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Asymptotic stability analysis with numerical confirmation of an axially accelerating beam constituted by the standard linear solid model

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ABSTRACT

Stability of an axially accelerating viscoelastic beam constituted by the standard linear solid model is investigated. The material time derivative is used in the viscoelastic constitutive relation. The instability condition is determined for combination and principal parametric resonances via the asymptotic analysis. The differential quadrature scheme is developed to solve numerically the partial differential equation governing transverse motion of axially accelerating viscoelastic beams. The stability boundaries are numerically located in the excitation amplitude and the excitation frequency plane. Numerical simulation demonstrates the effects of the stiffness, the viscosity and constant mean speed of beam. The numerical calculations validate the analytical results in the principal parametric resonance.

1. Introduction

Axially moving beams can represent many engineering devices [1–3]. As parametric vibration excited by the variation of the beam tension or the beam axial speed, large transverse motion of axially moving beams may occur under certain conditions.

Transverse parametric vibration of axially accelerating elastic beams has been extensively analyzed since first study by Pasin [4]. Öz et al. [5] employed the method of multiple scales to study dynamic stability of an axially accelerating beam with small bending stiffness. Özkaya and Pakdemirli [6] combined the method of multiple scales and the method of matched asymptotic expansions to construct nonresonant boundary layer solutions for an axially accelerating beam with small bending stiffness. Öz and Pakdemirli [7] and Öz [8] applied the method of multiple scales to calculate analytically the stability boundaries of an axially accelerating beam under pinned-pinned and clamped-clamped conditions, respectively. Parker and Lin [9] adopted a 1-term Galerkin discretization and the perturbation method to study dynamic stability of an axially accelerating beam subjected to a tension fluctuation. Özkaya and Öz [10] used an artificial neural network algorithm to determine stability boundary of an axially accelerating beam. Suweken and Horssen [11] applied the method of multiple scales to a discretized system via the Galerkin method to study the dynamic stability of an axially accelerating beam with pinned–pinned ends. Pakdemirli and Öz [12] employed the method of multiple scales to analyze the stability in the resonances involved up to four modes.

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In addition to elastic beams, axially accelerating viscoelastic beams have recently been investigated. Chen et al. [13] applied the averaging method to a discretized system via the Galerkin method to present analytically the stability boundaries of axially accelerating viscoelastic beams with clamped–clamped ends. Chen and Yang [14] applied the method of multiple scales without discretization to obtain analytically the stability boundaries of axially accelerating viscoelastic beams with pinned–pinned or clamped–clamped ends. Yang and Chen [15] applied the method of multiple scales to present analytically vibration and stability of an axially moving beam constituted by the viscoelastic constitutive law of an integral type. Chen and Yang [16] investigated an axially accelerating viscoelastic beam constrained by simple supports with rotational springs. In Chen et al. [13], Chen and Yang [14], and Chen and Yang [16], the Kelvin model containing the partial time derivative was used to describe the viscoelastic behavior of beam materials. Compared with Kelvin model, the standard linear solid model is more typical and representative, meanwhile this model can degenerate to the Kevin or Maxwell model by varying alternative of the stiffness of beam. In addition, if the viscoelastic materials are constituted by Botlzmann's superposition principle with the relaxation modulus expressed by the exponential function, the governing equation has the similar form as the standard linear solid model [15]. Mockensturm and Guo [17] convincingly argued that the Kelvin model generalized to axially moving materials should contain the material time derivative to account for the energy dissipation in steady motion. Based on the Kelvin model containing the material time derivative, Ding and Chen [18] employed the method of multiple scales to study the stability of an axially accelerating viscoelastic beam. Chen and Wang [19] revisited the problem in [18] via an asymptotic approach proposed by Maccari [20] and yield the same outcomes. The present investigation performs an asymptotic analysis for an axially accelerating viscoelastic beam based on the standard linear solid model with the material time derivative to represent the beam viscoelastic material property.

In spite of the fact that there have been many approximately analytical investigations on stability of axially accelerating beams, there are very limited researches on the topic to confirm the analytical results via the numerical solutions to the governing equations. Ding and Chen [18] studied the stability in principal parametric resonance of an axially accelerating viscoelastic beam via the finite difference scheme. Chen and Wang [19] presented the comparison between the analytical results and the numerical results in both summation and principal parametric resonances via the differential quadrature scheme. However, only the Kelvin model was considered in [18,19]. In the present investigation, the authors develop a differential quadrature scheme for an axially accelerating viscoelastic beam constituted by the standard linear solid model and contrast the approximately analytical results with the numerical ones.

The present paper is organized as follows. Section 2 presents the mathematical model. Section 3 proposes an asymptotic analysis approach to investigate stability in the model presented in Section 2. Section 4 develops a differential quadrature scheme to solve the governing equation in Section 2. Section 5 presents numerical examples to demonstrate the effects of some parameters on the stability boundaries in the summation and principal parametric resonances, and compares the analytical and numerical results. Section 6 ends the paper with the concluding remarks.

2. The governing equation

A uniform axially moving viscoelastic beam, with density ρ, cross-sectional area A, moment of inertial I and initial tension P_0, travels at time-dependent axial transport speed $\gamma(t)$ between two transversely motionless ends separated by distance L. Consider only the bending vibration described by the transverse displacement $v(x, t)$, where t is the time and x is the axial coordinate. The physical model is shown in Fig. 1. Newton's second law of motion yields

$$\rho A \frac{\mathrm{d}^2 v}{\mathrm{d}t^2} - P_0 v_{,xx} + M_{,xx} = 0 \tag{1}$$

where the material time derivative is introduced by defining differential operator d/dt as

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \gamma \frac{\partial}{\partial x} \tag{2}$$

where speed $\gamma(t)$ is equal to dx/dt, and $M(x, t)$ is the bending moment given by

$$M(x,t) = -\int_A z\sigma(x,z,t)\,\mathrm{d}A \tag{3}$$

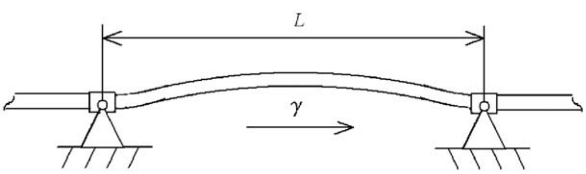

Fig. 1. The physical model of an axially accelerating beam.

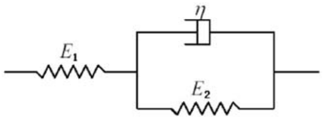

Fig. 2. The standard linear solid model.

where the z–x plane is the principal plane of bending, and $\sigma(x, z, t)$ is the disturbed normal stress. The viscoelastic material of the beam obeys the standard linear solid model shown in Fig. 2, which contains three parameters. For one-dimensional problem, the stress–strain relationship of the model is expressed in a differential form as

$$(E_1+E_2)\sigma+\eta\frac{\mathrm{d}\sigma}{\mathrm{d}t}=E_1E_2\varepsilon+E_1\eta\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} \tag{4}$$

where E_1 and E_2 are the stiffness of the beam, η is the viscosity of dashpot and $\varepsilon(x\ ,z,\ t)$ is the axial strain. The standard linear solid model can be used to describe the behavior of linear viscoelastic materials of solid type with limited creep deformation. It can reduce to Kelvin model ($E_1\to\infty$ and $E_2\neq 0$) or Maxwell model (E_2=0 and $E_2\neq 0$). The material time derivative is employed in standard linear solid model by substituting Eq. (2) into Eq. (4), and the resulting equation is

$$(E_1+E_2)\sigma+\eta\sigma_{,t}+\eta\gamma\sigma_{,x}=E_1E_2\varepsilon+E_1\eta\varepsilon_{,t}+E_1\eta\gamma\varepsilon_{,x} \tag{5}$$

Substituting Eqs. (2) and (3) into Eq. (1) leads to

$$\rho A(v_{,tt}+2\gamma v_{,xt}+\dot{\gamma}v_{,x}+\gamma^2v_{,xx})-P_0v_{,xx}-\frac{\partial^2}{\partial x^2}\int_A z\sigma(x,z,t)\,\mathrm{d}A=0 \tag{6}$$

Introduce the dimensionless variables and parameters

$$v\leftrightarrow\frac{v}{L},\quad x\leftrightarrow\frac{x}{L},\quad t\leftrightarrow t\sqrt{\frac{P_0}{\rho AL^2}},\quad \gamma\leftrightarrow\gamma\sqrt{\frac{\rho A}{P_0}},\quad \varsigma(x,t)=\frac{1}{P_0L}\int_A z\sigma(x,z,t)\,\mathrm{d}A \tag{7}$$

and then Eq. (6) can be cast into the dimensionless form

$$v_{,tt}+2\gamma v_{,xt}+\dot{\gamma}v_{,x}+(\gamma^2-1)v_{,xx}-\varsigma_{,xx}=0 \tag{8}$$

For small deflections, the strain–displacement relation is

$$\varepsilon(x,z,t)=-z\frac{\partial^2v(x,t)}{\partial x^2} \tag{9}$$

Substituting Eq. (6) into Eq. (5) leads to

$$(E_1+E_2)\sigma+\eta\sigma_{,t}+\eta\gamma\sigma_{,x}=-z(E_1E_2v_{,xx}+E_1\eta v_{,xxt}+E_1\eta\gamma v_{,xxx}) \tag{10}$$

In order to nondimensionalize and to eliminate σ, multiplying the both sides of Eq. (10) with z/P_0L and then integrating the resulting equation yield

$$\frac{1}{P_0L}\left[(E_1+E_2)\int_{\mathrm{A}}z\sigma\,\mathrm{d}A+\int_{\mathrm{A}}z\sigma_{,t}\,\mathrm{d}A+\eta\gamma\int_{\mathrm{A}}z\sigma_{,x}\,\mathrm{d}A\right]=-\frac{1}{P_0L}\int_{\mathrm{A}}z^2\,\mathrm{d}A(E_1E_2v_{,xx}+E_1\eta v_{,xxt}+E_1\eta\gamma v_{,xxx}) \tag{11}$$

Substituting (9) into Eq. (11) leads to

$$(E_1+E_2)\varsigma+\eta\varsigma_{,t}+\eta\gamma\varsigma_{,x}=-\frac{1}{P_0L}(E_1E_2Iv_{,xx}+E_1I\eta v_{,xxt}+E_1I\eta\gamma v_{,xxx}) \tag{12}$$

where I is the moment of inertial and expressed as

$$I=\int_A z^2\,\mathrm{d}A \tag{13}$$

Introduce the dimensionless variables and parameters

$$\varepsilon\eta\leftrightarrow\frac{\eta}{E_1+E_2}\sqrt{\frac{P_0}{\rho AL^2}},\quad a=\frac{E_1E_2I}{P_0L^2(E_1+E_2)},\quad b=\frac{E_1I}{P_0L^2} \tag{14}$$

here b can be also expressed as b=a(1+E_1/E_2). Then Eq. (12) can be cast into the dimensionless form

$$\varsigma+\varepsilon\eta(\varsigma_{,t}+\gamma\varsigma_{,x})=-av_{,xx}-\varepsilon\eta b(v_{,xxt}+\gamma v_{,xxx}) \tag{15}$$

where bookkeeping device ε is a small dimensionless parameter accounting for the fact that the viscosity is very small.

Assume that the beam is with simple supports at both ends. Then the boundary conditions in dimensionless form are

$$v(0,t)=0,\quad v_{,xx}(0,t)=0,\quad v(1,t)=0,v_{,xx}(1,t)=0 \tag{16}$$

In the present investigation, the axial speed is assumed to be a small simple harmonic variation about the constant mean speed:

$$\gamma(t) = \gamma_0 + \varepsilon\gamma_1 \sin\omega t \tag{17}$$

where γ_0 is the constant mean speed, and $\varepsilon\gamma_1$ and ω are, respectively, the amplitude and the frequency of the axial speed variation, all in the dimensionless form. Here the bookkeeping device ε is used to indicate the fact that fluctuation amplitude is small, with the same order as the dimensionless viscosity. Substituting Eqs. (17) and (15) into Eq. (9) and neglect higher order ε terms in the resulting equation yield

$$Mv_{,tt} + Gv_{,t} + Kv = -\varepsilon[\gamma_1(2\sin\omega t v_{,xt} + \omega\cos\omega t v_{,x} + 2\gamma_0 \sin\omega t v_{,xx}) - \eta(\varsigma_{,xxt} + \gamma_0\varsigma_{,xxx} - bv_{,xxxxt} - b\gamma_0 v_{,xxxxx})] \tag{18}$$

where the mass, gyroscopic, and linear stiffness operators are, respectively, defined as

$$M = I, \quad G = 2\gamma_0\frac{\partial}{\partial x}, \quad K = (\gamma_0^2 - 1)\frac{\partial^2}{\partial x^2} + a\frac{\partial^4}{\partial x^4} \tag{19}$$

3. Asymptotic analyses on stability

Under certain conditions, the straight configuration of the beam may become unstable. The conditions will be located via the analysis on the stability of the zero solution if Eqs. (15) and (18). If ε=0 in Eq. (18), under boundary conditions (16), the natural frequencies of the undisturbed gyroscopic continuous system

$$Mv_{,tt} + Gv_{,t} + Kv = 0 \tag{20}$$

can be determined. Previous studies found that, for elastic beams [12] and the Kelvin viscoelastic beams [18,19], if the axial speed variation frequency ω approaches the sum of any two natural frequencies of Eq. (20), the summation parametric resonance may occur. Therefore it can be expected that the summation parametric resonance occurs for the viscoelastic beams constituted by the standard linear solid model. A detuning parameter μ is introduced to quantify the deviation of ω from ω_m+ω_n ($m \leq n$), and ω is described by

$$\omega = \omega_m + \omega_n + \varepsilon\mu \tag{21}$$

If $\varepsilon \neq 0$, as ε is rather small to investigate the summation parametric resonance, it is usually assumed that the response is mainly influenced by two corresponding modes and thus the effects of other modes can be neglected. Therefore, the solution to Eq. (18) may take the following form:

$$v(x,t) = \psi_m(x,\tau;\varepsilon)\,\mathrm{e}^{\mathrm{i}\omega_m t} + \psi_n(x,\tau;\varepsilon)\,\mathrm{e}^{\mathrm{i}\omega_n t} + cc \tag{22}$$

where τ=εt is the slow time scale and *cc* denotes the complex conjugate of all preceding terms on the right hand side of an equation. Functions $\psi_m(x,\tau,\varepsilon)$ and $\psi_n(x,\tau,\varepsilon)$ can be expanded in the power series of ε

$$\psi_k = \psi_{0k} + \varepsilon\psi_{1k} + (\varepsilon^2) \quad (k = m, n) \tag{23}$$

The chain rule of partial derivatives leads to

$$\frac{\partial}{\partial t}[\psi_k \mathrm{e}^{\pm\mathrm{i}\omega_k t}] = (\pm\mathrm{i}\omega_k\psi_k + \varepsilon\psi_{k,\tau})\,\mathrm{e}^{\pm\mathrm{i}\omega_k t} \quad (k = m, n) \tag{24}$$

Substituting Eq. (23) into Eq. (22) leads to

$$v(x,t) = (\psi_{0m} + \varepsilon\psi_{1m})\,\mathrm{e}^{\mathrm{i}\omega_m t} + (\psi_{0n} + \varepsilon\psi_{1n})\,\mathrm{e}^{\mathrm{i}\omega_n t} + cc + O(\varepsilon^2) \tag{25}$$

Expanding ς in power series of ε

$$\varsigma = \varsigma_0 + \varepsilon\varsigma_1 + O(\varepsilon^2) \tag{26}$$

Inserting Eqs. (17), (21), (25) and (26) into Eq. (15) equating the coefficients at the order ε^0 and ε^1 in the resulting equation yield, at order ε^0

$$\varsigma_0 = -a(\psi''_{0m}\ \mathrm{e}^{\mathrm{i}\omega_m t} + \psi''_{0n}\ \mathrm{e}^{\mathrm{i}\omega_n t} + cc) \tag{27}$$

at order ε^1

$$\varsigma_1 + \eta\varsigma_{0,t} + \gamma_0\eta\varsigma_{0,x} = (-\mathrm{i}b\eta\omega_m\psi''_{0m} - a\psi''_{1m} - b\gamma_0\eta\psi'''_{0m})\mathrm{e}^{\mathrm{i}\omega_m t} + (-\mathrm{i}b\eta\omega_n\psi''_{0n} - a\psi''_{1n} - b\gamma_0\eta\psi'''_{0n})\mathrm{e}^{\mathrm{i}\omega_n t} + cc + O(\varepsilon) \tag{28}$$

Substituting Eq. (27) into Eq. (28) leads to

$$\varsigma_1 = [\eta(a-b)(\mathrm{i}\omega_m\psi''_{0m} + \gamma_0\psi'''_{0m}) - a\psi''_{1m}]\mathrm{e}^{\mathrm{i}\omega_m t} + [\eta(a-b)(\mathrm{i}\omega_n\psi''_{0n} + \gamma_0\psi'''_{0n}) - a\psi''_{1n}]\mathrm{e}^{\mathrm{i}\omega_n t} + cc + O(\varepsilon) \tag{29}$$

Substituting Eqs. (27) and (29) into Eq. (26) leads to

$$\begin{aligned}\varsigma &= \varsigma_0 + \varepsilon\varsigma_1 \\ &= -a(\psi''_{0m}\,\mathrm{e}^{\mathrm{i}\omega_m t} + \psi''_{0n}\,\mathrm{e}^{\mathrm{i}\omega_n t}) + \varepsilon[\eta(a-b)(\mathrm{i}\omega_m\psi''_{0m} + \gamma_0\psi'''_{0m}) - a\psi''_{1m}]\mathrm{e}^{\mathrm{i}\omega_m t} \\ &\quad + \varepsilon[\eta(a-b)(\mathrm{i}\omega_n\psi''_{0n} + \gamma_0\psi'''_{0n}) - a\psi''_{1n}]\mathrm{e}^{\mathrm{i}\omega_n t} + cc + O(\varepsilon^2)\end{aligned} \tag{30}$$

Inserting Eqs. (25) and (30) into Eqs. (18) and (16) equating the coefficients of $\mathrm{e}^{\mathrm{i}\omega_k t}$ (k=m, n) at the order ε^0 and ε^1 in the resulting equation yield, at order ε^0

$$-\omega_k^2 M\psi_{0k} + \mathrm{i}\omega_k G\psi_{0k} + K\psi_{0k} = 0 \quad (k = m, n) \tag{31}$$

$$\psi_{0k}(0,\tau) = \psi_{0k}(1,\tau) = \psi''_{0k}\,(0,\tau) = \psi''_{0k}\,(1,\tau) = 0 \quad (k = m, n) \tag{32}$$

at order ε^1

$$\begin{aligned}-\omega_m^2 M\psi_{1m} + \mathrm{i}\omega_m G\psi_{1m} + K\psi_{1m} = &-[\tfrac{1}{2}(\omega_m - \omega_n)\overline{\psi}'_{0n} - \mathrm{i}\gamma_0\overline{\psi}''_{0n}\,]\gamma_1\mathrm{e}^{\mathrm{i}\mu\tau} + \eta(a-b)(\mathrm{i}\omega_m\psi_{0k}^{(4)} + \gamma_0\psi_{0m}^{(5)}) \\ &- 2(\mathrm{i}\omega_m\dot{\psi}_{0m} - \gamma_0\dot{\psi}'_{0m}\,)\end{aligned} \tag{33}$$

$$\begin{aligned}-\omega_n^2 M\psi_{1n} + \mathrm{i}\omega_n G\psi_{1n} + K\psi_{1n} = &-[\tfrac{1}{2}(\omega_n - \omega_m)\overline{\psi}'_{0m} - \mathrm{i}\gamma_0\overline{\psi}''_{0m}\,]\gamma_1\mathrm{e}^{\mathrm{i}\mu\tau} + \eta(a-b)(\mathrm{i}\omega_n\psi_{0n}^{(4)} + \gamma_0\psi_{0n}^{(5)}) \\ &- 2(\mathrm{i}\omega_n\dot{\psi}_{0n} - \gamma_0\dot{\psi}'_{0n}\,)\end{aligned} \tag{34}$$

$$\psi_{1k}(0,\tau) = 0,\;\; \psi''_{1k}\,(0,\tau) = 0,\; \psi_{1k}(1,\tau) = 0,\;\; \psi''_{1k}\,(1,\tau) = 0(k = m, n) \tag{35}$$

Assume the solution to Eq. (31) is in the following form

$$\psi_{0k}(x,\tau) = q_k(\tau)\phi_k(x)(k = m, n) \tag{36}$$

then

$$-\omega_k^2 M\varphi_k + \mathrm{i}\omega_k G\varphi_k + K\varphi_k = 0 \quad (k = m, n) \tag{37}$$

$$\phi_k(0) = 0, \quad \phi''_k\,(0) = 0, \quad \phi_k(1) = 0, \quad \phi''_k\,(1) = 0. \tag{38}$$

Under boundary (38), Eq. (37) has the solution [7,16]

$$\begin{aligned}\phi_k(x) = \mathrm{e}^{\mathrm{i}\beta_{1k}x} &- \frac{(\beta_{4k}^2 - \beta_{1k}^2)(\mathrm{e}^{\mathrm{i}\beta_{3k}} - \mathrm{e}^{\mathrm{i}\beta_{1k}})}{(\beta_{4k}^2 - \beta_{2k}^2)(\mathrm{e}^{\mathrm{i}\beta_{3k}} - \mathrm{e}^{\mathrm{i}\beta_{2k}})}\mathrm{e}^{\mathrm{i}\beta_{2k}x} - \frac{(\beta_{4k}^2 - \beta_{1k}^2)(\mathrm{e}^{\mathrm{i}\beta_{2k}} - \mathrm{e}^{\mathrm{i}\beta_{1k}})}{(\beta_{4k}^2 - \beta_{3k}^2)(\mathrm{e}^{\mathrm{i}\beta_{2k}} - \mathrm{e}^{\mathrm{i}\beta_{3k}})}\mathrm{e}^{\mathrm{i}\beta_{3k}x} \\ &- \left[1 - \frac{(\beta_{4k}^2 - \beta_{1k}^2)(\mathrm{e}^{\mathrm{i}\beta_{3k}} - \mathrm{e}^{\mathrm{i}\beta_{1k}})}{(\beta_{4k}^2 - \beta_{2k}^2)(\mathrm{e}^{\mathrm{i}\beta_{3k}} - \mathrm{e}^{\mathrm{i}\beta_{2k}})} - \frac{(\beta_{4k}^2 - \beta_{1k}^2)(\mathrm{e}^{\mathrm{i}\beta_{2k}} - \mathrm{e}^{\mathrm{i}\beta_{1k}})}{(\beta_{4k}^2 - \beta_{3k}^2)(\mathrm{e}^{\mathrm{i}\beta_{2k}} - \mathrm{e}^{\mathrm{i}\beta_{3k}})}\right]\mathrm{e}^{\mathrm{i}\beta_{4k}x}\end{aligned} \tag{39}$$

where β_{jk} (j=1,2,3,4; k=m, n) are four roots of the following four-order algebraic equation

$$-\omega_k^2 - 2\gamma_0\omega_k\beta_k - (\gamma_0^2 - 1)\beta_k^2 + a\beta_k^4 = 0 \tag{40}$$

Substituting Eq. (36) into Eqs. (33) and (34) leads to

$$\begin{aligned}-\omega_m^2 M\psi_{1m} + \mathrm{i}\omega_m G\psi_{1m} + K\psi_{1m} = &-[\tfrac{1}{2}(\omega_m - \omega_n)\overline{\varphi}'_n - \mathrm{i}\gamma_0\overline{\varphi}''_n\,]\overline{q}_n\gamma_1\mathrm{e}^{\mathrm{i}\mu\tau} + \eta(a-b)(\mathrm{i}\omega_m\varphi_m^{(4)} + \gamma_0\varphi_m^{(5)})q_m \\ &- 2(\mathrm{i}\omega_m\varphi_m + 2\gamma_0\varphi'_m\,)\dot{q}_m\end{aligned} \tag{41}$$

$$\begin{aligned}-\omega_n^2 M\psi_{1n} + \mathrm{i}\omega_n G\psi_{1n} + K\psi_{1n} = &-[\tfrac{1}{2}(\omega_n - \omega_m)\overline{\varphi}'_m - \mathrm{i}\gamma_0\overline{\varphi}''_m\,]\overline{q}_m\gamma_1\mathrm{e}^{\mathrm{i}\mu\tau} + \eta(a-b)(\mathrm{i}\omega_n\varphi_n^{(4)} + \gamma_0\varphi_n^{(5)})q_n \\ &- 2(\mathrm{i}\omega_n\varphi_n + \gamma_0\varphi'_n\,)\dot{q}_n\end{aligned} \tag{42}$$

Introduce an inner product

$$\langle f_1, f_2 \rangle = \int_0^1 f_1(x)\overline{f}_2(x)\,\mathrm{d}x \tag{43}$$

for complex functions f_1 and f_2 defined on [0,1]. Under the boundary conditions of vanishing the function values and the second-order x-partial derivatives, both M and K are symmetric in the sense

$$\langle Mf_1, f_2 \rangle = \langle f_1, Mf_2 \rangle, \quad \langle Kf_1, f_2 \rangle = \langle f_1, Kf_2 \rangle \tag{44}$$

G is skew symmetric in the sense

$$\langle Gf_1, f_2 \rangle = -\langle f_1, Gf_2 \rangle \tag{45}$$

For function $\phi_k(x)$ satisfying Eq. (39), the distribution law of the inner product, and Eq. (45) with Eqs. (44) and (43) yield

$$\langle -\omega_k^2 M\psi_{1k} + \mathrm{i}\omega_k G\psi_{1k} + K\psi_{1k}, \phi_k \rangle = \langle \psi_{1k}, -\omega_k^2 M\phi_k + \mathrm{i}\omega_k G\phi_k + K\phi_k \rangle = 0 \quad (k = m, n) \tag{46}$$

Taking both sides of Eqs. (41) and (42) inner product with $\phi_k(x)$ (k=m, n) and using Eq. (46) give

$$\dot{q}_m+\eta(b-a)c_{mm}q_m+\gamma_1 d_{mn}\overline{q}_n \mathrm{e}^{\mathrm{i}\mu\tau}=0$$

$$\dot{q}_n+\eta(b-a)c_{nn}q_n+\gamma_1 d_{nm}\overline{q}_m \mathrm{e}^{\mathrm{i}\mu\tau}=0 \tag{47}$$

where

$$c_{kk}=\frac{\langle \mathrm{i}\omega_k\phi_k^{(4)}+\gamma_0\phi_k^{(5)},\phi_k\rangle}{2\langle \mathrm{i}\omega_k\phi_k+\gamma_0\phi_k',\phi_k\rangle}\quad (k=m,n)$$

$$d_{mn}=-\frac{\langle(\omega_n-\omega_m)\overline{\phi}_n'+2\mathrm{i}\gamma_0\overline{\phi}_n'',\phi_m\rangle}{4\langle \mathrm{i}\omega_m\phi_m+\gamma_0\phi_m',\phi_m\rangle},$$

$$d_{nm}=-\frac{\langle(\omega_m-\omega_n)\overline{\phi}_m'+2\mathrm{i}\gamma_0\overline{\phi}_m'',\phi_n\rangle}{4\langle \mathrm{i}\omega_n\phi_n+\gamma_0\phi_n',\phi_n\rangle} \tag{48}$$

The coefficients d_{mn}, d_{nm}, and c_{kk} (k=m, n) are determined by the natural frequencies and the modal function (39) of linear system (20) under boundary condition, which are independent of the viscosity and the axial speed variation.

Express the solutions to Eq. (47) in polar form

$$q_k=S_k(\tau)\,\mathrm{e}^{\mathrm{i}\mu\tau/2}\quad (k=m,n) \tag{49}$$

Substituting Eq. (49) into (47) yields

$$\dot{S}_m+\mathrm{i}\frac{\mu}{2}S_m+\eta(b-a)c_{mm}S_m+\gamma_1 d_{mn}\overline{S}_n=0$$

$$\dot{S}_n+\mathrm{i}\frac{\mu}{2}S_n+\eta(b-a)c_{nn}S_n+\gamma_1 d_{nm}\overline{S}_m=0 \tag{50}$$

Suppose that the solutions of Eq. (50) take the form

$$S_m=s_m\mathrm{e}^{\lambda\tau}, S_n=s_n\mathrm{e}^{\overline{\lambda}\tau} \tag{51}$$

where s_m and s_n are real coefficients, and λ is a complex to be determined later. Substituting Eq. (51) into Eq. (50) and taking the complex conjugation of the second resulting equation yields

$$\left[\lambda+\mathrm{i}\frac{\mu}{2}+\eta(b-a)c_{mm}\right]s_m+\gamma_1 d_{mn}s_n=0$$

$$\gamma_1\overline{d}_{nm}s_m+\left[\lambda-\mathrm{i}\frac{\mu}{2}+\eta(b-a)\overline{c}_{nn}\right]s_n=0 \tag{52}$$

As a set of homogeneous linear algebraic equations of s_m and s_n, Eq. (52) possesses nontrivial solutions if and only if its determinant of coefficient vanishes. That is

$$\lambda^2+\eta(b-a)(c_{mm}+\overline{c}_{nn})\lambda+\left[\mathrm{i}\frac{\mu}{2}+\eta(b-a)c_{mm}\right]\left[-\mathrm{i}\frac{\mu}{2}+\eta(b-a)\overline{c}_{nn}\right]-\gamma_1^2 d_{nm}\overline{d}_{mn}=0 \tag{53}$$

For the roots of Eq. (53) with respect to λ, if either of them has a positive real part, then the system is unstable. On the contrary if both of them have negative real parts the system is stable. It is numerically demonstrated that c_{kk} is a positive real number. Separating real and imaginary parts in Eq. (53) can lead to two new equations. Then the instability condition of the summation parametric resonance can obtained as

$$\mu^2<4(c_{mm}+c_{nn})^2\frac{\gamma_1^2\mathrm{Re}(d_{nm}\overline{d}_{mn})-[\eta(b-a)]^2c_{mm}c_{nn}}{(c_{mm}-c_{nn})^2+(c_{mm}+c_{nn})^2} \tag{54}$$

after some algebraic manipulations. Namely, if the positive square root of the right term of inequality (54) is more than modulus of μ, the system is unstable. Based on inequality (54), one can develop the analytical expression of the instability boundary in summation parametric resonance

$$\left[1+\frac{(c_{mm}-c_{nn})^2}{(c_{mm}+c_{nn})^2}\right]\mu^2+4[\eta(b-a)]^2c_{mm}c_{nn}=4\gamma_1^2 d_{nm}\overline{d}_{mn} \tag{55}$$

More details have been presented by Chen and Yang [14].

If the axial speed variation frequency ω approaches two times the any natural frequency of Eq. (20), the principal parametric resonance may occur, herein ω is described by

$$\omega=2\omega_k+\varepsilon\mu \tag{56}$$

Let m=n=k in Eq. (55), then stability boundary in the kth principal parametric resonance is described by

$$\mu^2+4[\eta(b-a)]^2c_{kk}^2=4\gamma_1^2|d_{kk}|^2 \tag{57}$$

where c_{kk} is expressed in Eq. (48), and

$$d_{kk} = -\frac{\langle \mathrm{i}\gamma_0 \overline{\phi}''_k \, , \phi_k \rangle}{2\langle \mathrm{i}\omega_k \phi_k + \gamma_0 \phi'_k \, , \phi_k \rangle} \tag{58}$$

4. Differential quadrature investigations on stability

The differential quadrature scheme will be employed to solve numerically equation

$$v_{,tt} + 2\gamma v_{,xt} + \dot{\gamma} v_{,x} + \gamma^2 v_{,xx} - v_{,xx} - \varsigma_{,xx} = 0 \tag{59}$$

$$\varsigma + \eta(\varsigma_{,t} + \gamma \varsigma_{,x}) = -a v_{,xx} - \eta b(v_{,xxt} + \gamma v_{,xxx}) \tag{60}$$

Eq. (60) is the same as Eq. (15) with the exception that ε=1 here. Other numerical methods such as the Galerkin finite-element method [21] and the finite difference method [18] may also serve the proposition.

Introduce N sampling points as

$$x_i = \frac{1}{2}\left[1 - \cos\frac{(i-1)\pi}{N-1}\right] \quad (i = 1, 2, \ldots, N) \tag{61}$$

The quadrature rules for the derivatives of a function at the sampling points yield [22,23]

$$v_{,x}(x_i, t) = \sum_{j=1}^{N} A_{ij}^{(1)} v(x_j, t), \quad v_{,xx}(x_i, t) = \sum_{j=1}^{N} A_{ij}^{(2)} v(x_j, t), \quad v_{,xxx}(x_i, t) = \sum_{j=1}^{N} A_{ij}^{(3)} v(x_j, t) \tag{62}$$

$$\varsigma_{,x}(x_i, t) = \sum_{j=1}^{N} A_{ij}^{(1)} \varsigma(x_j, t), \quad \varsigma_{,xx}(x_i, t) = \sum_{j=1}^{N} A_{ij}^{(2)} \varsigma(x_j, t) \quad (i, j = 1, 2, \ldots, N) \tag{63}$$

where the weighting coefficients are the expression

$$A_{ij}^{(1)} = \frac{\prod_{k=1,k\neq i}^{N}(x_i - x_k)}{(x_i - x_j)\prod_{k=1,k\neq j}^{N}(x_j - x_k)} \quad (i, j = 1, 2, \ldots, N; \;\; j \neq i) \tag{64}$$

and the recurrence relationship

$$A_{ij}^{(r)} = r\left[A_{ii}^{(r-1)} A_{ij}^{(1)} - \frac{A_{ij}^{(r-1)}}{x_i - x_j}\right] \quad (r = 2, 3, 4, 5; \;\; i, j = 1, 2, \ldots, N; \;\; j \neq i)$$

$$A_{ii}^{(r)} = -\sum_{k=1,k\neq i}^{N} A_{ik}^{(r)} \quad (r = 1, 2, 3, 4, 5; \;\; i = 1, 2, \ldots, N) \tag{65}$$

Substituting Eqs. (62) and (63) into Eqs. (59) and (60) leads to

$$\ddot{v}_i + 2\gamma \sum_{j=1}^{N} A_{ij}^{(1)} \dot{v}_j + \dot{\gamma} \sum_{j=1}^{N} A_{ij}^{(1)} v_j + (\gamma^2 - 1)\sum_{j=1}^{N} A_{ij}^{(2)} v_j - \sum_{j=1}^{N} A_{ij}^{(2)} \varsigma_j = 0 \quad (i = 1, 2, \ldots, N) \tag{66}$$

$$\eta \dot{\varsigma}_i + \varsigma_i + \eta\gamma \sum_{j=1}^{N} A_{ij}^{(1)} \varsigma_j = -a \sum_{j=1}^{N} A_{ij}^{(2)} v_j - \eta b \sum_{j=1}^{N} A_{ij}^{(2)} \dot{v}_j - \eta b \gamma \sum_{j=1}^{N} A_{ij}^{(3)} v_j \quad (i = 1, 2, \ldots, N) \tag{67}$$

where

$$v_i(t) = v(x_i, t), \quad \varsigma_i(t) = \varsigma(x_i, t) \tag{68}$$

In order to overcome difficulties in the implementation of the boundary conditions, the idea of incorporating the boundary conditions into the weighting coefficient matrices [24] is adopted. The simplest of the boundary conditions to invoke in Eqs. (66) and (67) is the condition of zero displacement (v) at a simply supported edge. This is done by simply ignoring the corresponding grid points in Eqs. (66) and (67). For the boundary condition (16), consider the DQ analogue of the second derivative with respect to x at the grid points on a line parallel to the x-axis.

$$[A^{(2)}] = \begin{bmatrix} A_{11}^{(2)} & A_{12}^{(2)} & \cdots & A_{1,N-1}^{(2)} & A_{1N}^{(2)} \\ A_{21}^{(2)} & A_{22}^{(2)} & \cdots & A_{2,N-1}^{(2)} & A_{2N}^{(2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{N-1,1}^{(2)} & A_{N-1,2}^{(2)} & \cdots & A_{N-1,N-1}^{(2)} & A_{N-1,N}^{(2)} \\ A_{N1}^{(2)} & A_{N2}^{(2)} & \cdots & A_{N,N-1}^{(2)} & A_{NN}^{(2)} \end{bmatrix} \tag{69}$$

Let the modified weighting coefficient matrix in Eq. (69) now be written as

$$[\tilde{A}^{(2)}] = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ A_{21}^{(2)} & A_{22}^{(2)} & \cdots & A_{2,N-1}^{(2)} & A_{2N}^{(2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{N-1,1}^{(2)} & A_{N-1,2}^{(2)} & \cdots & A_{N-1,N-1}^{(2)} & A_{N-1,N}^{(2)} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{70}$$

and let the weighting coefficient matrix of the third-order derivatives be modified as

$$[\tilde{A}^{(3)}] = [A^{(1)}][\tilde{A}^{(2)}] \tag{71}$$

Let Eqs. (66) and (67) be rewritten replacing the original x-derivative weighting coefficients by the modified coefficients

$$\ddot{v}_i + 2\gamma \sum_{j=1}^{N} A_{ij}^{(1)} \dot{v}_j + \dot{\gamma} \sum_{j=1}^{N} A_{ij}^{(1)} v_j + (\gamma^2 - 1) \sum_{j=1}^{N} \tilde{A}_{ij}^{(2)} v_j - \sum_{j=1}^{N} \tilde{A}_{ij}^{(2)} \varsigma_j = 0 \quad (i = 1, 2, \ldots, N) \tag{72}$$

$$\eta \dot{\varsigma}_i + \varsigma_i + \eta\gamma \sum_{j=1}^{N} A_{ij}^{(1)} \varsigma_j = -a \sum_{j=1}^{N} \tilde{A}_{ij}^{(2)} v_j - \eta b \sum_{j=1}^{N} \tilde{A}_{ij}^{(2)} \dot{v}_j - \eta b\gamma \sum_{j=1}^{N} \tilde{A}_{ij}^{(3)} v_j \quad (i = 1, 2, \ldots, N) \tag{73}$$

From the aforementioned discussion, the DQ analogue equations (72) and (73) may be written in terms of modified weighting coefficients. Thus

$$\ddot{v}_i + 2\gamma \sum_{j=2}^{N-1} A_{ij}^{(1)} \dot{v}_j + \dot{\gamma} \sum_{j=2}^{N-1} A_{ij}^{(1)} v_j + (\gamma^2 - 1) \sum_{j=2}^{N-1} \tilde{A}_{ij}^{(2)} v_j - \sum_{j=2}^{N-1} \tilde{A}_{ij}^{(2)} \varsigma_j = 0 \quad (i = 2, 3, \ldots, N-1) \tag{74}$$

$$\eta \dot{\varsigma}_i + \varsigma_i + \eta\gamma \sum_{j=2}^{N-1} A_{ij}^{(1)} \varsigma_j = -a \sum_{j=2}^{N-1} \tilde{A}_{ij}^{(2)} v_j - \eta b \sum_{j=2}^{N-1} \tilde{A}_{ij}^{(2)} \dot{v}_j - \eta b\gamma \sum_{j=2}^{N-1} \tilde{A}_{ij}^{(3)} v_j \quad (i = 2, 3, \ldots, N-1) \tag{75}$$

In the present investigation, the fourth-order Runge-Kutta method was used to integrate ordinary differential equations and analyze the stability of system. The initial conditions for Eqs. (74) and (75) are chosen as

$$v(x,0) = 0.0001x(1-x), \quad v_{,t}(x,0) = 0, \quad \varsigma(x,0) = 0 \tag{76}$$

In the differential quadrature method, let N=7. To decide the stability, choose T_1=20, T_2=2T_1 and T=60. In the first principal parametric resonance, μ=2$\omega_1-\omega$. In the second principal parametric resonance, μ=2$\omega_2-\omega$. In the summation parametric resonance, μ=ω_1+$\omega_2-\omega$. For the given parameters and initial conditions, Eqs. (74) and (75) can be numerically solved via the fourth-order Runge-Kutta. After a time interval [0, T_1] to remove the transient response, the maximum beam center displacements V_1 and V_2 are, respectively, recorded for time intervals [T_1, T_2] and [T_2, T]. If V_1 is bigger than V_2, the parametric resonance is stable. If V_1 is smaller than V_2, the parametric resonance is unstable. Varying the parameters, one can locate the stability boundary in the parameter space.

5. Numerical examples

In this paper, the stiffness of an axially moving beam a=0.64 is specified. Eqs. (14), (39), (40), (48), and (56) are helpful for analytical computation. Actually, E_1 and E_2 should be given an actual physical value, e.g., 1×10^{10}(N/m^2). Some other values of E_1 and E_2 are specified to examine the changing tendencies, even if the parameter values cannot depict the actual physical meanings.

Next to demonstrate the effects of stiffness, viscosity, and constant mean speed on the instability boundary in summation and principal parametric resonances. Fig. 3 shows the effects of stiffness E_1 and E_2, respectively, in summation resonance. Figs. 4 and 5 show the effects of the viscosity and the mean speed in summation and principal parametric resonances, respectively. Fig. 6 compares the instability boundaries in summation parametric resonances with those in the

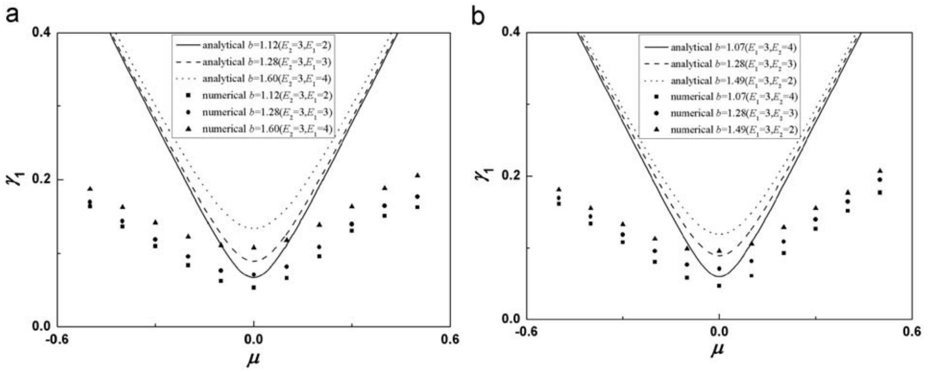

Fig. 3. The effects of stiffness in summation parametric resonance (η=0.0003 and γ_0=2.0): (a) the effects of stiffness E_1 on stability boundaries and (b) the effects of stiffness E_2 on stability boundaries.

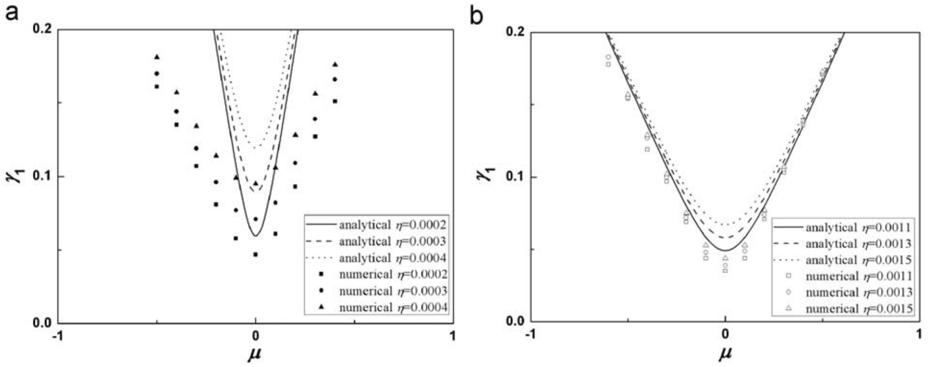

Fig. 4. The effects of viscosity on instability boundaries (b=1.28 and γ_0=2.0): (a) the summation parametric resonance and (b) the first principal parametric resonance.

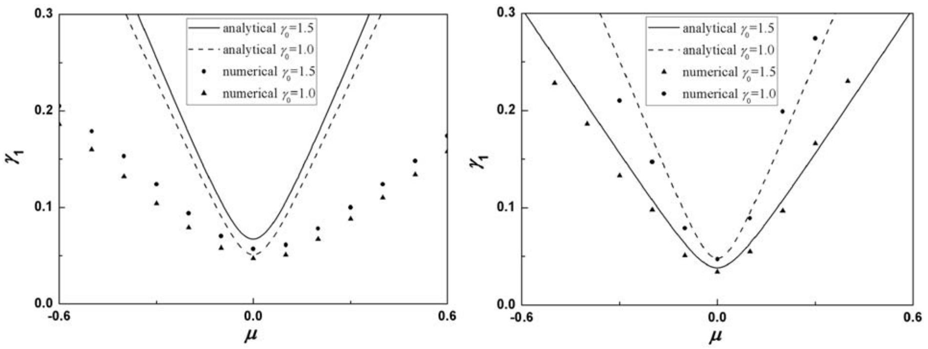

Fig. 5. The effects of constant mean speed on instability boundaries (b=1.28): (a) the summation parametric resonance (η=0.0003) and (b) the first principal parametric resonance (η=0.0007).

first principle parametric resonances. Both analytical (in line) and numerical (in symbol markers) results are given in Figs. 3–6.

In Figs. 3–5, the instability boundaries in the summation and the first principal parametric resonance have the same changing trend. The increasing stiffness E_1 makes the instability boundaries move towards the increasing direction of γ_1 in plane μ–γ_1 and the instability regions become narrow. However, the stiffness E_2 has an opposite effect on the instability boundaries. The relationship between b and E_1 (E_2), that b increases with the increase of E_1 and b decreases with the increase of E_2. Apparently, the direct effects on the instability boundaries are the coefficient b and instability regions become narrow with the coefficient b increasing. The viscosity η increasing makes instability regions become narrow in

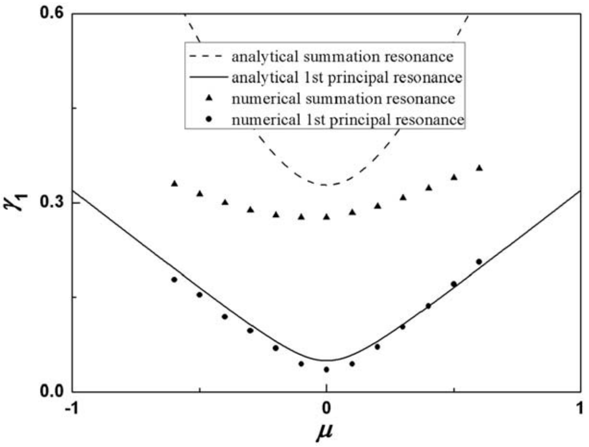

Fig. 6. The comparison of instability boundaries between summation and first principle parametric resonance (b=1.28, γ_0=2.0 and η=0.0011).

both summation resonance and principal parametric resonances, as shown in Fig. 4. The conclusions are accordant with the foregoing statements just under Eq. (55).

There is implicit effect of the constant mean speed on instability boundary, which cannot be directly achieved from expression (55) or (57). Numerical examples show that the instability boundaries move towards the decreasing direction of γ_1 in plane μ–γ_1 and the instability regions become narrow in the summation resonance with constant mean speed γ_0 increasing, but there is an opposite effect in the principal resonance in Fig. 5. Fig. 6 indicates that instability region of summation resonance is dramatically smaller than those of the first principal resonance under the same conditions.

The numerical examples show that changing trends predicted by both methods are qualitatively same. It demonstrates that the difference is very small in the first principal parametric resonances, but the difference in the summation parametric resonance is rather large.

6. Conclusions

This paper is devoted to parametric vibration of an axially accelerating beam constituted by the standard linear solid model using the material time derivative. The beam moves at an axial speed fluctuating harmonically about a constant mean speed. An asymptotic analysis is proposed to determine the stability condition, which is the same as that derived from the method of multiple scales. The differential quadrature scheme is developed to locate the stability boundary numerically. The analytical results are compared with the numerical calculations:

(1) Based on analytical expressions (55) and (57), with the stiffness coefficient E_1 increasing, the instability regions will become narrow. On the contrary, the decreasing stiffness coefficient E_2 leads the instability regions to become narrow. In addition, the instability regions will become narrow with the increase of viscosity η.
(2) When the material time derivative is used in the constitutive relation, the increasing constant mean speed leads to the additional viscosity. This conclusion cannot be achieved directly via the analytical conditions. Based on numerical results, with the increasing constant mean speed, the instability regions become narrow in the summation parametric resonance but there is an opposite effect in the principal resonance for standard linear solid model.
(3) Both analytical and numerical results indicate that instability region of summation resonance is dramatically smaller than those of the first principal resonance under the same conditions.
(4) The results compared indicate that the changing trend predicted by the numerical simulations is qualitatively same as by the analytical analysis.
(5) Quantitatively, the analytical results are validated by the numerical calculations in the principal parametric resonance, while there are differences in the summation parametric resonance is rather large.

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