Journal of Sound and Vibration 329 (2010) 2291–2303

Contents lists available at ScienceDirect

Journal of Sound and Vibration

journal homepage: www.elsevier.com/locate/jsvi

A new approach for free vibration of axially functionally graded beams with non-uniform cross-section

Yong Huang, Xian-Fang Li *

School of Civil Engineering and Architecture, Central South University, Changsha 410075, China

ARTICLE INFO

Article history:
Received 10 August 2009
Received in revised form
2 December 2009
Accepted 25 December 2009
Handling editor: L.G. Tham
Available online 27 January 2010

ABSTRACT

This paper studies free vibration of axially functionally graded beams with non-uniform cross-section. A novel and simple approach is presented to solve natural frequencies of free vibration of beams with variable flexural rigidity and mass density. For various end supports including simply supported, clamped, and free ends, we transform the governing equation with varying coefficients to Fredholm integral equations. Natural frequencies can be determined by requiring that the resulting Fredholm integral equation has a non-trivial solution. Our method has fast convergence and obtained numerical results have high accuracy. The effectiveness of the method is confirmed by comparing numerical results with those available for tapered beams of linearly variable width or depth and graded beams of special polynomial non-homogeneity. Moreover, fundamental frequencies of a graded beam combined of aluminum and zirconia as two constituent phases under typical end supports are evaluated for axially varying material properties. The effects of the geometrical and gradient parameters are elucidated. The present results are of benefit to optimum design of non-homogeneous tapered beam structures.

1. Introduction

Functionally graded materials (FGMs) have striking advantages over traditional homogeneous materials due to continuous transition of material properties, which avoids shortage resulting from the mismatch of material properties at distinct interfaces between two dissimilar materials. For instance, FGMs made of ceramic and metal are capable of both suffering from high-temperature environment because of better thermal resistance of the ceramic phase and exhibiting stronger mechanical performance of metal phase to guarantee the structural integrity of FGMs. Such excellent performances make FGMs to be widely used in thermal and structural fields as very promising new materials. With the development of advanced techniques, FGMs may be fabricated into various structures including beams, plates and shells [1,2].

For functionally graded beams, gradient variation may be orientated in the cross-section or/and in the axial direction. For the former, there have been a large number of researches devoted to bending, vibration and stability (e.g. [3–8]). For axially graded beams, similar problems become more complicated because of the governing equation with variable coefficients. So far, few analytical solutions are found for arbitrary gradient change due to the difficulty of mathematical treatment of the problem save certain special gradients. For example, by making use of the semi-inverse method, Elishakoff and co-workers treated a large class of problems involving graded beams of special forms such as polynomials and obtained explicit fundamental frequency of free vibration [9–14]. However, the semi-inverse method cannot apply for graded beams of any axial non-homogeneity. In addition, with the aid of special functions, Li solved some free vibration

* Corresponding author.
E-mail address: xfli@mail.csu.edu.cn (X.-F. Li).

0022-460X/$ - see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.jsv.2009.12.029

and buckling problems of axially graded or non-uniform beams [15,16]. Nevertheless, the assumption of non-homogeneity or non-uniformity still has special requirements, and is not arbitrary.

On the other hand, a large number of investigations on free vibration of beams with non-uniform cross-section have been studied. For this problem, the governing equation is still with varying flexural rigidity and distributed mass. There have been a lot of analytical and numerical approaches presented for dealing with such problems. For example, Laura et al. [17] employed Rayleigh's optimization technique to obtain an approximate solution of natural frequencies and buckling loads for a non-uniform beam subject to a $p_0\cos\omega t-$type loading excitation. By using the Rayleigh–Ritz and Lagrange multiplier method, Abrate [18] has investigated the vibration of a class of tapered non-uniform rods or/and beams. In [19–22], exact analytical solutions of the longitudinal and transverse vibration of rods/beams have been obtained by transforming the equation of motion to a differential equation which is analytically solvable in terms of special functions such as Bessel functions. Based on the fact that a non-uniform beam can be partitioned into multi-homogeneous uniform sub-beams, Singh et al. [23] developed a numerical method for determining natural frequencies of a non-uniform beam. Considering material properties being stochastic non-homogeneous properties, the functional perturbation method has been exploited to calculate natural frequencies and mode shapes of non-homogeneous rods and beams [24]. Although the integral equation approach according to Green's functions has been proposed to cope with free vibration of Euler–Bernoulli beams [25,26], the determination of Green's functions, which depend on the governing equation itself as well as the end supports, is often cumbersome. Here we will give a unified approach based on the integral equation method to treat free vibration of Euler–Bernoulli beams with variable flexural rigidity and mass density. This method does not require to know Green's function or influence function, and in contrary, a direct integration of the governing equation together with the end supports can get a corresponding Fredholm integral equation.

The objective of this paper is to present a novel approach for analyzing free vibration of axially graded and non-uniform beams. We transform the governing differential equation with variable coefficients in connection with appropriate end supports to Fredholm integral equations. Then by expanding the mode shapes as power series, the resulting Fredholm integral equations are reduced to a system of algebraic equations in unknown coefficients. Natural frequencies can be determined from the existence condition of a non-trivial solution in the resulting system. Obtained results are compared with those solutions available, and high accuracy can be achieved. Finally, we evaluate the fundamental frequencies of a graded beam made of Al and ZrO_2 under several typical end supports. The effects of the gradient parameter on the fundamental frequencies are elucidated.

2. Statement of the problem

Consider an axially graded and non-uniform beam of length L which is subject to the action of transverse loading. In the present study, the material properties and cross-section of the beam are assumed to vary continuously along the length direction. According to the Euler–Bernoulli beam theory, the governing differential equation reads [27]

$$\frac{\partial^2}{\partial x^2}\left[D(x)\frac{\partial^2 w}{\partial x^2}\right]+m(x)\frac{\partial^2 w}{\partial t^2}=q(x,t), \tag{1}$$

where x is axial coordinate, w the deflection and q the distributed transverse loading; $D(x)=E(x)I(x)$ is flexural rigidity which is a function of the axial coordinate x and depends upon both Young's modulus $E(x)$ and the inertial moment of cross-sectional area $I(x)$; $m(x)=\rho(x)A(x)$ is the mass of the beam per unit length which depends upon cross-sectional area $A(x)$ and mass density $\rho(x)$. In order to treat free vibration of axially graded and non-uniform beams, it is necessary to take

$$q(x,t)=0 \tag{2}$$

and

$$w(x,t)=W(x)e^{i\omega t}, \tag{3}$$

where ω is angular frequency. Putting the above into Eq. (1) one can get

$$\frac{d^2}{dx^2}\left[D(x)\frac{d^2W}{dx^2}\right]-m(x)\omega^2W=0, \quad 0\le x\le L. \tag{4}$$

Here L is the beam length. For later convenience, we introduce the following variables:

$$\xi=\frac{x}{L}, \quad k=\omega^2L^4, \tag{5}$$

and Eq. (4) can be rewritten as

$$\frac{d^2}{d\xi^2}\left[D(\xi)\frac{d^2W}{d\xi^2}\right]-km(\xi)W=0, \quad 0\le\xi\le 1, \tag{6}$$

where we still denote $D(x), m(x)$ and $W(x)$ as $D(\xi), m(\xi)$ and $W(\xi)$, respectively, without confusion.

Since natural frequencies are closely related to the end supports of beams, it is instructive to give explicit expressions for relevant physical quantities such as bending moment M, shear force Q, and rotation θ in terms of $W(\xi)$ and its derivatives. For convenience, suppressing the time factor $e^{i\omega t}$, we have

$$\theta = \frac{dW}{L\,d\xi}, \tag{7}$$

$$M = -\frac{D(\xi)}{L^2}\frac{d^2W}{d\xi^2}, \tag{8}$$

$$Q = -\frac{d}{d\xi}\left[\frac{D(\xi)}{L^3}\frac{d^2W}{d\xi^2}\right]. \tag{9}$$

It is worth noting that the relationship (9) between deflection w and shear force Q is different from the classical counterpart, $Q = -Dd^3W/L^3d\xi^3$, unless flexural rigidity $D(\xi)$ is a constant, corresponding to homogeneous and uniform beams.

3. Derivation of Fredholm integral equations

In this section, we introduce a novel method to solve free vibration of axially graded and non-uniform beams. That is, the resulting differential equation (6) with varying coefficients subject to various boundary conditions is converted to an integral equation. To this end, we integrate both sides of Eq. (6) twice with respect to ξ from 0 to ξ, yielding

$$\frac{d}{d\xi}\left[D(\xi)\frac{d^2W}{d\xi^2}\right] - k\int_0^{\xi} m(s)W(s)\,ds = C_1, \tag{10}$$

$$D(\xi)\frac{d^2W}{d\xi^2} - k\int_0^{\xi}(\xi - s)m(s)W(s)\,ds = C_2 + C_1\xi. \tag{11}$$

Furthermore, we repeat to integrate both sides of Eq. (11) twice with respect to ξ from 0 to ξ, yielding

$$D(\xi)\frac{dW}{d\xi} - D'(\xi)W(\xi) + \int_0^{\xi}\left[D''(s) - \frac{1}{2}km(s)(\xi - s)^2\right]W(s)\,ds = \frac{C_1}{2}\xi^2 + C_2\xi + C_3 \tag{12}$$

and

$$D(\xi)W(\xi) + \int_0^{\xi}\left[D''(s)(\xi - s) - 2D'(s) - \frac{1}{6}km(s)(\xi - s)^3\right]W(s)\,ds = \frac{C_1}{6}\xi^3 + \frac{C_2}{2}\xi^2 + C_3\xi + C_4, \tag{13}$$

where the prime stands for the derivative of a function with respect to the argument. In the above integration, C_j $(j = 1, \ldots, 4)$ are integration constants, which can be determined below through given boundary conditions of both ends of the beams. The above four relations are frequently used for various end supports for the determination of C_j. Once these four constants C_j can be uniquely solved and expressed in terms of unknown $W(\xi)$, we substitute these obtained C_j into Eq. (13) and derive an integral equation in $W(\xi)$.

3.1. Simply supported (S–S) beams

We first consider an axially graded and non-uniform beam with simply supported ends. For this case, the corresponding boundary conditions can be stated below:

$$W = 0, \quad M = 0, \quad \xi = 0, 1. \tag{14}$$

Bearing Eq. (8) in mind, application of the conditions $M = 0$ at $\xi = 0, 1$ in (14) to (11) leads to

$$C_2 = 0, \tag{15}$$

$$C_1 + C_2 = -k\int_0^1 (1 - s)m(s)W(s)\,ds. \tag{16}$$

On the other hand, setting $\xi = 0, 1$, respectively, in (13), using the conditions $W = 0$ in (14) one has, respectively,

$$C_4 = 0, \tag{17}$$

$$\int_0^1\left[D''(s)(1 - s) - 2D'(s) - \frac{1}{6}k(1 - s)^3 m(s)\right]W(s)\,ds = \frac{C_1}{6} + \frac{C_2}{2} + C_3 + C_4. \tag{18}$$

Therefore, $C_2 = C_4 = 0$ and C_1, C_3 can be obtained by solving a system of algebraic equations (16) and (18). After getting C_j $(j = 1, \ldots, 4)$, we then substitute them back into (13), and after collection get a Fredholm integral equation as follows:

$$D(\xi)W(\xi) + \int_0^1 K_1(\xi, s)W(s)\,ds + k\int_0^1 K_2(\xi, s)W(s)\,ds = 0, \tag{19}$$

where

$$K_1(\xi,s)=\begin{cases}(\xi-1)[D''(s)s+2D'(s)], & 0\le s\le\xi,\\ \xi[(s-1)D''(s)+2D'(s)], & \xi<s\le 1,\end{cases} \tag{20}$$

$$K_2(\xi,s)=\begin{cases}\dfrac{1}{6}m(s)s(1-\xi)(\xi^2+s^2-2\xi), & 0\le s\le\xi,\\ \dfrac{1}{6}m(s)\xi(1-s)(\xi^2+s^2-2s), & \xi<s\le 1.\end{cases} \tag{21}$$

3.2. Clamped or clamped–clamped (C–C) beams

In this subsection, we will establish an integral equation for an axially graded and non-uniform clamped beam. For this case, the corresponding boundary conditions are

$$W=0,\quad \theta=0,\quad \xi=0,1. \tag{22}$$

In view of $W=0$ and $\theta=0$ at $\xi=0$, from (12) and (13) we have

$$C_3=0,\quad C_4=0. \tag{23}$$

Keeping these results in mind, because of $W=0$ and $\theta=0$ at $\xi=1$, from (12) and (13) we further get

$$\frac{C_1}{2}+C_2=\int_0^1\left[D''(s)-\frac{1}{2}km(s)(1-s)^2\right]W(s)\,\mathrm{d}s, \tag{24}$$

$$\frac{C_1}{6}+\frac{C_2}{2}=\int_0^1\left[D''(s)(1-s)-2D'(s)-\frac{1}{6}km(s)(1-s)^3\right]W(s)\,\mathrm{d}s. \tag{25}$$

We therefore determine all of C_j $(j=1,\ldots,4)$ in terms of $W(s)$ and then insert them back into (13). After some manipulations, we get a Fredholm integral equation for clamped beams as follows:

$$D(\xi)W(\xi)+\int_0^1 K_1(\xi,s)W(s)\,\mathrm{d}s+k\int_0^1 K_2(\xi,s)W(s)\,\mathrm{d}s=0, \tag{26}$$

where

$$K_1(\xi,s)=\begin{cases}(1-\xi)^2[\{D''(s)(\xi-s-2s\xi)-2D'(s)(2\xi+1)\}], & 0\le s\le\xi,\\ \xi^2[D''(s)(3s+\xi-2s\xi-2)-2D'(s)(2\xi-3)], & \xi<s\le 1,\end{cases} \tag{27}$$

$$K_2(\xi,s)=\begin{cases}\dfrac{1}{6}m(s)s^2(1-\xi)^2(s-3\xi+2s\xi), & 0\le s\le\xi,\\ \dfrac{1}{6}m(s)\xi^2(1-s)^2(\xi-3s+2s\xi), & \xi<s\le 1.\end{cases} \tag{28}$$

3.3. Cantilever or clamped–free (C–F) beams

Next, we consider an axially graded and non-uniform cantilever beam with a clamped end at $\xi=0$ and a free end at $\xi=1$, say, where the corresponding boundary conditions read

$$W=0,\quad \theta=0,\quad \xi=0, \tag{29}$$

$$M=0,\quad Q=0,\quad \xi=1. \tag{30}$$

For this case, taking into account (8) and (9), application of the free boundary conditions $M=0$ and $Q=0$ at $\xi=1$ to (10) and (11) leads to

$$C_1=-k\int_0^1 m(s)W(s)\,\mathrm{d}s, \tag{31}$$

$$C_1+C_2=-k\int_0^1(1-s)m(s)W(s)\,\mathrm{d}s, \tag{32}$$

which allow us to express C_1 and C_2 in terms of the integrals of W by solving the above linear equations.

On the other hand, using the clamped boundary condition at $\xi=0$, from Eqs. (12) and (13) in connection with the conditions in (29) we simply get

$$C_3=0,\quad C_4=0. \tag{33}$$

With these obtained C_j $(j=1,\ldots,4)$, from (13) we finally obtain a Fredholm integral equation for cantilever beams as follows:

$$D(\xi)W(\xi)+\int_0^1 K_1(\xi,s)W(s)\,\mathrm{d}s+k\int_0^1 K_2(\xi,s)W(s)\,\mathrm{d}s=0, \tag{34}$$

where

$$K_1(\xi,s)=\begin{cases} D''(s)(\xi-s)-2D'(s), & 0\le s\le \xi,\\ 0, & \xi<s\le 1,\end{cases} \tag{35}$$

$$K_2(\xi,s)=\begin{cases} \dfrac{1}{6}m(s)s^2(s-3\xi), & 0\le s\le \xi,\\ \dfrac{1}{6}m(s)\xi^2(\xi-3s), & \xi<s\le 1.\end{cases} \tag{36}$$

3.4. Clamped–pinned (C–P) beams

Here we consider an axially graded and non-uniform beam with a clamped end and a pinned end. As a representative, we assume the case where the end $\xi=0$ is clamped or fixed, while the other end $\xi=1$ is pinned or simply supported. Thus the corresponding boundary conditions are

$$W=0,\quad \theta=0,\quad \xi=0, \tag{37}$$

$$W=0,\quad M=0,\quad \xi=1. \tag{38}$$

Similar to the treatment of clamped beams, applying the boundary conditions in Eqs. (12) and (13) at $\xi=0$, we can get $C_3=C_4=0$. In addition, using the boundary conditions in (38) on Eqs. (13) and (11), one gets

$$C_1+C_2=-k\int_0^1(1-s)m(s)W(s)\,\mathrm{d}s, \tag{39}$$

$$\frac{C_1}{6}+\frac{C_2}{2}=\int_0^1\left[D''(s)(1-s)-2D'(s)-\frac{1}{6}km(s)(1-s)^3\right]W(s)\,\mathrm{d}s. \tag{40}$$

Solving the above resulting linear equations, we obtain all of C_j, which are then plugged into Eq. (13). After some simplification, the final Fredholm equation is derived as follows:

$$D(\xi)W(\xi)+\int_0^1 K_1(\xi,s)W(s)\,\mathrm{d}s+k\int_0^1 K_2(\xi,s)W(s)\,\mathrm{d}s=0, \tag{41}$$

where

$$K_1(\xi,s)=\begin{cases} \dfrac{1}{2}\xi^2(\xi-3)[D''(s)(1-s)-2D'(s)]+D''(s)(\xi-s)-2D'(s), & 0\le s\le \xi,\\ \dfrac{1}{2}\xi^2(\xi-3)[D''(s)(1-s)-2D'(s)], & \xi<s\le 1,\end{cases} \tag{42}$$

$$K_2(\xi,s)=\begin{cases} \dfrac{1}{12}m(s)s^2(\xi-1)[s(\xi^2-2\xi-2)-3(\xi^2-2\xi)], & 0\le s\le \xi,\\ \dfrac{1}{12}m(s)\xi^2(s-1)[\xi(s^2-2s-2)-3(s^2-2s)], & \xi<s\le 1.\end{cases} \tag{43}$$

3.5. Clamped–guided (C–G) beams

Here we consider an axially graded and non-uniform beam with a guided end and a clamped end. Without loss of generality, the end $\xi=0$ is assumed clamped and the other end $\xi=1$ is guided. That is, the corresponding boundary conditions are

$$W=0,\quad \theta=0,\quad \xi=0, \tag{44}$$

$$\theta=0,\quad Q=0,\quad \xi=1. \tag{45}$$

Applying Eqs. (12) and (13) to the boundary conditions at $\xi=0$ in (44), $C_3=C_4=0$ can be easily obtained. In order to get C_1 and C_2, setting $\xi=1$ in Eqs. (12) and (13) and considering the boundary condition $\theta=0$ in (45), one gets

$$\frac{C_1}{2}+C_2+D'(1)W(1)=\int_0^1\left[D''(s)-\frac{1}{2}km(s)(1-s)^2\right]W(s)\,\mathrm{d}s, \tag{46}$$

$$\frac{C_1}{6}+\frac{C_2}{2}-D(1)W(1)=\int_0^1\left[D''(s)(1-s)-2D'(s)-\frac{1}{6}km(s)(1-s)^3\right]W(s)\,\mathrm{d}s. \tag{47}$$

It is easily found that in the above two equations, $W(1)$ is also unknown. As a consequence, another independent equation is needed for uniquely determining C_1 and C_2. This is achieved by setting $\xi=1$ in Eq. (10). Thus the condition $Q=0$ in (45) allows us to derive

$$C_1=-k\int_0^1 m(s)W(s)\,\mathrm{d}s. \tag{48}$$

After putting the above into (46) and (47), we can determine all of C_j $(j=1,\dots,4)$. Finally, for this case, substituting C_j into Eq. (13), we obtain a Fredholm equation as follows:

$$D(\xi)W(\xi)+\int_0^1 K_1(\xi,s)W(s)\,\mathrm{d}s+k\int_0^1 K_2(\xi,s)W(s)\,\mathrm{d}s=0, \tag{49}$$

where

$$K_1(\xi,s)=\begin{cases}-\dfrac{\xi^2}{2D(1)+D'(1)}\{D''(s)[D(1)+D'(1)(1-s)]-2D'(1)D'(s)\}+D''(s)(\xi-s)-2D'(s), & 0\le s\le\xi,\\ -\dfrac{\xi^2}{2D(1)+D'(1)}\{D''(s)[D(1)+D'(1)(1-s)]-2D'(1)D'(s)\}, & \xi<s\le 1,\end{cases} \tag{50}$$

$$K_2(\xi,s)=\begin{cases}\dfrac{1}{6}m(s)\left\{\xi^3-(\xi-s)^3-\dfrac{\xi^2}{2D(1)+D'(1)}[3D(1)[1-(1-s)^2]+D'(1)[1-(1-s)^3]]\right\}, & 0\le s\le\xi,\\ \dfrac{1}{6}m(s)\left\{\xi^3-\dfrac{\xi^2}{2D(1)+D'(1)}[3D(1)[1-(1-s)^2]+D'(1)[1-(1-s)^3]]\right\}, & \xi<s\le 1.\end{cases} \tag{51}$$

4. Solution of the resulting integral equations

In the preceding section, for several typical beams of interest in practical applications, we have converted the governing differential equation (6) with variable coefficients to the corresponding Fredholm integral equation. For elastic beams of other end supports such as free–free ends, pinned–free ends, etc. the corresponding Fredholm integral equations can be similarly obtained, which are omitted here. It will be seen in the following that the advantage of such transformations lies in that natural frequencies of free vibration of beams with various ends can be exactly calculated. Therefore, the present approach can overcome the drawback of the method of directly solving the governing differential equation (6) with variable coefficients.

For the resulting Fredholm integral equation, many existing techniques may be employed to determine the numerical solution or the approximate solution. For the present problem, it is sufficient to determine characteristic values of the resulting Fredholm integral equation, which is related to natural frequencies of free vibration of beams via (5). Here one of the simplest methods to seek k is invoked. That is, we expand $W(\xi)$ as power series. Or rather, if neglecting sufficiently small error, the unknown $W(\xi)$ can be approximately expanded as

$$W(\xi)=\sum_{n=0}^{N}c_n\xi^n,\quad 0\le\xi\le 1, \tag{52}$$

where c_n are unknown coefficients and N is a certain positive integer, which is chosen large enough such that the rest of the terms have a negligible error. Inserting (52) into the resulting Fredholm integral equation for each case leads to

$$\sum_{n=0}^{N}c_n\xi^nD(\xi)+\sum_{n=0}^{N}c_n\int_0^1K_1(\xi,s)s^n\,\mathrm{d}s+k\sum_{n=0}^{N}c_n\int_0^1K_2(\xi,s)s^n\,\mathrm{d}s=0. \tag{53}$$

We multiply both sides of (53) by ξ^m and then integrate with respect to ξ between 0 and 1, yielding a system of linear algebraic equations in c_n:

$$\sum_{n=0}^{N}(d_{mn}+K_{1mn}+kK_{2mn})c_n=0,\quad m=0,1,2,\dots,N \tag{54}$$

with

$$d_{mn} = \int_0^1 \xi^{m+n} D(\xi)\,d\xi, \tag{55}$$

$$K_{jmn} = \int_0^1 \int_0^1 K_j(\xi, s)\xi^m s^n \,ds\,d\xi, \quad j = 1, 2. \tag{56}$$

To obtain a non-trivial solution of the resulting system, the determinant of the coefficient matrix of the system has to vanish. Accordingly, we obtain a characteristic equation in k

$$\det(d_{mn} + K_{1mn} + kK_{2mn}) = 0. \tag{57}$$

Once a non-trivial solution of the above algebraic equation (54) is sought, which is substituted into (52), we can obtain the corresponding mode shape of free vibration.

5. Numerical results and discussion

A theoretical model for determining natural frequencies of beams with variable flexural rigidity and mass density has been formulated in the preceding section. Here numerical computations are carried out to show the effectiveness of the proposed method.

5.1. A comparison of the results for uniform homogeneous beams

As the first example, we consider the case of a homogeneous beam with uniform cross-section. For this case, $D(\xi) = EI$ and $m(\xi) = \rho A$ are unchanged, and natural frequencies can be exactly calculated from the corresponding frequency equations listed in Appendix A. In order to check the convergence of the suggested method, we have calculated non-dimensional natural frequencies, $\Omega = \omega L^2 \sqrt{\rho A/EI}$, of cantilever beams by taking different N values in (52). Evaluated results of first four non-dimensional natural frequencies, Ω_n, and the exact ones are tabulated in Table 1. By comparison, from Table 1 we find that numerical results have rapid convergence. As N increases from 2 to 6, the numerical results of the first two natural frequencies are identical to the exact results up to four decimal places, which indicates that the present approach is very efficient. However, the accuracy of the results drops with the vibration modes increasing. For the fourth natural frequency, the numerical result with $N = 6$ deviates from the exact one by about 0.08 percent. However, if taking $N = 10$, we find that the numerical and exact results agree up to four decimal places. As a result, higher accuracy can be achieved through increasing N. In the following computations, we take $N = 6$ to calculate the first few order natural frequencies, unless otherwise stated.

5.2. Effects of variable cross-section

The second illustrative example is devoted to Euler–Bernoulli beams with non-uniform cross-section along the length direction. Here keeping the material properties E and ρ constant, two cases are considered. One is a beam with a cross-section of constant height and linearly variable width, which means $A/A_0 = I/I_0 = 1 + \alpha\xi$, α being a geometrical parameter. For this case, Hodges et al. [28] used a finite element-transfer matrix approach and gave natural frequencies of cantilever beams. By using the Rayleigh–Ritz method incorporating Laplace multiplier method, Abrate [18] gave a 10-term Rayleigh–Ritz solution of natural frequencies for such tapered beams. Our numerical results show excellent consistency with the above-mentioned results, and a comparison of the non-dimensional natural frequencies, $\Omega_n = \omega_n L^2 \sqrt{\rho A_0/EI_0}$, for cantilever beams with $\alpha = -0.5$ is given in Table 2. The second case is a beam with a cross-section of constant width and linearly varying height, i.e. $A/A_0 = 1 + \alpha\xi, I/I_0 = (1 + \alpha\xi)^3$. The corresponding non-dimensional natural frequencies are also calculated and listed in Table 3 for clamped–pinned and clamped–clamped beams. Other numerical results derived previously by different approaches including finite element method [29] and modified Rayleigh–Ritz method [18,29] are presented in Table 3. From Table 3, it is seen that our results agree very well with the existing results.

Table 1
First four non-dimensional natural frequencies Ω_n for uniform cantilever beams.

n	Excat [27]	$N = 2$	$N = 4$	$N = 6$	$N = 10$
1	3.5160	3.5171	3.5160	3.5160	3.5160
2	22.0345	22.2334	22.0351	22.0345	22.0345
3	61.6972	118.1444	63.2397	61.7151	61.6972
4	120.9019	–	128.5194	121.1184	120.9019

Table 2
Non-dimensional natural frequencies Ω_n for tapered cantilever beams with $A/A_0 = I/I_0 = 1\text{-}0.5\xi$.

n	[28]	[18]	Present ($N=6$)	Present ($N=10$)
1	4.31517029863	4.31517029864	4.31517029912	4.31517029863
2		23.5192566	23.51926282671	23.51925663968
3		63.199197	63.21647063580	63.19919650267

Table 3
Non-dimensional natural frequencies Ω_n for tapered beams with $A/A_0 = 1+\alpha\xi, I/I_0 = (1+\alpha\xi)^3$.

	α	n	Finite elements [29]	Rayleigh method [29]	Modified Rayleigh method [29]	Modified Rayleigh method [18]	Present ($N=10$)
C–P	−0.1	1	14.92	14.94	14.85	14.848896	14.84889605539
		2				47.637037	47.63703719174
		3				99.171635	99.17165323722
	0	1	15.418	15.45	15.41	15.418206	15.41820571698
		2				49.964862	49.96486203816
		3				104.24770	104.24770194514
	0.1	1	15.997	16.00	15.96	15.9687099	15.96870988416
		2				52.237227	52.23722689317
		3				109.20235	109.20235370558
	0.2	1	16.561	16.58	16.50	16.502899	16.50289889399
		2				54.4614625	54.46146253076
		3				114.051623	114.05163085534
C–C	−0.1	1					21.24097778688
		2					58.55005461550
		3					114.78027750905
	0	1	22.373	22.451	22.375	22.3732854	22.37328544806
		2				61.672823	61.67282294761
		3				120.903392	120.90340027002
	0.1	1	23.521	23.74	23.61	23.479607	23.47960724845
		2				64.721086	64.72106768601
		3				126.87804	126.87805071630
	0.2	1	24.647	25.5	25.13	24.563418	24.5634175326
		2				67.704755	67.7047553184
		3				132.72398	132.7240684027

5.3. *Effects of variable flexural rigidity and mass density*

In what follows, free vibration of axially graded beams with uniform cross-section is considered. For non-homogeneous beams, Elishakoff and Candan presented closed-form solutions to vibrating non-homogeneous beams by the inverse method [9]. We first take some special polynomials, that is, mass density $\rho(\xi)$ and Young's modulus $E(\xi)$ can be represented as polynomial functions:

$$\rho(\xi) = \rho_0 \sum_{j=0}^{J} a_j \xi^j, \quad E(\xi) = E_0 \sum_{h=0}^{H} b_h \xi^h, \tag{58}$$

where J and H are any non-negative integers, a_j $(0 \le j \le J)$ and b_h $(0 \le h \le H)$ are any constants with requirements $\rho(\xi) > 0, E(\xi) > 0$ for all $\xi \in [0, 1]$.

Here for simplicity, the mass density and Young's modulus are chosen such that they satisfy specified forms:

- *Case* A: constant density and specified Young's modulus

$$\rho(\xi)=\rho_0 a_0, \quad E(\xi)=E_0\sum_{h=0}^{H} b_{0h}\xi^h; \tag{59}$$

- *Case* B: linearly varying density and specified Young's modulus

$$\rho(\xi)=\rho_0(a_0+a_1\xi), \quad E(\xi)=E_0\sum_{h=0}^{H} b_{1h}\xi^h; \tag{60}$$

- *Case* C: parabolically varying density and specified Young's modulus

$$\rho(\xi)=\rho_0(a_0+a_1\xi+a_2\xi^2), \quad E(\xi)=E_0\sum_{h=0}^{H} b_{2h}\xi^h. \tag{61}$$

The objective of such a choice is two-fold: one being that the results to be obtained can be compared with the closed-form solution for the chosen polynomials under specified cases, and the other being to show that it is still capable of determining natural frequencies for other cases when the closed-form solution is not available. For comparison, in the present study, we take a_j $(j=0,1,2)$ to be arbitrary constants unless otherwise stated, b_{jh} $(j=0,1,2; h=0,\ldots,j+4)$ are related to a_j $(j=0,1,2)$, and given in Appendix B. For chosen constants a_j and b_{jh}, the closed-form solution of fundamental frequency can be obtained when the difference between the degree number of $E(\xi)$ and $\rho(\xi)$ is equal to 4 by the inverse method [9].

Here we consider the case of cantilever beams. Numerical results of natural frequency parameter $k_n, k_n=\omega^2\rho_0 AL^4/E_0 I$, can be evaluated for arbitrary combination of $\rho(\xi)$ and $E(\xi)$, and those for the first two modes are listed in Table 4. In particular, we find that the fundamental frequency for homogeneous cantilever beams (i.e. $\rho(\xi)=\rho_0, E(\xi)=E_0$) is the same as the exact one [27], whereas the second-order natural frequency is in good agreement with the exact one [27] with error less than 0.003 percent. In addition, for $H=J+4, J=0,1,2$, evaluated fundamental frequencies are identical to the closed-form solutions derived in [9]. Other natural frequencies of arbitrary order vibrational mode are also obtained, but they seem not to be derived in closed form by the inverse method in [9]. From Table 4, when $E(\xi)$ takes $E_0\sum_{h=0}^{H} b_{jh}\xi^h$, the obtained k values have only slight change when $H\geq 2$ for the same mass density, irrespective of the first-order or the second-order natural frequencies. This is attributed to the slight variation of $E(\xi)=E_0\sum_{h=0}^{H} b_{jh}\xi^h$ $(j=0,1,2)$ when $H\geq 2$. It is worth mentioning that in [24], an approximate result with high accuracy has been derived by using the perturbation functional method. Here for the above case we used the present method to obtain the exact result only if $N=6$.

Table 4
Evaluated natural frequency parameter k_n for cantilever beams.

$\rho(\xi)/\rho_0$	$D(\xi)/E_0 I=\sum_{h=0}^{H} b_{0h}\xi^h$	Present results		Exact results	
	H	1st	2nd	1st	2nd
1	0	321.4215	12623.84	321.4215 [27]	12623.5 [27]
	1	357.1973	15615.45		
	2	360.7842	16174.13		
	3	359.9009	15962.74		
	4	360.0000	15994.99	360 [9]	
$1+\xi$	0	443.3508	19861.39		
	1	497.2307	25081.87		
	2	504.2409	26363.87		
	3	504.0932	26321.93		
	4	503.9725	26274.88		
	5	504.0000	26288.47	504 [9]	
$1.5954+0.04\xi+\xi^2$	0	590.2220	26503.05		
	1	662.4000	33454.66		
	2	671.9104	35176.27		
	3	671.7862	35141.46		
	4	672.0588	35246.03		
	5	671.9882	35211.73		
	6	672.0000	35218.59	672 [9]	

The above examples are only related to polynomials. Next, we will demonstrate that the suggested method is also suitable for analyzing the free vibration related to flexural rigidity and mass density of continuously varying functions of ξ other than polynomials. For this purpose, let us consider the case of flexural rigidity and mass density of trigonometric functions:

$$D(\xi)=D_0[1+\alpha\cos(\pi\xi)], \quad \rho(\xi)=\rho_0[1+\beta\cos(\pi\xi)], \tag{62}$$

where α, β are parameters and must satisfy $|\alpha|<1, |\beta|<1$ to insure that $D(\xi)$ and $\rho(\xi)$ are positive. For the above assumptions, we calculate the first three non-dimensional natural frequencies, $\Omega_n=\omega_n L^2\sqrt{\rho_0 A/D_0}$, which are displayed in Tables 5 and 6. It is worth noting that in [13], the fundamental frequency of simply supported beams having the above axial gradient with $\beta=4\alpha$ has been obtained to be $\Omega_1=\pi^2$, which is independent of α values. This analytic result is identical to our result in Table 5. Nevertheless, for other cases of interest in practice, through the present method we also can determine natural frequencies of beams with various end supports for $\beta=4\alpha$ (see Table 5). Also, for the case of $\beta=\alpha$, the corresponding results are shown in Table 6. It is interesting to point out that natural frequencies are sensitive to α, except for the above-mentioned case (i.e. simply supported beams with $\beta=4\alpha$).

Table 5
Non-dimensional natural frequencies Ω_n for beams with $D(\xi)$ and $\rho(\xi)$ given by (62) with $\beta=4\alpha$.

	n	α						
		−0.2	−0.15	−0.1	0	0.1	0.15	0.2
C–F	1	2.5690	2.7572	2.9714	3.5160	4.3428	4.9699	5.8908
	2	20.5462	20.8498	21.1877	22.0345	23.4080	24.6452	27.0597
	3	64.1287	62.8517	62.0246	61.7151	63.5303	65.7357	70.1756
S–S	1	9.8696	9.8696	9.8696	9.8696	9.8696	9.8696	9.8696
	2	42.5405	41.1404	40.1979	39.4791	40.1979	41.1404	42.5522
	3	98.9439	92.4350	90.4469	88.8481	90.3370	92.4350	98.6659
C–P	1	14.2117	14.4917	14.7850	15.4182	16.1235	16.5065	16.9107
	2	51.5819	50.7722	50.2210	49.9742	51.1459	52.4695	54.4812
	3	112.9319	110.0300	108.2707	107.4485	110.4157	113.7307	119.2732
C–C	1	22.3700	22.3715	22.3726	22.3735	22.3726	22.3715	22.3700
	2	64.7658	63.3937	62.4327	61.6883	62.4330	63.3897	64.7668
	3	138.6441	132.6284	131.3240	129.2174	131.2343	132.3560	137.5759

Table 6
Non-dimensional natural frequencies Ω_n for beams with $D(\xi)$ and $\rho(\xi)$ given by (62) with $\beta=\alpha$.

	n	α						
		−0.2	−0.15	−0.1	0	0.1	0.15	0.2
C–F	1	3.0224	3.1416	3.2632	3.5160	3.7853	3.9277	4.0763
	2	21.2069	21.4179	21.6255	22.0345	22.4447	22.6534	22.8668
	3	61.2666	61.3741	61.4838	61.7151	61.9758	62.1192	62.2737
S–S	1	9.8395	9.8528	9.8622	9.8696	9.8622	9.8528	9.8395
	2	39.5239	39.5045	39.4905	39.4791	39.4905	39.5045	39.5239
	3	90.2491	90.2874	90.3149	88.8481	90.3149	90.2874	90.2491
C–P	1	14.9196	15.0527	15.1799	15.4182	15.6367	15.7389	15.8365
	2	49.6719	49.7506	49.8265	49.9742	50.1206	50.1944	50.2691
	3	107.5159	107.5407	107.5357	107.4485	107.2753	107.1613	107.0311
C–C	1	22.2984	22.3316	22.3549	22.3735	22.3549	22.3316	22.2984
	2	61.6542	61.6699	61.6804	61.6883	61.6804	61.6699	61.6542
	3	128.7765	128.9739	129.1098	129.2174	129.1098	128.9739	128.7780

5.4. Effects of axial gradient parameter

Finally we consider a functionally graded beam with axial non-homogeneity, where the cross-sectional area A and moment of inertia I are unchanged. This benefits the design of non-homogeneous beams for certain particular purposes. To show the effect of the non-homogeneity on the natural frequencies, instead of the usual power-law gradient assumption, here we take material properties (such as Young's modulus, mass density, etc.) as follows:

$$Y(\xi)=\begin{cases} Y_1\left(1-\dfrac{e^{\alpha\xi}-1}{e^{\alpha}-1}\right)+Y_2\dfrac{e^{\alpha\xi}-1}{e^{\alpha}-1}, & \alpha\neq 0,\\ Y_1(1-\xi)+Y_2\xi, & \alpha=0, \end{cases} \tag{63}$$

where Y_1 and Y_2 are the corresponding material properties at the ends $\xi=0,1$, respectively, and α is the gradient parameter describing the volume fraction change of both constituents involved. On taking the usual power-law gradient assumption, the calculation of K_{jmn} in (54) inevitably involves divergent integrals; so in the present study, we choose the above gradient form. The variation of $Y(\xi)$ against ξ is shown in Fig. 1 for $Y_2=3Y_1$. In the following calculations, two materials chosen are aluminum and zirconia, the material properties of which are [30]

$$\text{Al}: E_a=70\,\text{GPa}, \quad \rho_a=2702\,\text{kg/m}^3, \tag{64}$$

$$\text{ZrO}_2: E_z=200\,\text{GPa}, \quad \rho_z=5700\,\text{kg/m}^3. \tag{65}$$

Based on the characteristic equation (57), the influence of the gradient parameter α on dimensionless natural frequencies

$$\Omega=\omega L\sqrt{\frac{\rho_a A L^2}{E_a I}} \tag{66}$$

is displayed in Table 7 with $N=6$ for two different cases, one being Al rich near the end $\xi=0$ and ZrO_2 rich near the end $\xi=1$ (Case 1) and the other being Al rich near the end $\xi=1$ and ZrO_2 rich near the end $\xi=0$ (Case 2), respectively. By

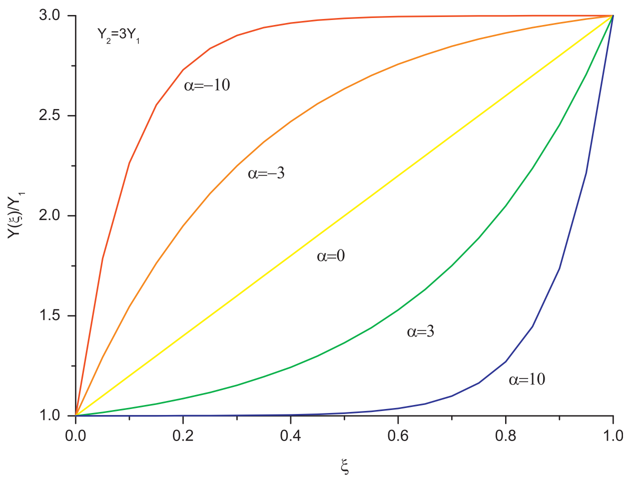

Fig. 1. Variation of the graded material properties governed by (63) with $Y_2=3Y_1$.

Table 7
Non-dimensional fundamental frequencies Ω_1 of axially graded beams.

α	C–F		C–G		S–S		C–P		C–C	
	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2
−10	3.5656	4.1800	5.8206	6.4845	11.4532	9.9358	16.4775	17.2993	24.0576	24.7949
−3	3.1421	4.8317	5.4433	7.0182	11.2443	10.3669	16.0307	17.8701	23.9456	24.9375
0	2.9256	5.0156	5.3275	7.0321	10.8663	10.8663	15.8734	17.9147	24.3752	24.3752
3	2.8544	4.8466	5.3198	6.8127	10.3669	11.2443	15.7171	17.8867	24.9375	23.9456
10	3.0431	4.4629	5.5135	6.5802	9.9358	11.4532	15.4930	17.9050	24.7949	24.0576
HB[a]	3.5160		5.5933		9.8696		15.4182		22.3733	

[a] HB means homogeneous beams.

comparing the results of cases 1 and 2 in Table 7, the gradient parameter plays a dominant role in determining natural frequencies although two constituent phases are unchanged. In addition, the fundamental frequencies are also sensitive to the end supports, as expected. Here for comparison, we also list numerical results of the first natural frequencies of homogeneous beams with the corresponding end conditions. It is noted that for simply supported and clamped beams, the natural frequencies for Case 1 are identical to those for Case 2 with the corresponding $-\alpha$. However, for clamped–pinned, or cantilever beams, such trends do not exist. Generally speaking, for the latter the fundamental frequencies for Case 2 are larger than those for Case 1 for the same gradient parameter α.

6. Conclusions

A new approach has been presented to solve free vibration of Euler–Bernoulli beams with continuously varying flexural rigidity and mass density. Instead of directly solving the fourth-order governing differential equation with variable coefficients, for various beams we transformed the corresponding problem to Fredholm integral equations. By expanding the mode shapes as power series, the resulting Fredholm integral equations were solved. The existence condition of a non-trivial solution permitted us to obtain natural frequencies. The effectiveness of the method has been confirmed by comparing our numerical results with those available for special cases, including rectangular Euler–Bernoulli beams with linearly varying width or depth and graded beams with polynomial and trigonometric function gradient. Our suggested approach is capable of treating arbitrarily axial gradient and varying cross-section. In particular, we illustrated the effects of the gradient parameter on the fundamental frequency of free vibration of an Al/ZrO_2 composite beam under various end supports.

Acknowledgment

This work was supported by the National Natural Science Foundation of China (Grant no. 10672189).

Appendix A

The characteristic equations of free vibration of uniform Euler–Bernoulli beams for various end supports are [27]

$$\text{simply supported beams}: \sin\sqrt{\Omega} = 0,$$

$$\text{clamped-clamped beams}: \cos\sqrt{\Omega}\cosh\sqrt{\Omega} - 1 = 0,$$

$$\text{cantilever beams}: \cos\sqrt{\Omega}\cosh\sqrt{\Omega} + 1 = 0,$$

$$\text{clamped-pinned beams}: \tan\sqrt{\Omega} - \tanh\sqrt{\Omega} = 0,$$

$$\text{clamped-guided beams}: \tan\sqrt{\Omega} + \tanh\sqrt{\Omega} = 0$$

with

$$\Omega = \omega L^2 \sqrt{\frac{\rho A}{EI}}.$$

Appendix B

The following constants are taken from [9] with only slight form change:

$$b_{00} = 26a_0, \quad b_{01} = 16a_0, \quad b_{02} = 6a_0, \quad b_{03} = -4a_0, \quad b_{04} = a_0,$$

$$b_{10} = \frac{2(71a_1 + 91a_0)}{5}, \quad b_{11} = \frac{2(51a_1 + 56a_0)}{5}, \quad b_{12} = \frac{2(31a_1 + 21a_0)}{5},$$

$$b_{13} = \frac{2(11a_1 - 14a_0)}{5}, \quad b_{14} = \frac{-18a_1 + 7a_0}{5}, \quad b_{15} = a_1,$$

$$b_{20} = \frac{465a_2 + 568a_1 + 728a_0}{15}, \quad b_{21} = \frac{2(181a_2 + 204a_1 + 224a_0)}{15},$$

$$b_{22} = \frac{259a_2 + 248a_1 + 168a_0}{15}, \quad b_{23} = \frac{4(39a_2 + 22a_1 - 28a_0)}{15},$$

$$b_{24} = \frac{53a_2 - 72a_1 + 28a_0}{15}, \quad b_{25} = \frac{2(2a_1 - 5a_2)}{3}, \quad b_{26} = a_2.$$

References

[1] M. Koizumi, The concept of FGM, *Ceramic Transactions on Functional Grade Materials* 34 (1993) 3–10.
[2] V. Birman, L.W. Byrd, Modeling and analysis of functionally graded materials and structures, *Applied Mechanics Review* 60 (2007) 195–216.
[3] B.V. Sankar, An elasticity solution for functionally graded beams, *Composite Science and Technology* 61 (2001) 689–696.
[4] S. Kapuria, M. Bhattacharyya, A.N. Kumar, Bending and free vibration response of layered functionally graded beams: a theoretical model and its experimental validation, *Composite Structures* 82 (2008) 390–402.
[5] X.-F. Li, A unified approach for analyzing static and dynamic behaviors of functionally graded Timoshenko and Euler–Bernoulli beams, *Journal of Sound and Vibration* 318 (2008) 1210–1229.
[6] M.A. Benatta, I. Mechab, A. Tounsi, E.A.A. Bedia, Static analysis of functionally graded short beams including warping and shear deformation effects, *Computational Materials Science* 44 (2008) 765–773.
[7] S.A. Sina, H.M. Navazi, H. Haddadpour, An analytical method for free vibration analysis of functionally graded beams, *Materials and Designs* 30 (2009) 741–747.
[8] Y.-A. Kang, X.-F. Li, Bending of functionally graded cantilever beam with power-law non-linearity subjected to an end force, *International Journal of Non-Linear Mechanics* 44 (2009) 696–703.
[9] I. Elishakoff, S. Candan, Apparently first closed-form solutions for vibrating inhomogeneous beams, *International Journal of Solids and Structures* 38 (2001) 3411–3441.
[10] I. Elishakoff, S. Candan, Apparently first closed-form solution for frequencies of deterministically and/or stochastically inhomogeneous simply supported beams, *Journal of Applied Mechanics* 68 (2001) 176–185.
[11] I. Elishakoff, Z. Guede, Analytical polynomial solutions for vibrating axially graded beams, *Mechanics of Advanced Materials and Structures* 11 (2004) 517–533.
[12] L. Wu, Q. Wang, I. Elishakoff, Semi-inverse method for axially functionally graded beams with an anti-symmetric vibration mode, *Journal of Sound and Vibration* 284 (2005) 1190–1202.
[13] I. Calio, I. Elishakoff, Closed-form trigonometric solutions for inhomogeneous beam-columns on elastic foundation, *International Journal of Structural Stability and Dynamics* 4 (2004) 139–146.
[14] I. Calio, I. Elishakoff, Closed-form solutions for axially graded beam-columns, *Journal of Sound and Vibration* 280 (2005) 1083–1094.
[15] Q.S. Li, A new exact approach for determining natural frequencies and mode shapes of non-uniform shear beams with arbitrary distribution of mass or stiffness, *International Journal of Solids and Structures* 37 (2000) 5123–5141.
[16] Q.S. Li, Classes of exact solutions for buckling of multi-step non-uniform columns with an arbitrary number of cracks subjected to concentrated and distributed axial loads, *International Journal of Engineering Science* 41 (2003) 569–586.
[17] P.A.A. Laura, B.V. De Greco, J. Utjes, R. Carnicer, Numerical experiments on free and forced vibrations of beams of non-uniform cross section, *Journal of Sound and Vibration* 120 (1988) 587–596.
[18] S. Abrate, Vibration of non-uniform rods and beams, *Journal of Sound and Vibration* 185 (1995) 703–716.
[19] H.H. Mabie, C.B. Rogers, Transverse vibrations of tapered cantilever beams with end loads, *Journal of the Acoustical Society of America* 36 (1964) 463–469.
[20] B.M. Kumar, R.I. Sujith, Exact solutions for the longitudinal vibration of non-uniform rods, *Journal of Sound and Vibration* 207 (1997) 721–729.
[21] S. Naguleswaran, Vibration of an Euler–Bernoulli beam of constant depth and with linearly varying breadth, *Journal of Sound and Vibration* 153 (1992) 509–522.
[22] S. Naguleswaran, A direct solution for the transverse vibration of Euler Bernoulli wedge and cone beams, *Journal of Sound and Vibration* 172 (1994) 289–304.
[23] K.V. Singh, G. Li, S.S. Pang, Free vibration and physical parameter identification of non-uniform composite beams, *Composite Structures* 74 (2006) 37–50.
[24] S. Nachum, E. Altus, Natural frequencies and mode shapes of deterministic and stochastic non-homogeneous rods and beams, *Journal of Sound and Vibration* 302 (2007) 903–924.
[25] L.M. Keer, B. Stahl, Eigenvalue problems of rectangular plates with mixed edge conditions, *ASME Journal of Applied Mechanics* 39 (1972) 513–520.
[26] Y.A. Melnikov, *Influence Functions and Matrices*, Marcel Dekker, New York, 1999.
[27] W. Weaver Jr., S.P. Timoshenko, D.H. Young, *Vibration Problems in Engineering*, Wiley-Interscience, New York, 1990.
[28] D.H. Hodges, Y.Y. Chung, X.Y. Shang, Discrete transfer matrix method for non uniform rotating beams, *Journal of Sound and Vibration* 169 (1994) 276–283.
[29] V.H. Cortinez, P.A.A. Laura, An extension of Timoshenko's method and its application to buckling and vibration problems, *Journal of Sound and Vibration* 169 (1994) 141–144.
[30] S.S. Vel, R.C. Batra, Three-dimensional exact solution for the vibration of functionally graded rectangular plates, *Journal of Sound and Vibration* 272 (2004) 703–730.