, Eq. (28). Table 7 lists the values of the fundamental frequencies k with first N partial sum, $N=3,6,12$, and various c for the polygonal plate with the CC boundary conditions. In practice the corners may not be mathematically sharp and a finite N in the summation Eqs. (76) would be desirable. For $N=12$, the boundary function $f(\theta)$ gives good approximation to a hexagon (given the corner curvature, see [13] for determination of N). In the polygonal case, we no longer have the same area for each M-sided polygon, since the Fourier coefficients c_n keep adding to the area. Therefore, the area of a polygon depends on the number of sides and it increases as M increases. Thus fundamental frequencies decrease as M increases for the same c since larger areas result in smaller frequencies.

4.2. Simply supported–clamped boundary condition (SC)

Solutions of the 0th, the 1st, and the 2nd order equations for SC boundary conditions are given in Section 3.2. The only contribution to the correction term of the fundamental frequency comes from the non-homogenous equation $\nabla^4U(r)-k_0^4U(r)=bk_0^4W_0(r)$, the boundary conditions of which are the θ-independent part of Eq. (74):

$$\begin{cases} U(1)=\varphi_0(1), & U(c)=0, \\ U_r(1)=\xi_0(1), & U_{rr}(c)+\dfrac{\nu}{c}U_r(c)=0, \end{cases} \tag{77}$$

where the constant terms $\varphi_0(1)$ and $\xi_0(1)$ are given by Eqs. (76). A unique solution for the correction term of the fundamental frequency, b, can be obtained from Eq. (35). Table 8 lists the values of the fundamental frequencies k with $N=3,6,12$ and various c for the polygonal plate with the SC boundary conditions. Frequency values in Table 8 are obtained down to the transition point $c=0.0042$. Results in Table 8 are normalized with respect to the averaging circle and we determine the radius of the inscribing circle a such that the mean radius of the polygon is 1, $a=[(1/\beta)\int_0^{\beta}1/\cos\theta\,d\theta]^{-1}$ where $\beta=\pi/M$. Frequency values in the literature are normalized with respect to the side of a polygon. The frequency values of square plates are compared with the existing literature in Table 9. Although our frequency values are in good agreement with those that are given in Table 9, our method gives a better approximation for polygons with $M\geq 5$. In the case of hexagonal plate with SC boundary conditions, Gutierrez et al. [16] reported 5.9250 ($=\sqrt{44.21}a$), 5.38585

Table 8
Fundamental frequency k of M-sided SC polygonal plate with core radius c and N iterations, $\nu = 0.3$.

	$N\backslash c$	0.3	0.2	0.1	0.01	0.0042
$M = 4$	3	6.12105	5.48565	5.05373	5.02268	5.03792
	6	6.14729	5.50587	5.07029	5.03838	5.05372
	12	6.15598	5.51251	5.07569	5.04350	5.05887
$M = 5$	3	6.01154	5.34938	4.92046	4.89518	4.91020
	6	6.02191	5.35750	4.92712	4.90147	4.91654
	12	6.02525	5.36009	4.92924	4.90347	4.91855
$M = 6$	3	5.93938	5.27977	4.85756	4.83538	4.85022
	6	5.94470	5.28393	4.86097	4.83859	4.85345
	12	5.94638	5.28525	4.86204	4.83960	4.85447
$M = 7$	3	5.89653	5.24170	4.82432	4.80389	4.81860
	6	5.89966	5.24415	4.82632	4.80577	4.82049
	12	5.90064	5.24491	4.82694	4.80636	4.82108
$M = 8$	3	5.87040	5.21938	4.80519	4.78581	4.80043
	6	5.87241	5.22094	4.80647	4.78701	4.80164
	12	5.87303	5.22143	4.80687	4.78738	4.80201
$M = 12$	3	5.82981	5.18591	4.77704	4.75925	4.77373
	6	5.83037	5.18634	4.77739	4.75957	4.77406
	12	5.83054	5.18647	4.77750	4.75967	4.77416
$\varepsilon \to 0$	N/A	5.81080	5.17080	4.76460	4.74753	4.76194

Table 9
Comparison of fundamental frequencies k of M-sided FC, SC, and SS, square plates with core radius c, $\nu = 0.3$.

	c				
	0.3	0.2	0.1	0.01	0.001
FC					
Present	3.70382	3.50078	3.44409	3.45211	3.45234
Grossi et al. [14]	3.58314	3.45201	–	–	–
Huang and Sakiyama (by extrapolation) [15]	–	3.50126	–	–	–
Huang and Sakiyama [15]	–	3.48499	–	–	–
SC					
Present	6.15598	5.51251	5.07569	5.04350	5.05887
Gutierrez et al. [16]	6.18127	5.64261	5.22908	–	–
Gutierrez et al. (finite element solution) [16]	6.05182	5.55772	5.17613	–	–
SS					
Present	5.18079	4.69140	4.38264	4.38975	4.40599
Gutierrez et al. [16]	5.05301	4.61673	4.27100	–	–
Gutierrez et al. (finite element solution) [16]	4.92363	4.51781	4.25993	–	–

$(=\sqrt{36.53}a)$, and 4.96867 $(=\sqrt{31.09}a)$ for the core radii 0.3, 0.2, and 0.1, respectively. Our frequency values are 5.94638, 5.28525, and 4.86204, respectively, for the same core radii given in Table 8.

4.3. Free–clamped boundary condition (FC)

The 0th, the 1st, and the 2nd order solutions for the FC polygonal plates are given in Section 3.3. The boundary conditions for the 2nd order equation are given by

$$\begin{cases} U(1) = \varphi_0(1), & \mathfrak{M}(U(c)) = 0, \\ U_r(1) = \xi_0(1), & \mathfrak{V}(U(c)) = 0. \end{cases} \tag{78}$$

Then the unique solution for the correction term of the fundamental frequency, b, can be obtained from Eq. (44). A small free inner boundary condition has little effect on the frequency. In the case of annular plate with FC boundary condition, the fundamental frequency is found to be 3.196, which is the frequency of the full clamped circular plate at $c = 0$ [3]. Table 10 lists the values of the fundamental frequencies k with $N = 3, 6, 12$ and various values of c for the polygonal plate with FC

Table 10
Fundamental frequency k of M-sided FC polygonal plate with core radius c and N iterations, $\nu = 0.3$.

	$N\backslash c$	0.3	0.2	0.1	0.01	0.001
$M=4$	6	3.69971	3.49723	3.44072	3.44873	3.44897
	12	3.70382	3.50078	3.44409	3.45211	3.45234
	24	3.70497	3.50178	3.44503	3.45306	3.45329
$M=5$	3	3.53237	3.35537	3.30801	3.31645	3.31667
	6	3.53774	3.35996	3.31232	3.32077	3.32099
	12	3.53943	3.36139	3.31367	3.32213	3.32235
$M=6$	3	3.46708	3.29983	3.25613	3.26471	3.26493
	6	3.46986	3.30219	3.25834	3.26693	3.26715
	12	3.47072	3.30292	3.25903	3.26762	3.26783
$M=7$	3	3.43445	3.27216	3.23030	3.23893	3.23915
	6	3.43608	3.27354	3.23159	3.24023	3.24044
	12	3.43658	3.27396	3.23199	3.24063	3.24084
$M=8$	3	3.41630	3.25681	3.21598	3.22463	3.22485
	6	3.41735	3.25769	3.21681	3.22546	3.22567
	12	3.41767	3.25797	3.21706	3.22572	3.22593
$M=12$	3	3.39064	3.23517	3.19579	3.20445	3.20466
	6	3.39092	3.23541	3.19601	3.20467	3.20488
	12	3.39101	3.23548	3.19608	3.20474	3.20495
$\varepsilon \to 0$	N/A	3.37991	3.22614	3.18736	3.19601	3.19622

boundary conditions. The comparison of the fundamental frequency values of FC square plates is given in Table 9. Our values are in excellent agreement with Grossi et al. [14] and Huang and Sakiyama [15].

4.4. Clamped–simply supported boundary conditions (CS)

The 0th and the 1st order solutions for the CS polygonal plates are given in Section 3.4. The 2nd order, $O(\varepsilon^2)$, equation with the corresponding 2nd order CS boundary conditions is given by Eq. (50) where

$$\begin{cases} \Phi_2(r,\theta) = \varphi_0(r) + \sum\limits_{n=1}^{\infty} \varphi_n(r)\cos nM\theta, \\ \Omega_2(r,\theta) = \omega_0(r) + \sum\limits_{n=1}^{\infty} \omega_n(r)\cos nM\theta. \end{cases} \tag{79}$$

Then the θ-independent part of the 2nd order boundary conditions for the CS polygonal plates are

$$\begin{cases} U(1) = \varphi_0(1), & U(c) = 0, \\ U_{rr}(1) + \nu U_r(1) = \omega_0(1), & U_r(c) = 0, \end{cases} \tag{80}$$

where the constant terms $\varphi_0(1)$ and $\omega_0(1)$ are

$$\varphi_0(1) = -\frac{1}{4} W_{0_{rr}}(1) \sum_{n=1}^{\infty} c_n^2 - \frac{1}{2} \sum_{n=1}^{\infty} c_n W_{1_r}^{nM}(1),$$

$$\begin{aligned} \omega_0(1) = {} & \frac{1}{2} W_{0_{rr}}(1) \sum_{n=1}^{\infty} c_n^2 [(nM)^2 + \nu(1-(nM)^2)] - \frac{\nu}{4} W_{0_{rrr}}(1) \sum_{n=1}^{\infty} c_n^2 \\ & - \frac{1}{4} W_{0_{rrrr}}(1) \sum_{n=1}^{\infty} c_n^2 - \frac{1}{2} \sum_{n=1}^{\infty} c_n (nM)^2 (2-\nu) W_1^{nM}(1) \\ & + \sum_{n=1}^{\infty} c_n \left[(nM)^2 + \frac{\nu}{2}(1-(nM)^2)\right] W_{1_r}^{nM}(1) - \frac{\nu}{2} \sum_{n=1}^{\infty} c_n W_{1_{rr}}^{nM}(1) \\ & - \frac{1}{2} \sum_{n=1}^{\infty} c_n W_{1_{rrr}}^{nM}(1). \end{aligned} \tag{81}$$

Similarly, the unique solution for the correction term of the fundamental frequency, b, can be obtained from Eq. (54). Table 11 lists the values of the fundamental frequencies k with $N = 3, 6, 12$ and various values of c for the polygonal plate with the CS boundary conditions.

Table 11
Fundamental frequency k of M-sided CS polygonal plate with core radius c and N iterations, $\nu = 0.3$.

	$N\backslash c$	0.3	0.2	0.1	0.01	0.00034
$M = 5$	3	5.67104	4.98180	4.43837	4.08340	4.06819
	6	5.74499	5.05492	4.51133	4.15606	4.14079
	12	5.81934	5.12869	4.58500	4.22940	4.21406
$M = 6$	3	5.63758	4.93225	4.38417	4.02879	4.01363
	6	5.68982	4.98474	4.43707	4.08175	4.06656
	12	5.74301	5.03828	4.49103	4.13574	4.12052
$M = 7$	3	5.60807	4.89925	4.35062	3.99572	3.98062
	6	5.64893	4.94058	4.39247	4.03775	4.02263
	12	5.69086	4.98305	4.43546	4.08087	4.06573
$M = 8$	3	5.58612	4.87664	4.32821	3.97378	3.95873
	6	5.61990	4.91094	4.36305	4.00882	3.99375
	12	5.65475	4.94636	4.39899	4.04495	4.02986
$M = 12$	3	5.54064	4.83174	4.28412	3.93054	3.91556
	6	5.56107	4.85263	4.30545	3.95207	3.93708
	12	5.58239	4.87441	4.32767	3.97448	3.95949
$\varepsilon \to 0$	N/A	5.47519	4.76597	4.21774	3.86398	3.84905

Table 12
Fundamental frequency k of M-sided SS polygonal plate with core radius c and N iterations, $\nu = 0.3$.

	$N\backslash c$	0.3	0.2	0.1	0.01	0.0013
$M = 4$	3	4.91759	4.43960	4.14090	4.15457	4.17029
	6	5.05028	4.56632	4.26241	4.27264	4.28847
	12	5.18079	4.69140	4.38264	4.38975	4.40570
$M = 5$	3	4.83190	4.33382	4.03353	4.05143	4.06731
	6	4.91114	4.41130	4.10849	4.12400	4.13991
	12	4.99077	4.48925	4.18399	4.19723	4.21317
$M = 6$	3	4.77499	4.27512	3.97686	3.99691	4.01276
	6	4.83225	4.33156	4.03163	4.04983	4.06569
	12	4.89047	4.38897	4.08738	4.10377	4.11964
$M = 7$	3	4.73816	4.23908	3.94260	3.96395	3.97975
	6	4.78342	4.28386	3.98613	4.00595	4.02176
	12	4.82979	4.32974	4.03074	4.04904	4.06486
$M = 8$	3	4.71319	4.21507	3.91989	3.94209	3.95786
	6	4.75084	4.25241	3.95622	3.97711	3.99289
	12	4.78961	4.29086	3.99363	4.01321	4.02899
$M = 12$	3	4.66385	4.16783	3.87509	3.89898	3.91470
	6	4.68687	4.19078	3.89745	3.92050	3.93622
	12	4.71085	4.21465	3.92071	3.94290	3.95862
$\varepsilon \to 0$	N/A	4.59121	4.09630	3.80588	3.83247	3.84819

4.5. Simply supported–simply supported boundary condition (SS)

The 0th, the 1st, and the 2nd order solutions for the SS polygonal plates are given in Section 3.5. The θ-independent part of the 2nd order boundary conditions for the SS polygonal plates are

$$\begin{cases} U(1) = \varphi_0(1), & U(c) = 0, \\ U_{rr}(1) + \nu U_r(1) = \omega_0(1), & U_{rr}(c) + \dfrac{\nu}{c} U_r(c) = 0, \end{cases} \tag{82}$$

where the constant terms $\varphi_0(1)$ and $\omega_0(1)$ are given by Eqs. (81). Then unique solution for the correction term of the fundamental frequency, b, can be obtained from Eq. (61). Table 12 lists the values of the fundamental frequencies k with $N = 3, 6, 12$ and various c for the polygonal plate with the SS boundary conditions. Frequency values in Table 12 are obtained down to the transition point $c = 0.0013$. The comparison of the fundamental frequency values of SS square plates is given in Table 9. Our values are in good agreement with Gutierrez et al. [16]. In the case of hexagonal plate with SS boundary conditions, Gutierrez et al. [16] reported 4.85223 ($= \sqrt{29.65a}$), 4.40535 ($= \sqrt{24.44a}$), and 4.06016 ($= \sqrt{20.76a}$)

Table 13
Fundamental frequency k of M-sided FS polygonal plate with core radius c and N iterations, $\nu = 0.3$.

	$N\backslash c$	0.3	0.2	0.1	0.01	0.001
$M = 4$	6	2.90008	2.83241	2.82605	2.83579	2.83595
	12	3.06285	2.98238	2.96957	2.97777	2.97792
	24	3.20856	3.11763	3.09964	3.10667	3.10681
$M = 5$	3	2.56477	2.52763	2.53503	2.54772	2.54790
	6	2.68765	2.63805	2.63918	2.65032	2.65048
	12	2.80353	2.74323	2.73900	2.74883	2.74898
$M = 6$	3	2.48054	2.45151	2.46254	2.47598	2.47616
	6	2.57677	2.53719	2.54292	2.55503	2.55520
	12	2.66902	2.62011	2.62117	2.63212	2.63228
$M = 7$	3	2.42737	2.40388	2.41750	2.43151	2.43169
	6	2.50751	2.47475	2.48373	2.49656	2.49674
	12	2.58524	2.54413	2.54891	2.56070	2.56086
$M = 8$	3	2.39036	2.37100	2.38659	2.40104	2.40123
	6	2.45958	2.43189	2.44332	2.45671	2.45689
	12	2.52732	2.49201	2.49960	2.51204	2.51221
$M = 12$	3	2.31069	2.30109	2.32138	2.33695	2.33715
	6	2.35681	2.34113	2.35840	2.37321	2.37340
	12	2.40295	2.38147	2.39586	2.40993	2.41012
$\varepsilon \to 0$	N/A	2.15965	2.17203	2.20301	2.22130	2.22152

for the core radii 0.3, 0.2, and 0.1, respectively. Our frequency values for the same core radii are 4.89047, 4.38897, and 4.08738, respectively, given in Table 12 and they are in excellent agreement (with the maximum difference 2.7 percent).

4.6. Free–simply supported boundary condition (FS)

The 0th, the 1st, and the 2nd order solutions for the FS polygonal plates are given in Section 3.6. The θ-independent part of the 2nd order boundary conditions for the FS polygonal plates are

$$\begin{cases} U(1) = \varphi_0(1), & \mathfrak{M}(U(c)) = 0, \\ \mathfrak{M}(U(1)) = \omega_0(1), & \mathfrak{V}(U(c)) = 0. \end{cases} \tag{83}$$

Similarly, the unique solution for the correction term of the fundamental frequency, b, can be obtained from Eq. (69). As in the case of the FC, a small free inner boundary condition has little effect on the frequency [3]. Table 13 lists the values of the fundamental frequencies k with $N = 3, 6, 12$ and various c for the polygonal plate with the FS boundary conditions.

6. Conclusions

Fundamental frequencies of clamped and simply supported circularly periodic plates with a core are now determined. A boundary perturbation method is developed to extract the fundamental eigenvalue of the biharmonic boundary value problem (BVP). The method is then applied to wavy and polygonal plates with clamped, simply supported, and free cores. For simplicity, we started with wavy boundary plates and generalized the boundary function to polygons. Approximate analytical solutions of the biharmonic BVP and formulations of the fundamental frequencies for the plates are obtained. Fundamental frequency values of rectangular plates with boundary conditions SC, FC, and SS are compared with the existing literature and found in good agreement with our results. The comparison is also made with hexagonal plates with SC and SS boundary conditions and the values are found to be in excellent agreement with the existing literature with the maximum difference being of the order of 2.7 percent.

In case of CC, SC, CS, and SS frequencies are calculated down to the transition radius of the core. When the radius c is very small (smaller then the transition radius), the approximation to the frequency should start from the mode $n = 1$, in which fundamental frequency occurs. Since there is no singular rise of the frequency in the case of FC, FS, the approximate solutions (44) and (83) are valid for all values of c. The smaller cores result in smaller frequencies, since the plates become less rigid with the exceptions of FS and FC polygonal and wavy plates. Our tables would be useful in the design of vibrating clamped and simply supported plates with a core and the parameters of the approximate formulations in Eqs. (28), (35), (44), (54), (61), and (69) can be modified to fit the specifics and the needs of the applications. The present analysis can be used to obtain the higher frequencies by simply applying the perturbation method to higher modes and frequencies about the circular state. It is worth to study ε^4 correction of the fundamental frequency, only if ε is large. Otherwise, $O(\varepsilon^4)$ is negligible.

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Appendix A. Perturbed boundary condition coefficients

Perturbed clamped and simply supported boundary conditions are given by the following functions:

$$\Phi_1(1,\theta) = -f(\theta)W_{0_r}(1,\theta), \tag{A.1}$$

$$\Phi_2(1,\theta) = -f(\theta)W_{1_r}(1,\theta) - \tfrac{1}{2}f^2(\theta)W_{0_{rr}}(1,\theta), \tag{A.2}$$

$$\Psi_1(1,\theta) = -f(\theta)W_{0_{rr}}(1,\theta), \tag{A.3}$$

$$\Psi_2(1,\theta) = f'(\theta)W_{1_\theta}(1,\theta) - f(\theta)W_{1_{rr}}(1,\theta) - \tfrac{1}{2}f^2(\theta)W_{0_{rrr}}(1,\theta), \tag{A.4}$$

$$\Omega_1(1,\theta) = \nu f(\theta)W_{0_r}(1,\theta) + \nu f''(\theta)W_{0_r}(1,\theta) - \nu f(\theta)W_{0_{rr}}(1,\theta) - f(\theta)W_{0_{rrr}}(1,\theta), \tag{A.5}$$

$$\begin{aligned}\Omega_2(1,\theta) = {} & (\nu-2)f'(\theta)W_{1_\theta}(1,\theta) - \nu(f^2(\theta) + 2f(\theta) + f''(\theta))W_{0_r}(1,\theta) \\ & + \nu(f(\theta) + f''(\theta))W_{1_r}(1,\theta) + 2f'(\theta)W_{1_{r\theta}}(1,\theta) \\ & + (\nu f^2(\theta) + (f'(\theta))^2 + \nu f(\theta)f''(\theta))W_{0_{rr}}(1,\theta) - \nu f(\theta)W_{1_{rr}}(1,\theta) \\ & - \frac{\nu}{2}f^2(\theta)W_{0_{rrr}}(1,\theta) - f(\theta)W_{1_{rrr}}(1,\theta) - \frac{1}{2}f^2(\theta)W_{0_{rrrr}}(1,\theta).\end{aligned} \tag{A.6}$$

Appendix B. Plate coefficients

B.1. Coefficients of the CC plates

The coefficients of the 0th order solution (16) are

$$\alpha_{01}^{cc} = \frac{\beta_{01}K_0(k_0) - \beta_{02}K_0(ck_0) - \beta_{03}K_1(k_0)}{\beta_{04}Y_0(k_0) - \beta_{05}Y_0(ck_0) - \beta_{06}Y_1(k_0)}, \tag{B.1}$$

$$\alpha_{02}^{cc} = \frac{1}{\beta_{05}}\{J_1(k_0)K_0(k_0) - J_0(k_0)K_1(k_0) - \alpha_{01}^{cc}(K_1(k_0)Y_0(k_0) - K_0(k_0)Y_1(k_0))\}, \tag{B.2}$$

$$\alpha_{03}^{cc} = -\frac{1}{K_0(k_0)}[J_0(k_0) + \alpha_{01}^{cc}Y_0(k_0) + \alpha_{02}^{cc}I_0(k_0)], \tag{B.3}$$

where

$$\begin{cases}\beta_{01} = I_1(k_0)J_0(ck_0) + I_0(ck_0)J_1(k_0), \\ \beta_{02} = I_1(k_0)J_0(k_0) + I_0(k_0)J_1(k_0), \\ \beta_{03} = I_0(ck_0)J_0(k_0) - I_0(k_0)J_0(ck_0), \\ \beta_{04} = I_1(k_0)K_0(ck_0) + I_0(ck_0)K_1(k_0), \\ \beta_{05} = I_1(k_0)K_0(k_0) + I_0(k_0)K_1(k_0), \\ \beta_{06} = I_0(ck_0)K_0(k_0) - I_0(k_0)K_0(ck_0).\end{cases} \tag{B.4}$$

The coefficients of the first-order solution (20) can be determined by Cramer's rule. Let

$$\mathbf{A}_n^{cc} = \begin{bmatrix} J_{nM}(k_0) & Y_{nM}(k_0) & I_{nM}(k_0) & K_{nM}(k_0) \\ J'_{nM}(k_0) & Y'_{nM}(k_0) & I'_{nM}(k_0) & K'_{nM}(k_0) \\ J_{nM}(ck_0) & Y_{nM}(ck_0) & I_{nM}(ck_0) & K_{nM}(ck_0) \\ J'_{nM}(ck_0) & Y'_{nM}(ck_0) & I'_{nM}(ck_0) & K'_{nM}(ck_0) \end{bmatrix}, \tag{B.5}$$

$$\mathbf{B}_n^{cc} = [0 \;\; -c_nW_{0_{rr}}(1) \;\; 0 \;\; 0]^{\mathrm{T}}, \tag{B.6}$$

and let $\mathbf{A}_{ni}^{cc}$ be the matrix obtained by replacing the i-th column of $\mathbf{A}_{n}^{cc}$ by $\mathbf{B}_{n}^{cc}$. Then the coefficients are given by

$$\delta_{ni}^{cc} = \frac{\det(A_{ni}^{cc})}{\det(A_{n}^{cc})} \tag{B.7}$$

for $i = 1, 2, 3, 4$ and $n = 1, 2, 3, \ldots$.

B.2. Coefficients of the SC plates

The coefficients of the 0th order solution are the same as in CC case i.e. $\alpha_{01}^{sc} = \alpha_{01}^{cc}$, $\alpha_{02}^{sc} = \alpha_{02}^{cc}$, $\alpha_{03}^{sc} = \alpha_{03}^{cc}$. The coefficients of the first-order solution can be determined by Cramer's rule. Let

$$\mathbf{A}_{n}^{sc} = \begin{bmatrix} J_{nM}(k_0) & Y_{nM}(k_0) & I_{nM}(k_0) & K_{nM}(k_0) \\ J'_{nM}(k_0) & Y'_{nM}(k_0) & I'_{nM}(k_0) & K'_{nM}(k_0) \\ J_{nM}(ck_0) & Y_{nM}(ck_0) & I_{nM}(ck_0) & K_{nM}(ck_0) \\ \mathfrak{M}(J_{nM}(ck_0)) & \mathfrak{M}(Y_{nM}(ck_0)) & \mathfrak{M}(I_{nM}(ck_0)) & \mathfrak{M}(K_{nM}(ck_0)) \end{bmatrix}, \tag{B.8}$$

$$\mathbf{B}_{n}^{sc} = [0 \;\; -c_n W_{0_{rr}}(1) \;\; 0 \;\; 0]^{\mathrm{T}}, \tag{B.9}$$

and let $\mathbf{A}_{ni}^{sc}$ be the matrix obtained by replacing the i-th column of $\mathbf{A}_{n}^{sc}$ by $\mathbf{B}_{n}^{sc}$. Then the coefficients are given by

$$\delta_{ni}^{sc} = \frac{\det(A_{ni}^{sc})}{\det(A_{n}^{sc})} \tag{B.10}$$

for $i = 1, 2, 3, 4$ and $n = 1, 2, 3, \ldots$.

B.3. Coefficients of the FC plates

The coefficients of the 0th order solution are

$$\alpha_{01}^{fc} = \frac{\beta_{11}\mathfrak{M}(I_0(ck_0)) - \beta_{12}\mathfrak{M}(J_0(ck_0)) + \beta_{13}\mathfrak{M}(K_0(ck_0))}{\beta_{12}\mathfrak{M}(Y_0(ck_0)) - \beta_{14}Y_0(k_0) + \beta_{15}Y_1(k_0)}, \tag{B.11}$$

$$\alpha_{02}^{fc} = \frac{1}{\beta_{12}}\{J_1(k_0)K_0(k_0) - J_0(k_0)K_1(k_0) - \alpha_{01}^{fc}(K_1(k_0)Y_0(k_0) - K_0(k_0)Y_1(k_0))\}, \tag{B.12}$$

$$\alpha_{03}^{fc} = -\frac{1}{K_0(k_0)}[J_0(k_0) + \alpha_{01}^{fc}Y_0(k_0) + \alpha_{02}^{fc}I_0(k_0)], \tag{B.13}$$

where

$$\begin{cases} \beta_{11} = J_0(k_0)K_1(k_0) - J_1(k_0)K_0(k_0), \\ \beta_{12} = I_1(k_0)K_0(k_0) + I_0(k_0)K_1(k_0), \\ \beta_{13} = I_1(k_0)J_0(k_0) + I_0(k_0)J_1(k_0), \\ \beta_{14} = K_1(k_0)\mathfrak{M}(I_0(ck_0)) + I_1(k_0)\mathfrak{M}(K_0(ck_0)), \\ \beta_{15} = K_0(k_0)\mathfrak{M}(I_0(ck_0)) - I_0(k_0)\mathfrak{M}(K_0(ck_0)). \end{cases} \tag{B.14}$$

The coefficients of the 1st order solution can be determined by Cramer's rule. Let

$$\mathbf{A}_{n}^{fc} = \begin{bmatrix} J_{nM}(k_0) & Y_{nM}(k_0) & I_{nM}(k_0) & K_{nM}(k_0) \\ J'_{nM}(k_0) & Y'_{nM}(k_0) & I'_{nM}(k_0) & K'_{nM}(k_0) \\ \mathfrak{M}(J_{nM}(ck_0)) & \mathfrak{M}(Y_{nM}(ck_0)) & \mathfrak{M}(I_{nM}(ck_0)) & \mathfrak{M}(K_{nM}(ck_0)) \\ \mathfrak{V}(J_{nM}(ck_0)) & \mathfrak{V}(Y_{nM}(ck_0)) & \mathfrak{V}(I_{nM}(ck_0)) & \mathfrak{V}(K_{nM}(ck_0)) \end{bmatrix}, \tag{B.15}$$

$$\mathbf{B}_{n}^{fc} = [0 \;\; -c_n W_{0_{rr}}(1) \;\; 0 \;\; 0]^{\mathrm{T}}, \tag{B.16}$$

and let $\mathbf{A}_{ni}^{fc}$ be the matrix obtained by replacing the i-th column of $\mathbf{A}_{n}^{fc}$ by $\mathbf{B}_{n}^{fc}$. Then the coefficients are given by

$$\delta_{ni}^{fc} = \frac{\det(A_{ni}^{fc})}{\det(A_{n}^{fc})} \tag{B.17}$$

for $i = 1, 2, 3, 4$ and $n = 1, 2, 3, \ldots$.

B.4. Coefficients of the CS plates

The coefficients of the 0th order solution are

$$\alpha_{01}^{cs} = \frac{\beta_{21}K_0(k_0) - \beta_{22}K_0(ck_0) + \beta_{23}K_1(ck_0)}{\beta_{24}Y_0(k_0) - \beta_{25}Y_0(ck_0) + \beta_{26}Y_1(ck_0)}, \tag{B.18}$$

$$\alpha_{02}^{cs} = -\frac{1}{\beta_{26}}\{-J_0(ck_0)K_0(k_0) + J_0(k_0)K_0(ck_0) + \alpha_{01}^{cs}[Y_0(k_0)K_0(ck_0) - Y_0(ck_0)K_0(k_0)]\}, \tag{B.19}$$

$$\alpha_{03}^{cs} = -\frac{1}{K_0(k_0)}[J_0(k_0) + \alpha_{01}^{cs}Y_0(k_0) + \alpha_{02}^{cs}I_0(k_0)], \tag{B.20}$$

where

$$\begin{cases} \beta_{21} = I_1(ck_0)J_0(ck_0) + I_0(ck_0)J_1(ck_0), \\ \beta_{22} = I_1(ck_0)J_0(k_0) + I_0(k_0)J_1(ck_0), \\ \beta_{23} = -I_0(ck_0)J_0(k_0) + I_0(k_0)J_0(ck_0), \\ \beta_{24} = I_1(ck_0)K_0(ck_0) + I_0(ck_0)K_1(ck_0), \\ \beta_{25} = I_1(ck_0)K_0(k_0) + I_0(k_0)K_1(ck_0), \\ \beta_{26} = -I_0(ck_0)K_0(k_0) + I_0(k_0)K_0(ck_0). \end{cases} \tag{B.21}$$

The coefficients of the 1st order solution can be determined by Cramer's rule. Let

$$\mathbf{A}_n^{cs} = \begin{bmatrix} J_{nM}(k_0) & Y_{nM}(k_0) & I_{nM}(k_0) & K_{nM}(k_0) \\ \mathfrak{M}(J_{nM}(k_0)) & \mathfrak{M}(Y_{nM}(k_0)) & \mathfrak{M}(I_{nM}(k_0)) & \mathfrak{M}(K_{nM}(k_0)) \\ J_{nM}(ck_0) & Y_{nM}(ck_0) & I_{nM}(ck_0) & K_{nM}(ck_0) \\ J'_{nM}(ck_0) & Y'_{nM}(ck_0) & I'_{nM}(ck_0) & K'_{nM}(ck_0) \end{bmatrix}, \tag{B.22}$$

$$\mathbf{B}_n^{cs} = \begin{bmatrix} -c_nW_{0_r}(1) \\ \nu c_nW_{0_r}(1) - \nu n^2M^2W_{0_r}(1) - \nu c_nW_{0_{rr}}(1) - c_nW_{0_{rrr}}(1) \\ 0 \\ 0 \end{bmatrix}, \tag{B.23}$$

and let $\mathbf{A}_{ni}^{cs}$ be the matrix obtained by replacing the i-th column of $\mathbf{A}_n^{cs}$ by $\mathbf{B}_n^{cs}$. Then the coefficients are given by

$$\delta_{ni}^{cs} = \frac{\det(A_{ni}^{cs})}{\det(A_n^{cs})} \tag{B.24}$$

for $i = 1, 2, 3, 4$ and $n = 1, 2, 3, \ldots$.

B.5. Coefficients of the SS plates

The coefficients of the 0th order solution are

$$\alpha_{01}^{ss} = \frac{\beta_{31}\mathfrak{M}(I_0(k_0)) - \beta_{32}\mathfrak{M}(J_0(k_0)) + \beta_{33}\mathfrak{M}(K_0(k_0))}{\beta_{32}\mathfrak{M}(Y_0(k_0)) + \beta_{34}Y_0(k_0) - \beta_{35}Y_0(ck_0)}, \tag{B.25}$$

$$\alpha_{02}^{ss} = \frac{1}{\beta_{32}}\{-J_0(ck_0)K_0(k_0) + J_0(k_0)K_0(ck_0) + \alpha_{01}^{ss}[Y_0(k_0)K_0(ck_0) - Y_0(ck_0)K_0(k_0)]\}, \tag{B.26}$$

$$\alpha_{03}^{ss} = -\frac{1}{K_0(k_0)}[J_0(k_0) + \alpha_{01}^{ss}Y_0(k_0) + \alpha_{02}^{ss}I_0(k_0)], \tag{B.27}$$

where

$$\begin{cases} \beta_{31} = J_0(ck_0)K_0(k_0) - J_0(k_0)K_0(ck_0), \\ \beta_{32} = I_0(ck_0)K_0(k_0) - I_0(k_0)K_0(ck_0), \\ \beta_{33} = I_0(ck_0)J_0(k_0) - I_0(k_0)J_0(ck_0), \\ \beta_{34} = K_0(ck_0)\mathfrak{M}(I_0(k_0)) - I_0(ck_0)\mathfrak{M}(K_0(k_0)), \\ \beta_{35} = K_0(k_0)\mathfrak{M}(I_0(k_0)) - I_0(k_0)\mathfrak{M}(K_0(k_0)). \end{cases} \tag{B.28}$$

The coefficients of the 1st order solution can be determined by Cramer's rule. Let

$$\mathbf{A}_n^{ss} = \begin{bmatrix} J_{nM}(k_0) & Y_{nM}(k_0) & I_{nM}(k_0) & K_{nM}(k_0) \\ \mathfrak{M}(J_{nM}(k_0)) & \mathfrak{M}(Y_{nM}(k_0)) & \mathfrak{M}(I_{nM}(k_0)) & \mathfrak{M}(K_{nM}(k_0)) \\ J_{nM}(ck_0) & Y_{nM}(ck_0) & I_{nM}(ck_0) & K_{nM}(ck_0) \\ \mathfrak{M}(J_{nM}(ck_0)) & \mathfrak{M}(Y_{nM}(ck_0)) & \mathfrak{M}(I_{nM}(ck_0)) & \mathfrak{M}(K_{nM}(ck_0)) \end{bmatrix}, \tag{B.29}$$

$$\mathbf{B}_n^{ss} = \begin{bmatrix} -c_n W_{0_r}(1) \\ \nu c_n W_{0_r}(1) - \nu n^2 M^2 W_{0_r}(1) - \nu c_n W_{0_{rr}}(1) - c_n W_{0_{rrr}}(1) \\ 0 \\ 0 \end{bmatrix}, \tag{B.30}$$

and let $\mathbf{A}_{ni}^{ss}$ be the matrix obtained by replacing the i-th column of $\mathbf{A}_n^{ss}$ by $\mathbf{B}_n^{ss}$. Then the coefficients are given by

$$\delta_{ni}^{ss} = \frac{\det(A_{ni}^{ss})}{\det(A_n^{ss})} \tag{B.31}$$

for $i = 1, 2, 3, 4$ and $n = 1, 2, 3, \ldots$.

B.6. Coefficients of the FS plates

The coefficients of the 0th order solution are

$$\alpha_{01}^{fs} = \frac{\beta_{41} K_0(k_0) + \beta_{42}\mathfrak{M}(K_0(k_0)) - \beta_{43}\mathfrak{M}(K_0(ck_0))}{\beta_{44}\mathfrak{M}(Y_0(k_0)) - \beta_{45}\mathfrak{M}(Y_0(ck_0)) - \beta_{46} Y_0(k_0)}, \tag{B.32}$$

$$\alpha_{02}^{fs} = \frac{1}{\beta_{45}}\{J_0(k_0)\mathfrak{M}(K_0(k_0)) - \mathfrak{M}(J_0(k_0))K_0(k_0) - \alpha_{01}^{fs}[K_0(k_0)\mathfrak{M}(Y_0(k_0)) - \mathfrak{M}(K_0(k_0))Y_0(k_0)]\}, \tag{B.33}$$

$$\alpha_{03}^{fs} = -\frac{1}{K_0(k_0)}[J_0(k_0) + \alpha_{01}^{fs} Y_0(k_0) + \alpha_{02}^{fs} I_0(k_0)], \tag{B.34}$$

where

$$\begin{cases} \beta_{41} = \mathfrak{M}(I_0(k_0))\mathfrak{M}(J_0(ck_0)) - \mathfrak{M}(I_0(ck_0))\mathfrak{M}(J_0(k_0)), \\ \beta_{42} = J_0(k_0)\mathfrak{M}(I_0(ck_0)) - I_0(k_0)\mathfrak{M}(J_0(ck_0)), \\ \beta_{43} = \mathfrak{M}(I_0(k_0))J_0(k_0) - I_0(k_0)\mathfrak{M}(J_0(k_0)), \\ \beta_{44} = K_0(k_0)\mathfrak{M}(I_0(ck_0)) - I_0(k_0)\mathfrak{M}(K_0(ck_0)), \\ \beta_{45} = K_0(k_0)\mathfrak{M}(I_0(k_0)) - I_0(k_0)\mathfrak{M}(K_0(k_0)), \\ \beta_{46} = \mathfrak{M}(I_0(ck_0))\mathfrak{M}(K_0(k_0)) - \mathfrak{M}(I_0(k_0))\mathfrak{M}(K_0(ck_0)). \end{cases} \tag{B.35}$$

The coefficients of the 1st order solution can be determined by Cramer's rule. Let

$$\mathbf{A}_n^{fs} = \begin{bmatrix} J_{nM}(k_0) & Y_{nM}(k_0) & I_{nM}(k_0) & K_{nM}(k_0) \\ \mathfrak{M}(J_{nM}(k_0)) & \mathfrak{M}(Y_{nM}(k_0)) & \mathfrak{M}(I_{nM}(k_0)) & \mathfrak{M}(K_{nM}(k_0)) \\ \mathfrak{M}(J_{nM}(ck_0)) & \mathfrak{M}(Y_{nM}(ck_0)) & \mathfrak{M}(I_{nM}(ck_0)) & \mathfrak{M}(K_{nM}(ck_0)) \\ \mathfrak{V}(J_{nM}(ck_0)) & \mathfrak{V}(Y_{nM}(ck_0)) & \mathfrak{V}(I_{nM}(ck_0)) & \mathfrak{V}(K_{nM}(ck_0)) \end{bmatrix}, \tag{B.36}$$

$$\mathbf{B}_n^{fs} = \begin{bmatrix} -c_n W_{0_r}(1) \\ \nu c_n W_{0_r}(1) - \nu n^2 M^2 W_{0_r}(1) - \nu c_n W_{0_{rr}}(1) - c_n W_{0_{rrr}}(1) \\ 0 \\ 0 \end{bmatrix}, \tag{B.37}$$

and let $\mathbf{A}_{ni}^{fs}$ be the matrix obtained by replacing the i-th column of $\mathbf{A}_n^{fs}$ by $\mathbf{B}_n^{fs}$. Then the coefficients are given by

$$\delta_{ni}^{fs} = \frac{\det(A_{ni}^{fs})}{\det(A_n^{fs})} \tag{B.38}$$

for $i = 1, 2, 3, 4$ and $n = 1, 2, 3, \ldots$.

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