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Vibration modes of planetary gears with unequally spaced planets and an elastic ring gear

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ABSTRACT

The highly structured modal properties of planetary gears having diametrically opposed planets and an elastic ring gear are illustrated and mathematically proved in this work. Two types of modes are found: rotational and translational modes. The properties of each mode type are given mathematically. A rule for how the modes of planetary gears having equally spaced planets evolve as the planets deviate to diametrically opposed is presented and discussed.

1. Introduction

While planetary gearboxes usually have equally spaced planets, some planetary gears have unequally spaced planets due to the limitations of assembly conditions or special applications. If the first planet is located at angle zero, the possible locations of the remaining planets are at $\psi = 2\pi j/(z_s + z_r)$, where j is an arbitrary integer and z_s, z_r are the tooth numbers of the sun and ring, respectively. In the case that equal spacing is not achievable, such as when $(z_s + z_r)/N$ is not an integer where N is the number of planets, the planets are diametrically opposed in almost all applications. This is because diametrically opposed planets have the benefits of reducing the bearing force, maintaining static and dynamic balance, and improving the load sharing. Several helicopter planetary gears have diametrically opposed planets, as do some automotive transmission planetary gears.

Ring gears have the largest radii in planetary gears. Often for purposes of weight reduction and increased power density, ring gears are designed to be thin in the radial direction. Some planetary gears are designed with thin rings to introduce compliance that improves load sharing among the planets. Thin rings with large radius deform elastically in applications like helicopters, wind turbines, and cars. This is evident from measured data on an OH-58 helicopter planetary gear [1], cracks observed in ring gears, and finite element analyses that show elastic deformation of the ring under operating loads.

In the literature, many studies exist on the vibration of elastic rings [2–6] and the free vibration of planetary gears with lumped-parameter models and equally spaced planets [7–15]. Lin and Parker [12,14] analytically studied the modal properties of planetary gears using a lumped-parameter model with consideration of both translational and rotational motions. Wu and Parker [16] analytically investigated the vibration of planetary gears with equally spaced planets based on an elastic-discrete model in which the ring gear was modeled as an elastic body and the remaining components were modeled as rigid bodies. The well-defined modal properties were characterized for all possible modes. Guo and Parker [17] studied the modal properties of compound planetary gears using a model with only rotational degrees of

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freedom. Parker and Ambarisha [18,19] showed how these modal properties are important for dynamic response suppression. Eritenel and Parker [20] derived the modal properties for helical planetary gears including three-dimensional motions of all gear bodies. Kiracofe and Parker [21] mathematically modeled and derived the structured modal properties of general compound planetary gears with equally spaced or diametrically opposed planets. Recently, Bartelmus and Zimroz examined planetary gear condition monitoring [22], which can take advantage of the modal properties derived in this work.

The natural frequencies and vibration modes are fundamental when dealing with an existing vibration problem or designing new systems to avoid resonant vibration, as gear engineers routinely need to do. This paper provides detailed and rigorously derived properties of the natural frequencies and modes for planetary gears with unequally spaced planets. The work provides knowledge engineers can use in practice as well as modal properties critical to further research on resonant vibration response, nonlinearity, diagnostics, and the like.

2. Modeling and equations of motion

Details of the model in Fig. 1 including nomenclature, dimensionless variables, extended operators, and equations of motion are given in [16] and adopted here. The arrows on the sun, ring, carrier, and planets shown in Fig. 1 define the positive directions of vibration (deviations from each gear's nominal rotation), which is not necessarily the positive direction of nominal rotation. The motion of the ring $u(\theta,t)$ is separated into the rigid body motion (x_r, y_r, u_r) and the elastic tangential deformation $v(\theta,t)$. The elastic radial deflection is $w(\theta,t) = -\partial v(\theta,t)/\partial\theta$.

The deflection of the whole system represented in the vector $\mathbf{a}$ is the combination of the elastic deformation of the ring $v(\theta,t)$ and the discrete body deflections $\mathbf{q}$ as

$$\mathbf{a}^{\mathrm{T}} = [v, \mathbf{q}^{\mathrm{T}}], \quad \mathbf{q} = [\underbrace{x_r, y_r, u_r}_{\mathbf{p}_r}, \underbrace{x_c, y_c, u_c}_{\mathbf{p}_c}, \underbrace{x_s, y_s, u_s}_{\mathbf{p}_s}, \underbrace{\xi_1, \eta_1, u_1}_{\mathbf{p}_1}, \ldots, \underbrace{\xi_N, \eta_N, u_N}_{\mathbf{p}_N}]^{\mathrm{T}}, \tag{1}$$

where $\mathbf{a}$ is referred to as an *extended variable*, $\mathbf{p}_j$, $(j = r, c, s, 1, \ldots, N)$ represent the deflections of the ring rigid motion, carrier, sun, and planets, and N is the number of planets.

The dimensionless eigenvalue problem for planetary gears having an elastic ring is written in extended operator form as

$$-\omega^2 M\mathbf{a} + K\mathbf{a} = 0, \tag{2}$$

where ω is a natural frequency, and M, K are extended inertia and stiffness operators defined by their action on elements of the space of extended variables (see details in [16]). Equations associated with the individual components in the sequence

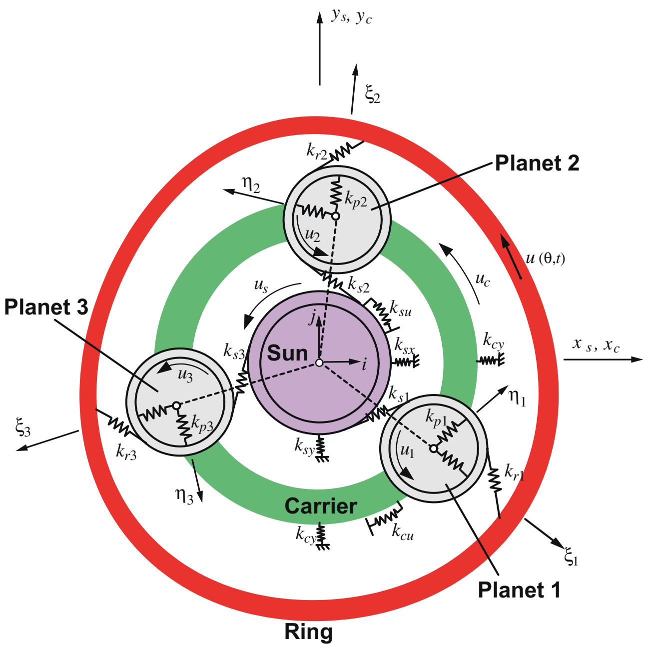

Fig. 1. Elastic-discrete model of a planetary gear and corresponding system coordinates. The distributed springs around the ring circumference are not shown.

of the elastic ring, ring rigid motion, carrier, sun, and planets are

$$-\frac{\omega^2}{2\pi}\left(1-\frac{\partial^2}{\partial\theta^2}\right)v+k_{bend}L_1v+L_2v+L_3\mathbf{q}=0, \tag{3}$$

$$-\omega^2\mathbf{M}_r\mathbf{p}_r+(\mathbf{K}_{rb}+\sum_n\mathbf{K}_{r1}^n)\mathbf{p}_r+\sum_n\mathbf{K}_{r2}^n\mathbf{p}_n+\sum_n(\mathbf{b}_r\chi|_{\theta=\psi_n})=\mathbf{0}, \tag{4}$$

$$-\omega^2\mathbf{M}_c\mathbf{p}_c+(\mathbf{K}_{cb}+\sum_n\mathbf{K}_{c1}^n)\mathbf{p}_c+\sum_n\mathbf{K}_{c2}^n\mathbf{p}_n=\mathbf{0}, \tag{5}$$

$$-\omega^2\mathbf{M}_s\mathbf{p}_s+(\mathbf{K}_{sb}+\sum_n\mathbf{K}_{s1}^n)\mathbf{p}_s+\sum_n\mathbf{K}_{s2}^n\mathbf{p}_n=\mathbf{0}, \tag{6}$$

$$-\omega^2\mathbf{M}_p\mathbf{p}_n+(\mathbf{K}_{c2}^n)^{\mathrm{T}}\mathbf{p}_c+(\mathbf{K}_{r2}^n)^{\mathrm{T}}\mathbf{p}_r+(\mathbf{K}_{s2}^n)^{\mathrm{T}}\mathbf{p}_s+\mathbf{K}_{pp}\mathbf{p}_n+\mathbf{b}_p\chi|_{\theta=\psi_n}=\mathbf{0},\quad n=1,\ldots,N. \tag{7}$$

The operators L_1, L_2, L_3, vectors $\mathbf{b}_r$, $\mathbf{b}_p$, all the inertia and stiffness matrices, and variables k_{bend} and χ are given in [16]. k_{bend} is the dimensionless bending stiffness of the ring gear.

3. Diametrically opposed planet pair modal properties

The natural frequencies of planetary gears with equally spaced planets and an elastic ring gear are either distinct or degenerate with multiplicity two. When the planets are diametrically opposed, it destroys the cyclic symmetry of the equal spacing. The asymmetry of diametrically opposed planets causes all the repeated natural frequencies to split into distinct ones. Certain modal properties remain, however, and all modes are classified into two types: rotational and translational modes.

Fourier expansion of the elastic deformation of the ring gives

$$v(\theta)=\sum_{m=2}^{JN}V_m\mathrm{e}^{\mathrm{i}m\theta}+cc, \tag{8}$$

where $J\geq 1$ is an arbitrarily large integer, and $m=0,\pm 1$ are contained in the rigid body motion of the ring $\mathbf{p}_r$. cc denotes the complex conjugate of all proceeding terms and $V_{-m}=\overline{V}_m$.

3.1. Rotational modes

Rotational modes for diametrically opposed planets contain only even numbered nodal diameter components of the elastic ring, and the translations (but not rotations) of the ring rigid motion, sun, and carrier are zero. A candidate rotational mode has the form

$$\mathbf{a}=\left[\sum_{m=2,4,\cdots}^{JN}V_m\mathrm{e}^{\mathrm{i}m\theta}+cc,\mathbf{p}_r,\mathbf{p}_c,\mathbf{p}_s,\mathbf{p}_1,\ldots,\mathbf{p}_{N/2},\mathbf{p}_1,\ldots,\mathbf{p}_{N/2}\right]^{\mathrm{T}}, \tag{9}$$

$$\mathbf{p}_h=[0,0,u_h]^{\mathrm{T}},\quad h=r,c,s,\quad \mathbf{p}_{z+N/2}=\mathbf{p}_z,\quad z=1,\ldots,N/2. \tag{10}$$

Compared to rotational modes of equally spaced planets, deflections of the planets are no longer identical for all the planets. Instead, they are identical for the two planets of every diametrically opposed pair. Furthermore, the elastic deformation of the ring contains all even numbered nodal diameter components, where the equally spaced case contains only the JN nodal diameter components.

Substituting (9) and (10) into (3), multiplying by $\mathrm{e}^{-\mathrm{i}j\theta}$, and integrating from 0 to 2π yields

$$-\frac{1+j^2}{2}\omega^2V_j+\frac{c_j}{2}V_j+2\sum_{m=\pm 2,\pm 4,\cdots}^{\pm JN}\sum_{n=1}^{N/2}(\cos\alpha_r+im\sin\alpha_r)(\cos\alpha_r-ij\sin\alpha_r)V_m\mathrm{e}^{\mathrm{i}(m-j)\psi_n}$$
$$+2(\cos\alpha_r-ij\sin\alpha_r)\sum_{n=1}^{N/2}(\xi_n\sin\alpha_r-\eta_n\cos\alpha_r-u_n)\mathrm{e}^{-\mathrm{i}j\psi_n}=0,\quad j=\pm 2,\pm 4,\ldots,\pm JN, \tag{11}$$

$$c_j=2\pi k_{bend}j^2(j^2-1)^2+2\pi k_{rus}+2\pi j^2k_{rbs}, \tag{12}$$

where k_{rus} and k_{rbs} are the tangential and radial distributed stiffnesses per unit length around the circumference of the ring. Use of the specified modal properties (9) and (10) to reduce (4) yields only one equation for the ring rigid motion

$$(2\pi k_{rus}/\cos^2\alpha_r+N-\omega^2/\cos^2\alpha_r)u_r+2\sum_{n=1}^{N/2}(\xi_n\sin\alpha_r-\eta_n\cos\alpha_r-u_n)+2\sum_{m=2,4,\cdots}^{JN}\sum_{n=1}^{N/2}[(\cos\alpha_r+im\sin\alpha_r)V_m\mathrm{e}^{im\psi_n}+cc]=0. \tag{13}$$

The remaining equations in (4) vanish. Similarly, (5) and (6) reduce to the following equations, respectively:

$$(k_{cu}+Nk_p-\omega^2I_c)u_c-k_p\sum_{n=1}^{N}\eta_n=0, \tag{14}$$

$$(k_{su}+Nk_{sp}-\omega^2I_s)u_s+k_{sp}\sum_{n=1}^{N}(-\xi_n\sin\alpha_s-\eta_n\cos\alpha_s+u_n)=0. \tag{15}$$

With the modal expressions (9) and (10) and straight forward manipulation, (7) becomes

$$(\mathbf{K}_{c2}^1)^{\mathrm{T}}\mathbf{p}_c+(\mathbf{K}_{r2}^1)^{\mathrm{T}}\mathbf{p}_r+(\mathbf{K}_{s2}^1)^{\mathrm{T}}\mathbf{p}_s+(\mathbf{K}_{pp}-\omega^2\mathbf{M}_p)\mathbf{p}_n+\mathbf{b}_p\sum_{m=2,4,\cdots}^{JN}[(\cos\alpha_r+im\sin\alpha_r)V_m\mathrm{e}^{im\psi_n}+cc]=\mathbf{0},\quad n=1,\ldots,N/2. \tag{16}$$

Eqs. (11)–(16) form a reduced eigenvalue problem of order $JN+3N/2+3$ as: JN equations from (11) for the ring elastic deformation, three equations from (13) to (15) for the ring, carrier, and sun rotations, and $3N/2$ equations from (16) for the planet motions. Thus, one can construct $JN+3N/2+3$ homogeneous equations with undetermined eigenvalue ω^2. This algebraic eigenvalue problem yields $JN+3N/2+3$ rotational modes when each eigenvector is substituted into (9) and (10). Compared to the rotational modes of planetary gears with equally spaced planets [16], the number of rotational modes increases from $6+J$ to $JN+3N/2+3$. Where these additional rotational modes come from is discussed subsequently.

3.2. Translational modes

Translational modes for diametrically opposed planet pairs contain only odd numbered nodal diameter components of the elastic ring, and the rotations (but not translations) of the ring rigid motion, sun, and carrier are zero. A candidate translational mode has the form

$$\mathbf{a}=\left[\sum_{m=3,5,\cdots}^{JN-1}V_m\mathrm{e}^{im\theta}+cc,\mathbf{p}_r,\mathbf{p}_c,\mathbf{p}_s,\mathbf{p}_1,\ldots,\mathbf{p}_{N/2},-\mathbf{p}_1,\ldots,-\mathbf{p}_{N/2}\right]^{\mathrm{T}}, \tag{17}$$

$$\mathbf{p}_h=[x_h,y_h,0]^{\mathrm{T}},\quad h=r,c,s,\quad \mathbf{p}_{z+N/2}=-\mathbf{p}_z,\quad z=1,\ldots,N/2. \tag{18}$$

For translational modes with equally spaced planets, the nth planet's motion is calculable from the arbitrarily chosen first planet's motion. For diametrically opposed planets, only the deflections of a diametrically opposed pair of planets are related as $\mathbf{p}_{z+N/2}=-\mathbf{p}_z$. Furthermore, the elastic deformation of the ring contains all odd numbered nodal diameter components instead of only the $JN\pm1$ nodal diameter components for equally spaced planets.

Substituting (17) and (18) into (3), multiplying by $\mathrm{e}^{-\mathrm{i}j\theta}$, and integrating from 0 to 2π yields the equations governing V_j as

$$-\frac{1+j^2}{2}\omega^2V_j+\frac{c_j}{2}V_j+2\sum_{m=\pm3,\pm5,\cdots}^{\pm(JN-1)}\sum_{n=1}^{N/2}(\cos\alpha_r+im\sin\alpha_r)(\cos\alpha_r-ij\sin\alpha_r)V_m\mathrm{e}^{\mathrm{i}(m-j)\psi_n}$$
$$+2(\cos\alpha_r-ij\sin\alpha_r)\sum_{n=1}^{N/2}(\xi_n\sin\alpha_r-\eta_n\cos\alpha_r-u_n)\mathrm{e}^{-\mathrm{i}j\psi_n}=0,\quad j=\pm3,\pm5,\ldots,\pm(JN-1). \tag{19}$$

Substitution of (17) and (18) into (4)–(6) generates the following six equations:

$$(k_{rx}+2k_{rp}\sum_{n=1}^{N/2}\sin^2\psi_{rn}-\omega^2m_r)x_r+2\sum_{n=1}^{N/2}\sin\psi_{rn}\delta_n=0, \tag{20}$$

$$(k_{ry}+2k_{rp}\sum_{n=1}^{N/2}\cos^2\psi_{rn}-\omega^2m_r)y_r-2\sum_{n=1}^{N/2}\cos\psi_{rn}\delta_n=0, \tag{21}$$

$$(k_{cx}+Nk_{pn}-\omega^2m_c)x_c+2k_{pn}\sum_{n=1}^{N/2}(-\zeta_n\cos\psi_n+\eta_n\sin\psi_n)=0, \tag{22}$$

$$(k_{cy}+Nk_{pn}-\omega^2 m_c)y_c-2k_{pn}\sum_{n=1}^{N/2}(\zeta_n \sin\psi_n+\eta_n\cos\psi_n)=0, \tag{23}$$

$$(k_{sx}+2k_{sp}\sum_{n=1}^{N/2}\sin^2\psi_{sn}-\omega^2 m_s)x_s+2k_{sn}\sum_{n=1}^{N/2}\sin\psi_{sn}(\zeta_n\sin\alpha_s+\eta_n\cos\alpha_s-u_n)=0, \tag{24}$$

$$(k_{sy}+2k_{sp}\sum_{n=1}^{N/2}\cos^2\psi_{sn}-\omega^2 m_s)y_s-2k_{sn}\sum_{n=1}^{N/2}\cos\psi_{sn}(\zeta_n\sin\alpha_s+\eta_n\cos\alpha_s-u_n)=0, \tag{25}$$

where

$$\delta_n=-\zeta_n\sin\alpha_r+\eta_n\cos\alpha_r+u_n-\sum_{m=3,5,\cdots}^{JN-1}[(\cos\alpha_r+im\sin\alpha_r)V_m e^{im\psi_n}+cc]. \tag{26}$$

The specified modal expressions (17) and (18) and straightforward manipulation reduce (7) to

$$(\mathbf{K}_{c2}^1)^{\mathrm{T}}\mathbf{p}_c+(\mathbf{K}_{r2}^1)^{\mathrm{T}}\mathbf{p}_r+(\mathbf{K}_{s2}^1)^{\mathrm{T}}\mathbf{p}_s+(\mathbf{K}_{pp}-\omega^2\mathbf{M}_p)\mathbf{p}_n+\mathbf{b}_p\sum_{m=3,5,\cdots}^{JN-1}[(\cos\alpha_r+im\sin\alpha_r)V_m e^{im\psi_n}+cc]=\mathbf{0},\quad n=1,\ldots,N/2. \tag{27}$$

Eqs. (19)–(27) form a reduced eigenvalue problem of order $JN+3N/2+4$ as: $JN-2$ equations from (19) for the ring elastic deformation, six Eqs. (20)–(25) for the ring, carrier, and sun translations, and $3N/2$ Eqs. (27) for the planet motions. For each eigensolution of this reduced eigenvalue problem, the full system mode is constructed from (17) and (18). Generally, all the eigenvalues are distinct. Compared to the translational modes of planetary gears with equally spaced planets [16], the number of translational modes increases from $4J+10$ to $JN+3N/2+4$. These additional translational modes are discussed below.

4. Relationships between modes for equally spaced and diametrically opposed planets

Four types of modes exist for planetary gears with equally spaced planets and an elastic ring: rotational, translational, planet, and purely ring modes [16]. Rotational and purely ring modes have distinct eigenvalues. Translational modes have degenerate eigenvalues with multiplicity two. If the number of planets is odd, all the planet modes are degenerate with multiplicity two; otherwise, the planet modes are either degenerate with multiplicity two or distinct. Planet modes exist only when the number of planets $N\geq 4$, and distinct planet modes always contain the $jN+N/2$ ($j=0,1,\ldots$) nodal diameter components. For rotational modes, the translations of the ring rigid motion, sun, and carrier are zero. For translational modes, the rotations for the ring rigid motion, sun, and carrier are zero. The deflections of the ring rigid motion, sun, and carrier are zero for all planet modes. Rotational, translational, planet, and purely ring modes contain the jN, $jN\pm 1$, $jN\pm s$, and jN or $jN+N/2$ nodal diameter components of the ring, respectively, where s is selected from $2,3,\ldots,\mathrm{int}(N/2)$ and j is an integer.

An interesting question is: when the planet spacing changes from equally spaced (with an even number of planets) to diametrically opposed, how do the planet and purely ring modes, which exist only for equally spaced planets, evolve into rotational or translational modes? The rule is: *If a mode for equally spaced planets has odd nodal diameter ring components* (*and so* $\mathbf{p}_{z+N/2}=-\mathbf{p}_z$, $z=1,\ldots,N/2$), *it evolves into a translational mode when the planets are diametrically opposed*; *if a mode has even nodal diameter ring components* (*and so* $\mathbf{p}_{z+N/2}=\mathbf{p}_z$), *it evolves into a rotational mode*. To apply this rule, note that for equally spaced planets with even N, the nodal diameter components of any mode are all even or all odd; for diametrically opposed planets, translational modes have all odd nodal diameter components while rotational modes have all even nodal diameter components.

Some clues guide the justification of the above rule. Every mode for equally spaced planets must evolve into either a rotational mode or a translational mode as the planets deviate to the diametrically opposed case. All diametrically opposed modes satisfy either $\mathbf{p}_{z+N/2}=-\mathbf{p}_z$ (translational mode) or $\mathbf{p}_{z+N/2}=\mathbf{p}_z$ (rotational mode). Because equal spacing is a special case of diametrically opposed planets, all equally spaced modes also satisfy one of these two conditions. Because of the continuity of the modes for changes in planet spacing, equally spaced modes where $\mathbf{p}_{z+N/2}=-\mathbf{p}_z$ holds retain this property when the planet spacing changes to diametrically opposed (rather than discontinuously jumping to the alternate diametrically opposed possibility that $\mathbf{p}_{z+N/2}=\mathbf{p}_z$). One can imagine small deviations from equal spacing to clarify this continuity argument, but the conclusion is not restricted to that case; the properties established for small deviations must also hold for large deviations because the foregoing proof of the modal properties is not limited to small deviations. Similar arguments apply to equally spaced modes where $\mathbf{p}_{z+N/2}=\mathbf{p}_z$ holds. Because purely ring modes where $\mathbf{p}_z=\mathbf{0}$ satisfy both $\mathbf{p}_{z+N/2}=-\mathbf{p}_z$ and $\mathbf{p}_{z+N/2}=\mathbf{p}_z$, one cannot use the simpler criteria $\mathbf{p}_{z+N/2}=-\mathbf{p}_z$ or $\mathbf{p}_{z+N/2}=\mathbf{p}_z$ as the conditions for the mode evolution rule given above (as discussed subsequently).

A pair of degenerate planet modes for equally spaced planets [16] has the form

$$\mathbf{a}_{s1} = \left[\sum_{m=jN+s} V_m e^{im\theta} + cc, \widehat{\mathbf{q}}_{p,s}^{\mathrm{T}} \right]^{\mathrm{T}}, \tag{28}$$

$$\mathbf{a}_{s2} = \left[\sum_{m=jN+s} iV_m e^{im\theta} + cc, \hat{\mathbf{q}}_{p,s}^{\mathrm{T}} \right]^{\mathrm{T}}, \tag{29}$$

$$\widehat{\mathbf{q}}_{p,s}^{\mathrm{T}} = [\mathbf{0}, \mathbf{0}, \mathbf{0}, \cos s\psi_1 \mathbf{p}_1^{\mathrm{T}}, \ldots, \cos s\psi_N \mathbf{p}_1^{\mathrm{T}}], \quad \hat{\mathbf{q}}_{p,s}^{\mathrm{T}} = [\mathbf{0}, \mathbf{0}, \mathbf{0}, \sin s\psi_1 \mathbf{p}_1^{\mathrm{T}}, \ldots, \sin s\psi_N \mathbf{p}_1^{\mathrm{T}}], \tag{30}$$

where s is an integer selected from $2, 3, \ldots, \mathrm{int}[(N-1)/2]$ and j is any integer satisfying $jN \pm s \in \{-JN, -JN+1, \ldots, JN\}$. Notice that

$$\cos s\psi_{z+N/2} = \cos s(\psi_z + \pi) = \begin{cases} \cos s\psi_z & \text{for even } s \\ -\cos s\psi_z & \text{for odd } s \end{cases} \tag{31}$$

$$\sin s\psi_{z+N/2} = \sin s(\psi_z + \pi) = \begin{cases} \sin s\psi_z & \text{for even } s \\ -\sin s\psi_z & \text{for odd } s \end{cases} \tag{32}$$

Thus, according to (30)–(32), $\mathbf{p}_{z+N/2} = -\mathbf{p}_z$ holds for odd s, and $\mathbf{p}_{z+N/2} = \mathbf{p}_z$ holds for even s. For distinct planet modes, one can substitute $s = N/2$ into (28) and the first of (30) to find $\mathbf{p}_{z+N/2} = -\mathbf{p}_z$ holds for odd $N/2$, and $\mathbf{p}_{z+N/2} = \mathbf{p}_z$ holds for even $N/2$. This indicates planet modes, whether distinct or degenerate, having odd (even) nodal diameter components will evolve into translational (rotational) modes as the planets change from equally spaced to diametrically opposed.

As an example, Fig. 2a shows a planet mode having $jN \pm 2$ (even numbered) nodal diameter components with degenerate natural frequency $\omega_{4,5} = 0.5539$ for equally spaced planets. The system parameters are listed in Table 3. When the planets are diametrically opposed the natural frequency pair splits into two rotational modes. One of the split rotational modes is shown in Fig. 2b with natural frequency $\omega_5 = 0.5561$.

A purely ring mode for equally spaced planets has the form

$$\mathbf{a} = [(\cos \alpha_r \sin m\theta - m \sin \alpha_r \cos m\theta) V_m, \mathbf{0}]^{\mathrm{T}}, \quad m = \begin{cases} jN & \text{for odd or even } N \\ jN + N/2 & \text{for even } N \end{cases} \tag{33}$$

The deflections of the planets (and all other rigid components) are zero for a purely ring mode. Thus, using the deflections of the planets as the condition to determine which type of mode it will evolve into does not work. A purely ring mode has

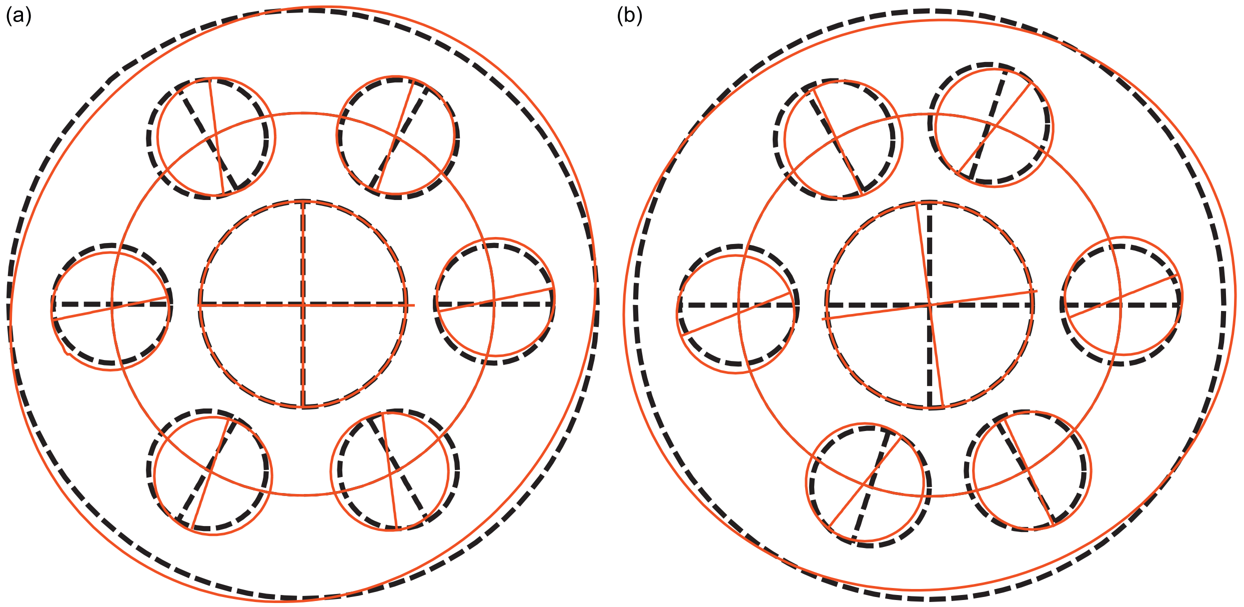

Fig. 2. Mode comparison of a planetary gear with: (a) a planet mode for equally spaced planets ($\omega_{4,5} = 0.5539$), and (b) the corresponding rotational mode for diametrically opposed planets ($\omega_5 = 0.5561$). Parameters are given in Table 3. For the diametrically opposed case, the positions of the planets are $\psi_1 = 0$, $\psi_2 = 2\pi/5$, $\psi_3 = 2\pi/3$, $\psi_4 = \pi$, $\psi_5 = \psi_2 + \pi$, $\psi_6 = \psi_3 + \pi$.

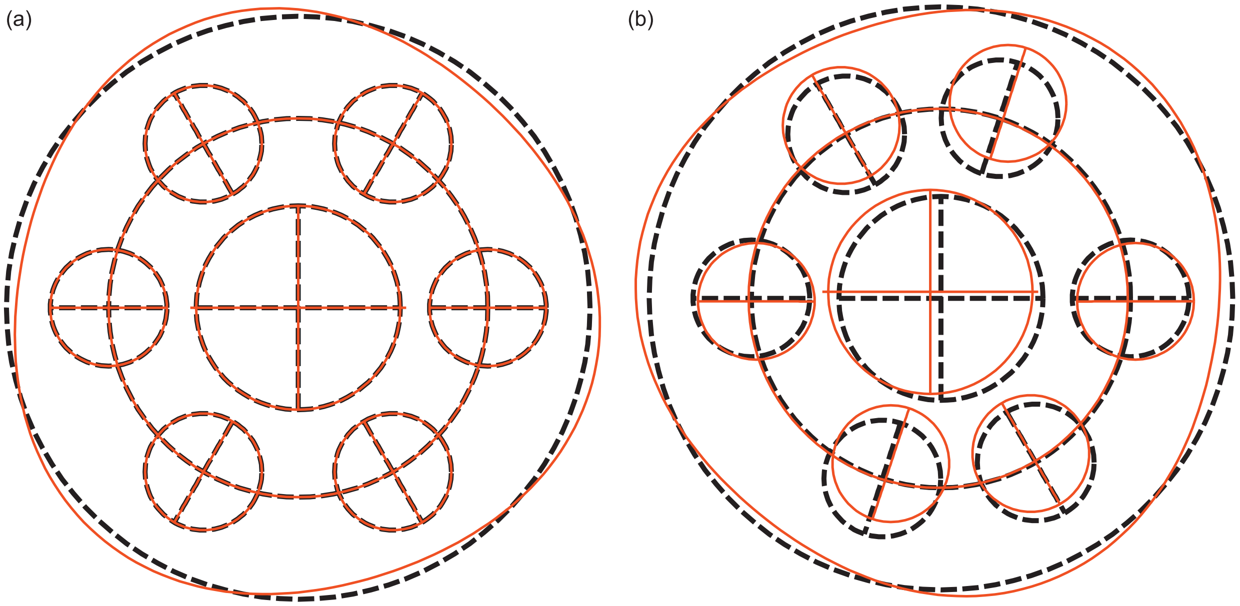

Fig. 3. Mode comparison of a planetary gear with: (a) a purely ring mode for equally spaced planets ($\omega_{15}=1.518$), and (b) the corresponding translational mode for diametrically opposed planets ($\omega_{15}=1.519$). Parameters are given in Table 3. For the diametrically opposed case, the positions of the planets are $\psi_1=0$, $\psi_2=2\pi/5$, $\psi_3=2\pi/3$, $\psi_4=\pi$, $\psi_5=\psi_2+\pi$, $\psi_6=\psi_3+\pi$.

one and only one nodal diameter component. For continuity of the modes, the mode that a purely ring mode evolves into should contain at least that specific nodal diameter component. Thus, if the purely ring mode has an odd nodal diameter component, the corresponding diametrically opposed mode will contain that (and other) odd nodal diameter components; this means the purely ring mode evolves into a translational mode. Similarly, if the purely ring mode has an even nodal diameter component, it evolves into a rotational mode. The nodal diameter component in a purely ring mode is jN or $jN+N/2$. Whether jN or $jN+N/2$ is odd or even for given j governs the mode type to which it will evolve.

Fig. 3a shows a three nodal diameter purely ring mode with $\omega_6=1.518$ for equally spaced planets. The system parameters are listed in Table 3. The corresponding mode for diametrically opposed planets is a translational mode with natural frequency $\omega_{15}=1.519$ (Fig. 3b). The deflections of the planets and sun are significant, and the dominant (but not only) elastic ring deformation is the three nodal diameter component.

For rotational modes of equally spaced planets, the deflections of all the planets are identical [16]. This guarantees $\mathbf{p}_{z+N/2}=\mathbf{p}_z$ for all z, so they are rotational modes when the planets deviate to the diametrically opposed case. Considering the ring nodal diameter components, a rotational mode of equally spaced planets contains only the jN nodal diameter components. Because N is even, all the numbers of nodal diameter components are even. Thus, one can identify the corresponding modes for diametrically opposed planets are rotational modes using the rule based on even/odd nodal diameter components, and this agrees with the conclusion immediately above from $\mathbf{p}_{z+N/2}=\mathbf{p}_z$.

For translational modes of equally spaced planets, the planet deflections for a pair of translational modes satisfy

$$\begin{bmatrix}\mathbf{p}_n\\ \hat{\mathbf{p}}_n\end{bmatrix}=\begin{bmatrix}\cos\psi_n\mathbf{I} & \sin\psi_n\mathbf{I}\\ -\sin\psi_n\mathbf{I} & \cos\psi_n\mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{p}_1\\ \hat{\mathbf{p}}_1\end{bmatrix}, \tag{34}$$

where $\mathbf{I}$ is a 3×3 identity matrix and $\psi_1=0$. Substitution of $n=N/2+z$ into (34) yields

$$\mathbf{p}_{z+N/2}=-\mathbf{p}_z,\quad \hat{\mathbf{p}}_{z+N/2}=-\hat{\mathbf{p}}_z. \tag{35}$$

According to (35), translational modes of equally spaced planets remain translational modes when the planets become diametrically opposed. Because translational modes of equally spaced planets contain the $jN\pm1$ nodal diameter components with even N, all the ring nodal diameter components are odd. Thus, one can identify it as a translational mode for diametrically opposed planets based on the even/odd condition as well.

In summary, for any mode of systems with equally spaced planets, whether the elastic ring nodal diameter components are even or odd determines the mode type to which it evolves as the planets deviate to diametrically opposed.

For equally spaced or diametrically opposed planets, the total number of degrees of freedom (i.e., modes) is the same: $(2J+3)N+7$, where J is the user-selected upper limit in (8). For equally spaced planets with even N, the numbers of modes for rotational, translational, planet and purely ring modes are J+6, 4J+10, (2$JN-7J$)+(3$N-9$), and 2J, respectively. For diametrically opposed planets (obviously with even N), all the modes fall into two types: JN+3N/2+3 rotational modes plus

Table 1
Modal property comparison of planetary gears for four different cases.

	Elastic-discrete model		Lumped-parameter model	
	Equally spaced (even N)	Diametrically opposed	Equally spaced (even or odd N)	Diametrically opposed
Rotational mode				
Multiplicity	1	1	1	1
Number	$6+J$	$JN+3N/2+3$	6	6
Ring nodal diameter components	jN	$2j$	0 nodal diameter (pure rotation)	0 nodal diameter (pure rotation)
Planet deflections	$\mathbf{p}_n=\mathbf{p}_1$	$\mathbf{p}_{n+N/2}=\mathbf{p}_n$	$\mathbf{p}_n=\mathbf{p}_1$	$\mathbf{p}_n=\mathbf{p}_1$
Translational mode				
Multiplicity	2	1	2	1
Number	$10+4J$	$JN+3N/2+4$	12	12
Ring nodal diameter components	$jN\pm1$	$2j+1$	1 nodal diameter (pure translation)	1 nodal diameter (pure translation)
Planet deflections	Eq. (34)	$\mathbf{p}_{n+N/2}=-\mathbf{p}_n$	Eq. (34)	$\mathbf{p}_n\sin\psi_2=\mathbf{p}_1\sin(\psi_2-\psi_n)+\mathbf{p}_1\sin\psi_n$
Planet mode				
Multiplicity	2 or 1		$N-3$	$N-3$
Number	$2JN-7J+3N-9$		$3N-9$	$3N-9$
Ring nodal diameter components	$jN\pm s$ $s=2,\dots,N/2$	×	×	×
Planet deflections	$\mathbf{p}_n=w_n\mathbf{p}_1$		$\mathbf{p}_n=w_n\mathbf{p}_1$	$\mathbf{p}_n=w_n\mathbf{p}_1$
Number of purely ring modes	$2J$	×	×	×

× denotes not applicable, and $j=0,1,2,\dots$.

Table 2
Number of planet and purely ring modes that evolve into rotational or translational modes when the planets deviate from equally spaced to diametrically opposed.

$N/2$	Planet modes	Purely ring modes
Even	$(N/2-2)(2J+3)\to t$ $(N/2-1)(2J+3)-J\to r$	$2J\to r$
Odd	$(N/2-2)(2J+3)-J\to t$ $(N/2-1)(2J+3)\to r$	$J\to t$ $J\to r$

Designations t, r denote translational and rotational modes for diametrically opposed planets.

$JN+3N/2+4$ translational modes. The total number of rotational and translational modes equals the total degrees of freedom, so no other mode types are possible.

Table 1 summarizes the modal properties of four cases of planetary gears with either equally spaced or diametrically opposed planets based on the elastic-discrete [16] and lumped-parameter models [12,14]. For the lumped-parameter model, when the planets change from equally spaced to diametrically opposed, most of the modal properties in Table 1 are retained. For the elastic-discrete model, the changes are more dramatic: the number of mode types reduces from four to two and the properties of each mode type change. Compared to the other three cases in Table 1, a planetary gear with an elastic-discrete model and diametrically opposed planets is the only case without any planet modes. In the lumped-parameter model, however, no matter if the planets are equally spaced or diametrically opposed $3N-9$ planet modes are always present.

The number of planet and purely ring modes that evolve into rotational or translational modes can be determined. According to [16], a pair of degenerate planet modes is given in (28)–(30) with s selected from $2,3,\dots,N/2-1$ when the number of planets N is even. For each s in $[2,N/2-1]$, there are $2J+3$ pairs of degenerate planet modes. For $s=N/2$, there are $J+3$ distinct planet modes [16]. To identify how many of these planet modes evolve into either rotational or translational modes, we must consider two cases.

Case 1: When $N/2$ is even, $N/2-1$ is odd. Accordingly, half of the s in $[2,N/2-1]$ are odd and the other half are even. Thus, $(N/2-2)(2J+3)$ planet modes contain odd nodal diameter components, and they evolve into translational modes; $(N/2-2)(2J+3)+J+3$ planet modes evolve into rotational modes, where the additional $J+3$ modes come from the distinct planet modes.

Case 2: When $N/2$ is odd, there are $(N-2)/4$ even s and $(N-6)/4$ odd s for s in $[2,N/2-1]$. Thus, $[(N-2)/4](2J+3)(2)=(N/2-1)(2J+3)$ degenerate planet modes contain even nodal diameter components, and they evolve

Table 3
Dimensional parameters of a planetary gear with six equally spaced planets.

Inertias (kg)	$I_r/r_r^2=8.891$, $I_c/r_c^2=6.000$, $I_s/r_s^2=2.500$, $I_p/r_p^2=2.000$
Masses (kg)	$m_r=7.350$, $m_c=5.430$, $m_s=0.400$, $m_p=1.000$
Stiffnesses (N/m)	$k_{rp}=k_{sp}=100\times10^6$, $k_{rbs}=k_{rus}=0$, $k_{bend}=4\times10^6$, $k_s=10\times10^6$, $k_{su}=50\times10^6$, $k_c=k_{cu}=500\times10^9$, $k_p=200\times10^6$
Pressure angle (deg)	$\alpha_r=\alpha_s=24.60$

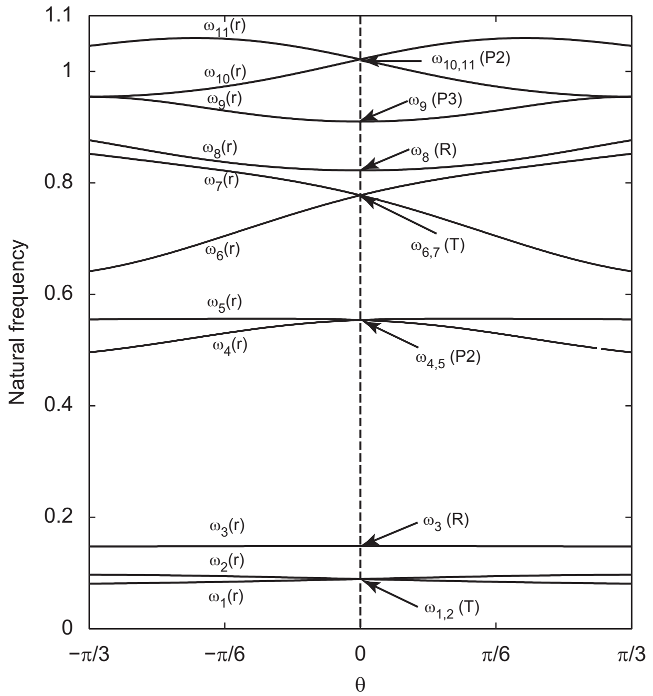

Fig. 4. Dimensionless natural frequencies of planetary gears when the position of one pair of diametrically opposed planets deviates an angle θ from the equally spaced position. The six-planet system is defined in Table 3. For equally spaced planets (θ=0), the designations R, T denote rotational and translational modes, and the designations P2, P3 denote planet modes having $jN\pm2$, $jN\pm3$ nodal diameter components for equally spaced planets. The designations r, t denote rotational and translational modes for diametrically opposed planets.

into rotational modes; $(N/2-3)(2J+3)+J+3$ planet modes evolve into translational modes, where J+3 modes come from the distinct planet modes.

According to (33), for even N there are $2J$ purely ring modes. When $N/2$ is even, all the purely ring modes have an even nodal diameter component, thus they evolve into rotational modes. When $N/2$ is odd, half of the purely ring mode evolve into rotational modes, and the other half evolve into translational modes.

Table 2 lists the number of planet and purely ring modes evolving into translational and rotational modes as the planets deviate from equally spaced to diametrically opposed. As indicated in Table 2, whether $N/2$ is even or odd, there are the same number, $(N/2-1)(2J+3)+J$, of total planet and purely ring modes evolving into rotational modes. Similarly, whether $N/2$ is even or odd, there are the same number, $(N/2-2)(2J+3)$, of total planet and purely ring modes evolving into translational modes.

5. Example

As an example, a planetary gear with six equally spaced planets is analyzed. The system parameters are given in Table 3. One of the three pairs of diametrically opposed planets deviates from the equally spaced position by an angle θ. Fig. 4 shows the effects of θ on the natural frequencies. The designations R, T denote rotational and translational modes, and P2, P3 denote planet modes having $jN\pm2$, $jN\pm3$ nodal diameter components for equally spaced planets, respectively. From Fig. 4 one can numerically verify how each type of mode evolves when the planets deviate from equally spaced to diametrically opposed. Natural frequency splitting is observed, such as for the translational modes $\omega_{6,7}$ and planet modes $\omega_{4,5}$ and $\omega_{10,11}$. Note that the limiting case of $\theta=\pm\pi/3$ is not practically meaningful because two pairs of diametrically opposed planets lie along the same diameter.

In the elastic-discrete model, the elastic deformation of the ring is highly coupled with the positions of the planets. Thus, deviation of planet positions from equal spacing yields significant changes to some natural frequencies (Fig. 4). For certain rotational modes with small elastic ring deformation, the natural frequencies are insensitive to θ (see ω_3 in Fig. 4). For rotational modes in the lumped-parameter, rigid ring model, *all* rotational modes are independent of the positions of the planets [14]. This is because changing planet positions does not change the projection of mesh stiffnesses in the direction tangent to the ring gear. For the elastic-discrete model, the ring-planet mesh stiffnesses have a radial component that couples to elastic ring deformation. Thus, rotational modes for the elastic-discrete model are affected by planet position, with greater effect for modes with large relative amplitude of ring deformation. Translational modes are, in general, more sensitive to planet position. They experience the same ring-planet mesh stiffness interaction with ring deformation as above for rotational modes. In addition, changing the positions of the planets alters the support and mesh stiffness forces between the sun, planets, carrier, and ring in the horizontal and vertical directions, and this directly affects the translational modes even if the elastic deformation of the ring is negligible.

6. Conclusions

This work analytically identifies the modal properties of planetary gears with diametrically opposed planets and an elastic ring gear. The elastic-discrete model represents the ring gear as an elastic body free to deform radially while the remaining components are rigid bodies. The elastic continuum ring model leads to an infinite-dimensional system. Relationships between the modal properties of planetary gears with equally spaced and diametrically opposed planets are examined in detail. The following conclusions are obtained:

1. All the modes are classified into rotational or translational modes with distinct natural frequencies. Closed-form expressions are provided for the structure of each mode type. A rotational mode contains only even numbered nodal diameter components of the elastic ring, and a translational mode contains only odd numbered nodal diameter components. The planet and purely ring modes present when the planets are equally spaced no longer exist.
2. For rotational modes, the translations for the ring rigid motion, sun, and carrier are zero. For translational modes, the rotations for the ring rigid motion, sun, and carrier are zero. The motions (displacements and rotation) of the two planets of every diametrically opposed pair are identical for rotational modes and opposite for translational modes.
3. All the planet and purely ring modes of equally spaced planets evolve into either rotational or translational modes when the planets change to diametrically opposed. The rule governing this modal evolution is: any mode for equally spaced planets having odd (even) nodal diameter components evolves into a translational (rotational) mode as the planets deviate to diametrically opposed. The exact numbers of planet and purely ring modes evolving to each of rotational and translational modes are given.

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